Action dimensions of some simple complexes of groups

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The *geometric dimension* of a discrete torsion-free group G is the smallest dimension of a model for BG by a CW complex. This number equals $\operatorname{cd} G$, the cohomological dimension of G, provided $\operatorname{cd} G \neq 2$. The *action dimension* of G, denoted $\operatorname{actdim} G$ is the smallest dimension of a model for BG by a manifold with boundary (possibly empty boundary). In other words, $\operatorname{actdim} G$ is the minimum dimension of a thickening of a model for BG to a manifold. It follows that $\operatorname{gdim} G \leq \operatorname{actdim} G$ with equality if and only if BG is homotopy equivalent to a closed manifold. On general principles, $\operatorname{actdim} G \leq 2 \operatorname{gdim} G$.

In [2], Bestvina, Kapovich, and Kleiner defined a number which is a lower bound for the action dimension of G called its "obstructor dimension," denoted $\operatorname{obdim} G$. It is based on the classical van Kampen obstruction for embedding a simplicial complex K into a euclidean space of given dimension. Let $\mathcal{C}(K)$ denote the configuration space of unordered pairs of distinct points in K, i.e., if $\Delta \subset K \times K$ is the diagonal, then $\mathcal{C}(K)$ is the quotient of $(K \times K) - \Delta$ by the free involution which interchanges the factors. Let $c : \mathcal{C}(K) \to \mathbf{R}P^\infty$ classify the double cover and let w_1 be the generator of $H^1(\mathbf{R}P^\infty; \mathbb{Z}/2)$. The *van Kampen obstruction* of K in degree m is the cohomology class $\operatorname{vk}^m(K) := c^*(w_1)^m \in H^m(\mathcal{C}(K); \mathbb{Z}/2)$. It is an obstruction to embedding K in $\mathbf{R}^m$. Let $\operatorname{Cone} K$ stand for the cone of infinite radius with base K, i.e., $K \times [0, \infty)/K \times 0$. The simplicial complex K is an *m-obstructor* for G if $\operatorname{Cone} K$ coarsely embeds in EG (or, more precisely, in some contractible geodesic space on which G acts properly) and if $\operatorname{vk}^m(K) \neq 0$. Define $\operatorname{obdim} G = 2 + \max\{m \mid \text{there is an } m\text{-obstructor } K \text{ for } G\}$. The main result of [2] is that $\operatorname{obdim} G \leq \operatorname{actdim} G$.

A *simple complex of groups* over a poset $\mathcal{Q}$ is a functor $G\mathcal{Q}$ from $\mathcal{Q}$ to the category of groups and monomorphisms. The group G_σ associated to the element $\sigma \in \mathcal{Q}$ is called the *local group* at σ. One can glue together the classifying spaces BG_σ, with $\sigma \in \mathcal{Q}$, to form a space $BG\mathcal{Q}$ called the *aspherical realization* of $G\mathcal{Q}$. Assume the geometric realization of $\mathcal{Q}$ is simply connected. Then van Kampen's Theorem implies that $\pi_1(BG\mathcal{Q})$ is the direct limit G of $G\mathcal{Q}$. The $K(G,1)$-Question for $G\mathcal{Q}$ asks if $BG\mathcal{Q}$ is aspherical, i.e., if it is homotopy equivalent to BG. If, whenever $\tau < \sigma$, we have $\dim BG_\tau < \dim BG_\sigma$, then $\dim BG\mathcal{Q} = \max\{\dim BG_\sigma \mid \sigma \in \mathcal{Q}\}$. If the answer to the $K(G,1)$-Question is affirmative, then, supposing each BG_σ has minimum dimension $\operatorname{gdim} G_\sigma$, we see $\operatorname{gdim} G = \max\{\operatorname{gdim} G_\sigma \mid \sigma \in \mathcal{Q}\}$. Given models for the BG_σ by manifolds with boundary M_σ and "dual disks" D_σ, one can glue together the $M_\sigma \times D_\sigma$ to get a manifold model for $BG\mathcal{Q}$. Hence, when the $K(G,1)$-Question has a positive answer, we get a manifold model for BG and hence, an upper bound for $\operatorname{actdim} G$. Note that these methods only work when the $K(G,1)$-Question has a positive answer.

In most applications $\mathcal{Q}$ will be a poset of the form $\mathcal{S}(L)$, where L is a simplicial complex and $\mathcal{S}(L)$ means the poset of simplices in L (including the empty simplex).

Our first two examples of such simple complexes of groups are "Artin complexes" and "graph product complexes".

