A Hyperbolic 4-manifold

by

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Abstract. There is a regular 4-dimensional polyhedron with 120 dodecahedra as 3-dimensional faces. (Coxeter calls it the "120-cell.") The group of symmetries of this polyhedron is the Coxeter group with diagram $\bullet\overset{5}{\text{———}}\bullet\text{———}\bullet\text{———}\bullet$. For each pair of opposite 3-dimensional faces of this polyhedron there is a unique reflection in its symmetry group which interchanges them. The result of identifying opposite faces by these reflections is a hyperbolic manifold M^4.

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0. Introduction. Let us first recall a well-known method of constructing a Riemann surface from a regular polygon. Suppose that m is an odd integer ≥ 3 and that T is a triangle in the Euclidean plane (if $m = 3$) or the hyperbolic plane (if $m > 3$) with angles $\pi/2$, π/m and $\pi/2m$ at its vertices A, B and C, respectively. (See Figure 1.) For any vertex v of T let r_v denote the orthogonal reflection across the line supported by the edge opposite v. Let G be the group generated by r_A, r_B, and r_C and let D_{2m} be the subgroup generated by r_A and r_B. The group D_{2m} is the dihedral group of order $4m$ and the translates of T under D_{2m} fit together at C to form a regular 2m-gon P with interior angle $2\pi/m$. Each pair of opposite edges in P is interchanged by a unique reflection in D_{2m}. If we identify the edges of P by these reflections, we obtain a closed surface. If $m = 3$, then P is a hexagon and the surface is a torus; otherwise, it is hyperbolic of Euler characteristic $3 - m$. In fact, this surface is the orbit space of a certain torsion-free normal subgroup of G acting on the appropriate plane, and G is the semi-direct product of D_{2m} with this normal subgroup. An explicit homomorphism from G to D_{2m} can be described as follows. Let s be the reflection across the line L through C which is orthogonal to AC. From the fact that BC and L make an angle of $\pi/2 - \pi/2m$, it is easy to see that the mapping $r_A \to r_A$, $r_B \to r_B$ and $r_C \to s$ extends to a surjective homomorphism $G \to D_{2m}$. It is also not hard to convince onself that the kernel is torsion-free and that it has orbit space P with opposite edges identified.

Reflection groups in hyperbolic n-space, $n > 2$, with fundamental chamber an n-simplex were classified by Lanner [7] in 1950 (see also exercise 15, pp. 113 in [3].) There are 9 such groups in dimension 3, 5 in dimension 4, and none in higher dimensions. For one of the 4-dimensional reflection groups there is a construction completely analogous to the 2-dimensional construction discussed above. The remainder of the paper is devoted to the description of this 4-dimensional example.

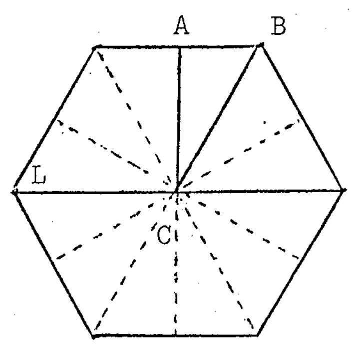
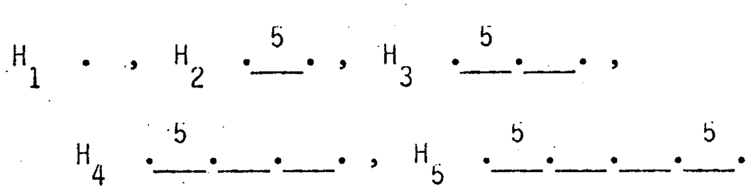


Figure 1.

1. <u>Some Coxeter groups</u>. For $1 \leq n \leq 5$, let H_n denote the Coxeter group of rank n with diagram as indicated below:



Obviously, $H_1 \subset H_2 \subset H_3 \subset H_4 \subset H_5$. The first four of these groups are finite and have canonical representations as subgroups of $O(n)$ (cf. [3]). In fact, for $1 \leq n \leq 3$, H_{n+1} is the group of isometries of S^n generated by the orthogonal reflections across the faces of a certain spherical n-simplex Δ^n (a "fundamental chamber"). The group H_5 can be represented as a discrete cocompact subgroup of $O(4, 1)$, the group of isometries of hyperbolic 4-space $\mathbb{H}^4$ (Ex. 15, p. 133 in [3]). Its fundamental chamber is a certain hyperbolic 4-simplex Δ^4. For $1 \leq n \leq 4$, let x_n be a vertex of Δ^n such that the isotropy subgroup of H_{n+1} at x_n is H_n. The translates of Δ^n under H_n fit together at x_n to give the barycentric subdivision of a convex polyhedron X^n (in S^n if $n \leq 3$ or in $\mathbb{H}^4$ if $n = 4$). The translates of X^n under H_{n+1} then give a tessellation of S^n $(n \leq 3)$ or $\mathbb{H}^4$ $(n = 4)$ by congruent copies of X^n. The full group of symmetries of this tessellation is H_{n+1}. For $1 \leq n \leq 3$, the convex hull of the vertex set of this tessellation of S^n is a convex polyhedron Y^{n+1} in $\mathbb{R}^{n+1}$. X^1 is a circular arc (of length $2\pi/5$), S^1 is tessellated by 5 copies of it, Y^2 is a pentagon. X^2 is a spherical pentagon, S^2 is tessellated by 12 copies of it, Y^3 is a dodecahedron. X^3 is a spherical dodecahedron, S^3 is tessellated by 120 copies of it, Y^4 is the 4-dimensional regular polyhedron called the "120-cell" in [5]. X^4 is a hyperbolic 120-cell and $\mathbb{H}^4$ is tessellated by an infinite number of copies of it (cf. [4]). The orders of these Coxeter groups are as follows: $|H_1| = 2$, $|H_2| = 10$, $|H_3| = 120$, $|H_4| = 14400$ $(= (120)^2)$, $|H_5| = \infty$.

The group H_{n+1} and its fundamental chamber Δ^n can be recovered from the tessellation as follows. Choose a chain of cells in the tessellation:

the tessellation as follows. Choose a chain of cells in the tessellation: $C^0 \subset C^1 \subset \ldots C^n = X^n$, where C^j has dimension j. Let v_j denote the center of C^j (so that $v_n = x_n$). The vertices $v_0, \ldots, v_n$ span an n-simplex which we can take to be Δ^n. For $0 \leq j \leq n$, let r_j denote the reflection across the "hyperplane" (i.e. great sub-sphere if $1 \leq n \leq 3$ or hyperbolic hyperplane if $n = 4$) supported by the face of Δ^n which is opposite to v_j. The family $(r_j)_{0<j<n}$ is denoted by $R(\Delta^n)$ and called a "fundamental system of reflections" for H_{n+1}. Its elements correspond to the nodes of the corresponding diagram (where the nodes are numbered from left to right).

2. <u>The tessellation of S^3 by dodecahedra and its symmetry group.</u> In this section we shall discuss several facts about H_4. A good general reference for Coxeter groups is [3]; in particular, Exercise 12 on p. 231 gives an interesting method of proving some properties of H_4.

Let $\mathcal{D}$ be the set of dodecahedra in the previously mentioned tessellation of S^3. For each $D \in \mathcal{D}$, let

$-D$ = the face opposite to D,

v_D = the center of D,

S_D = the great 2-sphere orthogonal to v_D,

s_D = the orthogonal reflection across S_D.

Clearly, $s_D = s_{D'}$ if and only if $D' = D$ or $-D$.

(2.1) For each $D \in \mathcal{D}$, the reflection s_D belongs to H_4 . Moreover, $(s_D)_{D \in \mathcal{D}}$ is the family of all reflections in H_4 .

The above fact is proved in [5], p.227. (Alternatively, it follows from the exercise in [3], mentioned above.)

Next suppose, as in Section 1, that $C^0 \subset C^1 \subset C^2 \subset C^3 = X^3$ is a chain of cells, and that for $0 \leq j \leq 3$, v_j is the center of C^j , S_j is the 2-sphere supported by the face of Δ^3 opposite to v_j , and r_j is the orthogonal reflection across S_j . Put $D = C^3$.

(2.2) For $0 \leq j \leq 2$, the 2-spheres S_D and S_j make a dihedral angle of $\pi/2$. The 2-spheres S_D and S_3 make a dihedral angle of $2\pi/5$.

Proof. For $0 \leq j \leq 2$, S_j contains v_3 ($= v_D$) ; hence, S_D and S_j intersect orthogonally. The 2-sphere S_3 is spanned by the spherical pentagon C^2 . The circular arc from v_3 to v_2 has length $\pi/10$ ([4], p. 35); hence, S_D and S_3 make an angle of $\pi/2 - \pi/10 = 2\pi/5$.

