Weighted $L^2$-cohomology of Coxeter groups

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Suppose $(W, S)$ is a Coxeter system and $q$ is a positive real number. $R^{(W)}$ denotes the Euclidean space of finitely supported real-valued functions on $W$ and $\{e_w\}_{w \in W}$ its standard basis. Define an inner product $\langle \cdot, \cdot \rangle_q$ by

$$\langle e_v, e_w \rangle_q = \begin{cases} q^{l(v)} & \text{if } v = w, \\ 0 & \text{if } v \neq w. \end{cases}$$

$L^2_q(W)$ denotes the Hilbert space completion of $R^{(W)}$ with respect to this inner product. Deform the multiplication in the group algebra $R^{(W)}$ by

$$e_s e_w = \begin{cases} e_{sw} & \text{if } l(sw) > l(w), \\ qe_{sw} + (q-1)e_w & \text{if } l(sw) < l(w). \end{cases}$$

This defines a new multiplication on $R^{(W)}$ making it into an associative algebra $R_q W$ called the Hecke algebra (see [1, Ex. 22, pp. 56–57]). Define a linear involution $x \mapsto x^*$ of $R_q W$ by $(e_w)^* := e_{w^{-1}}$. $R_q W$ acts on $L^2_q(W) = L^2_q(W)$ by either left or right multiplication. It is easily checked that $*$ is an anti-involution of the algebra $R_q W$ and that $\langle xy, z \rangle_q = \langle y, x^*z \rangle_q = \langle x, zy^* \rangle_q$. So, $R_q W$ defines two $C^*$-algebras of operators (by left or right multiplication); the weak closure of either one is denoted $\mathcal{N}_q$. It is a von Neumann algebra, called the Hecke - von Neumann algebra. The trace of an element $\phi \in \mathcal{N}_q$ is defined by the usual formula: $\text{tr} \phi := \langle \phi(e_1), e_1 \rangle_q$. This extends to definition of a trace for any $\mathcal{N}_q$-endomorphism of a finite direct sum of copies of $L^2_q$. Hence, if $V \subset \oplus L^2_q$ is any closed $\mathcal{N}_q$-submodule we can define its von Neumann dimension by $\text{dim} V := \text{tr} p_V$, where $p_V : \oplus L^2_q \to \oplus L^2_q$ is orthogonal projection onto $V$.

The von Neumann dimensions of some important $\mathcal{N}_q$-modules are tied to the growth series of $W$, i.e., to the power series defined by $W(t) := \sum_{w \in W} q^{l(w)}$. Its radius of convergence is denoted by $\rho$. $W(t)$ is a rational function of $t$. One way to see this is simply to prove a formula such as

$$1 \quad \frac{1}{W(t)} = \sum_{T \subset S} (-1)^{|T|} W_T(t^{-1}).$$

where for any $T \subset S$, $W_T := (T)$ is the special subgroup generated by $T$ and $S := \{ T \subset S \mid |W_T| < \infty \}$ is the poset of spherical subsets. Formula (1) shows that $W(t)$ is a rational function, since the growth series of any finite group, such as a spherical subgroup $W_T$, is a polynomial. Hecke algebras (resp. growth series) make sense when $q$ (resp. $t$) is replaced by a certain $1$-tuple $q = (q_i)$ (resp. $t = (t_i)$), where we are given a function $i : S \to I$ which is constant on conjugacy classes of elements of $S$. Given $s \in S$, write $q_s$ (resp. $t_s$) instead of $q_{i(s)}$ (resp. $t_{i(s)}$). Then $q^{l(w)}$ (resp. $t^{l(w)}$) is replaced by $q_w := q_{s_1} \cdots q_{s_n}$ (resp. $t_w$) where $w = s_1 \cdots s_n$ is any reduced expression for $w$. 


Two important self-adjoint idempotents in $N_q$ are

$$a_T := \frac{1}{W_T(q)} \sum_{w \in W_T} e_w \quad \text{and} \quad h_T := \frac{1}{W_T(q^{-1})} \sum_{w \in W_T} (-1)^{l(w)} q^{-l(w)} e_w.$$ 

Right multiplication by $a_T$ (resp. $h_T$) is a well-defined, bounded linear operator precisely when $q < \rho_T$ (resp. $q > \rho_T^{-1}$), where $\rho_T$ is the radius of convergence of $W_T(t)$. In particular, both are defined when $T$ is spherical. These idempotents define subspaces, $A_T := L^2_q a_T$ and $H_T := L^2_q h_T$ of dimension $1/W_T(q)$ and $1/W_T(q^{-1})$, respectively.