One associates to a Coxeter system (W, S) a simplicial complex L called its "nerve". The vertex set of L is S and its simplices are the subsets of S which generate finite subgroups (such subgroups are said to be "spherical" Coxeter groups). There is an Artin group A associated to (W, S) and for each $\sigma \in \mathcal{S}(L)$ there is an associated "spherical" Artin group A_σ. The functor $\sigma \mapsto A_\sigma$ defines a simple complex of groups $A\mathcal{S}(L)$ called the *Artin complex.* The direct limit of $A\mathcal{S}(L)$ is the *Artin group* A associated to (W, S). The $K(\pi, 1)$-Question for $A\mathcal{S}(L)$ was raised in [4]. Although the answer is not known in general, it was proved in [4] that the answer is positive whenever L is a flag complex. In particular, since L is a flag complex whenever A is right-angled (i.e., when A is a "RAAG"), the $K(\pi, 1)$-Question has a positive answer for RAAGs.

Given a simplicial complex L of dimension d, there is another simplicial complex OL of the same dimension called the *octahedralization* of L. It is formed formed by replacing each simplex of L by a join of 0-spheres. In the case of a RAAG A associated to a flag complex L, OL is the link of a vertex in the standard model for BA as a union of tori.

Lemma. ([1, Theorems 5.1 and 5.2]). *Suppose L is a flag complex of dimension d. Then $\mathrm{vk}^{2d}(OL) \neq 0$ if and only if $H_d(L; \mathbb{Z}/2) \neq 0$, . (Hence, OL is a $2d$-obstructor for the associated* RAAG.*)*

In the case of a general Artin group A, the relevant obstructor is defined by the simplicial complex $L_\oslash$ of standard abelian subgroups of A(cf. [6]). It is a certain subdivision of L. Moreover, the cone on its octahedralization coarsely embeds in EA. By the Lemma, if $H_d(L; \mathbb{Z}/2) \neq 0$, then $\mathrm{vk}^{2d}(OL_\oslash) \neq 0$ and hence, $\mathrm{obdim}\, A = 2d + 2$. Using this and the gluing method in Le's thesis [9] we get the following.

Theorem 1. *Suppose the $K(\pi, 1)$-Question has a positive answer for the Artin complex $A\mathcal{S}(L)$. Let d denote the dimension of L.*

(1) *(Proved in* [6]*). If $H_d(L; \mathbb{Z}/2) \neq 0$, then* $\mathrm{actdim}\, A = \mathrm{obdim}\, A = 2d + 2 = 2\, \mathrm{gdim}\, A$.
(2) *(Proved in* [9]*). If L embeds is some d-dimensional contractible complex (this implies $H_d(L; \mathbb{Z}/2) \neq 0$), then* $\mathrm{actdim}\, A \leq 2d + 1$.

Suppose $\{G_v\}_{v \in \mathrm{Vert}\, L^1}$ is a collection of groups indexed by the vertex set of a simplicial graph L^1. Let L be the flag complex (i.e., the clique complex) of L^1. For each $\sigma \in \mathcal{S}(L)$, G_σ denotes the product of the G_v, with $v \in \mathrm{Vert}\, \sigma$. There is simple complex of groups $G\mathcal{S}(L)$ over $\mathcal{S}(L)$, called the *graph product complex*, defined by $\sigma \mapsto G_\sigma$. The group $G := \lim G\mathcal{S}(L)$ is the *graph product* of the G_v. Note that G_v and G_w commute in G if and only if $\{v, w\}$ spans an edge of L^1. When each G_v is equal to $\mathbb{Z}/2$, then G is the *right-angled Coxeter group* associated to L^1. When each G_v is infinite cyclic, G is the RAAG associated to L^1.

Consider the case, where each BG_v is a closed aspherical m-manifold M_v. Provided that we also assume that the ideal boundary of the universal cover of each

M_v is S^{m-1}, then the appropriate candidate for an obstructor is the complex $\tilde{O}L$ formed by taking the polyhedral join of $(m-1)$-spheres over L. Its dimension is $\tilde{O}L = m(d+1)-2$. We show that $\tilde{O}L$ is an $((m+1)(d+1)-2)$-obstructor for G if and only if $H_d(L;\mathbb{Z}/2) \neq 0$. This gives the following.

Theorem 2. *Suppose G is a graph product of closed m-manifold groups as above. Then* $\operatorname{gdim} G = m(d+1)$*. Moreover,*

(1) *If $H_d(L;\mathbb{Z}/2) \neq 0$, then* $\operatorname{actdim} A = \operatorname{obdim} A = (m+1)(d+1)$*.*
(2) *If L embeds is some d-dimensional contractible complex, then* $\operatorname{actdim} G \leq (m+1)(d+1)-1$*.*

Details for these theorems and related results will appear in [7].

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