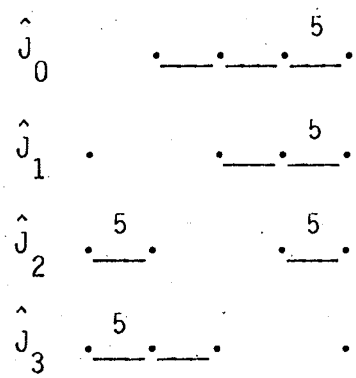
For any two reflections r , r' in a Coxeter group, let $m(r , r')$ denote the order of rr' . As an immediate corollary of (2.2) we have the following fact.

(2.3) With notation as above, put $s = s_D$. For $0 \leq j \leq 2$, $m(r_j , s) = 2$; while $m(r_3 , s) = 5$.

For $0 \leq i \leq 3$, let $T_i = (R(\Delta^3) - \{r_i\}) \cup \{s\}$ (where $R(\Delta^3) = \{r_0 , r_1 , r_2 , r_3\}$) and let J_i be the subgroup of H_4 generated by T_i .

(2.4) The pair (J_i , T_i) is a Coxeter system.

Sketch of Proof. Since J_i is generated by reflections, it is a Coxeter group (Théorème 1, p. 74 in [3]). For $0 \leq i \leq 3$, let $\hat{J}_i$ be the Coxeter group whose diagram is indicated below.



From (2.3) it is clear that there is a surjective homomorphism $\hat{J}_i \to J_i$. The meaning of (2.4) is that this map is an isomorphism. This can be proved on a case by case basis. The only two cases which present any difficulties are J_0 and J_1 . The first step is to show the subgroup J generated by

$\{r_2, r_3, s\}$, which is a quotient of H_3, is actually isomorphic to H_3. This follows from the fact that the orientation-preserving subgroup of H_3 is the alternating group on 5 letters (a simple group). Next one can show that J_0 is irreducible and conclude that $\hat{J}_0 \cong J_0$ by arguing that no other irreducible Coxeter group of rank 4 has J as an isotropy subgroup. Similarly, for J_1.

From (2.4) and Théorème 2(i), p.20 in [3], one can immediately deduce the following fact.

(2.5) Let T be any proper subset of $R(\Delta^3) \cup \{s\}$ and let J_T be the subgroup of H_4 generated by T. Then (J_T, T) is a Coxeter system.

3. A torsion-free subgroup of H_5. Let $\mathcal{D}(X^4)$ denote the set of 3-dimensional faces of the hyperbolic polyhedron X^4 (of course, $\mathcal{D}(X^4)$ can be identified with $\mathcal{D}$). For each $D \in \mathcal{D}(X^4)$, let r_D be the reflection of $\mathbb{H}^4$ across the hyperplane supported by D, let s_D be the reflection of $\mathbb{H}^4$ across the hyperplane through x_4 (the center of X^4) which is orthogonal to the geodesic ray from x_4 to the center of D, and let $t_D = r_D s_D$. It is clear that r_D belongs to H_5 and that s_D belongs to H_4 (the isotropy subgroup of H_5 at x_4) and hence, that t_D belongs to H_5. Let K denote the subgroup of H_5 generated by the family $(t_D)_{D \in \mathcal{D}(X^4)}$. The transformation t_D takes $-D$ to D and it takes X^4 to the adjacent 4-cell across D. Hence, X^4 is a fundamental domain for K. Since X^4 is the union of

14400 copies of Δ^4 , K has index 14400 in H_5 . The orbit space $M^4 = H^4/K$ is obviously the space formed from X^4 by identifying D with -D via s_D for each $D \in \mathcal{D}(X^4)$.

Let $C^0 \subset C^1 \subset C^2 \subset C^3 \subset C^4 = X^4$ be a chain of cells in the tesselation of H^4 and let $R(\Delta^4) = \{r_0, r_1, r_2, r_3, r_4\}$ be the corresponding set of fundamental reflections for H_5 . Define a map from $R(\Delta^4)$ to G_3 by sending r_i to itself for $0 \leq i \leq 3$ and by sending r_4 (= r_D) to s_D (where $D = C^3$) . It follows from (2.3) that this map extends to a homomorphism $f : H_5 \to H_4$ which restricts to the identity on H_4 . The kernel of f is K , since K is clearly contained in this kernel and since both groups have the same index in H_5 . Thus, H_5 is the semi-direct product of H_4 and K . Every finite subgroup of H_5 is conjugate to a subgroup of some "standard subgroup" of H_5 (where "standard subgroup" means, a subgroup generated by some proper subset of $R(\Delta^4)$. It follows from (2.5) that each standard subgroup of H_5 is mapped monomorphically by f . Hence, K is torsion-free. Consequently, K acts freely on H^4 , and M^4 is a hyperbolic 4-manifold.