Let $\Sigma$ be the standard piecewise Euclidean CAT(0) complex on which $W$ acts properly and isometrically (e.g., see [2]). It is defined by pasting together copies of a certain fundamental domain $K$, one for each element of $W$. $C^*(\Sigma)$ is its cellular cochain complex, i.e., $C^i(\Sigma) = \{ f : \{i\text{-cells}\} \to \mathbb{R} \}$ and $C^i_c(\Sigma) := \{\text{finitely supported} f\}$ is the subcomplex of finitely supported cochains. We can define a weighted inner product on $C^i_c(\Sigma)$ similar to the one discussed above. Given a cell $\sigma$ of $\Sigma$, let $w(\sigma)$ be the element $w$ of shortest length such that $w^{-1} \sigma \subset K$. Assign a weight to the characteristic function $e_\sigma$ of the cell so that its length is $q^{l(w(\sigma))}$. Its Hilbert space completion is denoted $L^2_q C^i_c(\Sigma)$. As $q$ varies between 0 and infinity it interpolates between $C^i(\Sigma)$ and $C^i_c(\Sigma)$. $L^2_q C^i(\Sigma)$ inherits the structure of an $N_q$-module in a straightforward fashion so that the coboundaries are maps of $N_q$-modules. Its (reduced) cohomology groups are denoted $L^2_q H^i(\Sigma)$. We compute them as $N_q$-modules for $q \leq \rho$ and for $q > \rho^{-1}$. The von Neumann dimension of $L^2_q H^i(\Sigma)$ is denoted $b^i_q(\Sigma)$ and called the $i^{th}$ $L^2_q$-Betti number. When $q$ is an integer (or when $q$ is an $I$-tuple of integers), these numbers are equal to the ordinary $L^2$-Betti numbers of any building of type $(W, S)$ and thickness $q$ (or $q I$) with a chamber transitive automorphism group. (Thickness $q$ means that $q + 1$ chambers meet along each mirror.) The $L^2_q$-Euler characteristic is the alternating sum of the $b^i_q(\Sigma)$.

**Theorem 1.** (Dymara [6].) $\chi_q(\Sigma) = 1/W(q)$.

This is proved in the usual fashion by calculating the alternating sum of the dimensions of the spaces of cochains on orbits of cells. There is a cellulation of $\Sigma$ with one orbit of cells for each $T \in \Sigma$; the dimension of the space of cochains on this orbit is $\dim H_T = 1/W_T(q^{-1})$; so, the theorem follows from (1). The next result of Dymara says that for $q < \rho$, $L^2_q$-cohomology behaves like ordinary cohomology.

**Theorem 2.** (Dymara [6].) For $q < \rho$, $L^2_q H^*(\Sigma)$ is concentrated in dimension 0 and is isomorphic to $\mathbb{R}$ as a vector space. Its von Neumann dimension is given by $b^0_q(\Sigma) = 1/W(q)$. Moreover, for $q > \rho$, $b^0_q(\Sigma) = 0$.

For each $T \in \mathcal{S}$, let $A_{> T}$ be the $N_q$-submodule of $A_T$ defined by $A_{> T} = \sum_{U \in S_{< T}} A_U$. Put $D_T := A_T/A_{> T}$. Using Theorem 2 we prove the following generalization of a result of L. Solomon [8] when $W$ is finite.
The Decomposition Theorem. ([3]). For $q > \rho^{-1}$ and any $T \in \mathcal{S}$, \[ A_T = \bigoplus_{U \in \mathcal{S}_{>T}} D_U. \]

This leads directly to the following.

The Main Theorem. ([3]). For $q > \rho^{-1}$, we have an isomorphism of $N_q$-modules \[ L^2_q H^\ast(\Sigma) \cong \bigoplus_{T \in \mathcal{S}} H^\ast(K, K^{S-T}) \otimes D_T. \]

Here the fundamental domain $K$ is contractible complex, it is equipped with a family $\{K_s\}_{s \in \mathcal{S}}$ of closed contractible subcomplexes ("mirrors") indexed by $\mathcal{S}$. $K^{S-T}$ denotes the union of those mirrors which are indexed by $S-T$. $H^\ast(K, K^{S-T})$ is ordinary cohomology.

In [2, 4] we carry out a similar calculation for $H^\ast_c(\Sigma)$. A consequence is that the inclusion $C^\ast_c(\Sigma) \hookrightarrow L^2_q C^\ast(\Sigma)$ induces an injection with dense image, $H^\ast_c(\Sigma) \hookrightarrow L^2_q H^\ast(\Sigma)$. So, in this sense weighted $L^2$-cohomology is like compactly supported cohomology in the range $q > \rho^{-1}$. The big question is what happens in the intermediate range $\rho < q < \rho^{-1}$?

References