Since K is normal in H_5 , the quotient group H_4 acts isometrically on M^4 . The orbit space of this action is Δ^4 (the orbit space of H_5 on $\mathbb{H}^4$). The fixed point set of any reflection s_D in H_4 on M^4 is a hyperbolic 3-manifold M^3 obtained by gluing D to the polyhedron formed by intersecting X^4 with the hyperplane in H^4 which is fixed by s_D . Hence, M^3 is 2-sided and connected. It is easy to see that the complement of M^3 in M^4 is connected. Therefore, M^3 represents a non-zero homology class. It follows that the first Betti number of M^4 is ≥ 1. (This Betti number is also ≤ 60, since K is generated by 60 elements and their inverses.)

The Euler characteristic of M^4 is 26. One way to see this is to compute the rational Euler characteristic of H_5, as in [9], p. 111, obtaining $\chi(H_5) = 26/14400$.

<u>Remarks</u>. 1) The group H_5 is an "arithmetic" subgroup of $O(4,1)$; more specifically, it is a subgroup of finite index in $O(4,1,A)$ where A is the ring of integers in $\mathbb{Q}(\sqrt{5})$ (cf. Exercise 19, p. 134 in [3]).

2) Any finitely generated Coxeter group has a torsion-free subgroup of finite index. (This follows from the fact that any such group has a faithful linear representation together with a well-known argument of Minkowski and Selberg.)

3) The tessellation of $\mathbb{H}^4$ by copies of X^4 is described by Coxeter in [4]. The apparently new fact is the existence of the torsion-free subgroup K with X^4 as its fundamental domain.

4) Coxeter also describes two other tessellations of $\mathbb{H}^4$ by regular 120-cells. There symmetry groups are $\bullet\overset{5}{\text{—}}\bullet\text{—}\bullet\text{—}\bullet\text{—}\bullet$ and $\bullet\overset{5}{\text{—}}\bullet\text{—}\bullet\text{—}\bullet\overset{4}{\text{—}}\bullet$. These have rational Euler characteristics 1/14400 and 17/28800, respectively (cf. [9] p. 111). Hence, the second one does not admit a torsion-free subgroup of index 14400; however, the first one quite possibly does. In this vein, it might be interesting to find further examples of hyperbolic 4-manifolds formed by

identifying faces of a regular 120-cell.

5) For $n > 3$, there is no known example of a closed aspherical n-manifold (e.g. a hyperbolic manifold) with the homology of S^n. The fact that the Euler characteristic of $\bullet\overset{5}{—}\bullet—\bullet—\bullet—\bullet$ is $1/14400$ suggests that we search for torsion-free subgroups of indes 28800. Such subgroups would give hyperbolic 4-manifolds of Euler characteristic 2, and hence, candidates for homology 4-spheres.

6) In [14], Seifert and Weber constructed a hyperbolic 3-manifold by identifying opposite faces of a dodecahedron. The fundamental group of this 3-manifold is a torsion-free subgroup of $\bullet\overset{5}{—}\bullet—\bullet\overset{5}{—}\bullet$ of index 120. (See [12].) There are several other ways to identify opposite faces of a dodecahedron to obtain a hyperbolic 3-manifold (and hence, several other torsion-free subgroups of $\bullet\overset{5}{—}\bullet—\bullet\overset{5}{—}\bullet$ of index 120). These are determined in [2] and [8].

7) The fundamental chamber of a hyperbolic reflection group need not be a simplex; other convex polyhedra can occur. The theory of 2-dimensional hyperbolic polygon groups is classical. In dimension 3, there is a beautiful and rich theory due to Andreev [1] (See also [10].) In dimension ≥ 4 and < 30, there are a few isolated examples of these more general reflection groups but no complete classification of the possibilities. However, Vinberg [13] has shown that in dimensions ≥ 30, there do not exist any cocompact hyperbolic reflection groups.

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