SMOOTH ACTIONS OF THE CLASSICAL GROUPS

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To my father

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0. Introduction

Let $G_n = O(n)$, $U(n)$ or $Sp(n)$ and let ρ_n denote the standard representation of G_n. This paper concerns smooth G_n-actions which locally resemble or are "modeled on" the linear action $k\rho_n$. More precisely, a G_n-action is said to be modeled on $k\rho_n$ if it has the same orbit types and slice representations as does $k\rho_n$.

Already in the case $k=2$, the study of such actions encompasses a rich variety of phenomena, e.g., the Brieskorn-Hirzebruch examples and the so-called knot manifolds are actions of this type. Actions modeled on $2\rho_n$ have been analyzed fairly completely by the Hsiangs, Jänich and Bredon (see [B1] and [B2] for a lucid account).

In this paper, we classify G_n-actions modeled on $k\rho_n$, for an arbitrary $k \leq n$ (throughout virtually the entire paper the technical assumption, $k \leq n$, is needed). We show that any G_n-manifold which is modeled on $k\rho_n$ and which has trivial principal orbit bundle is equivariantly diffeomorphic to a pullback of the linear model via some map from the orbit space of the given G_n-manifold to the orbit space of $k\rho_n$. Roughly speaking, over a given orbit space B, such actions are classified by homotopy classes of maps from B to the linear orbit space (where the phrase "homotopy classes of maps" is interpreted in the appropriate category).

In Chapter IV this result is combined with P. A. Smith theory to show that the orbit space of a modeled G_n-action on a homology sphere has the same cohomological structure as does the orbit space of the restriction of a linear action to the unit sphere (in particular, the orbit space will be acyclic for $n \geq k$). In Theorem 15.1 some of the homotopy theoretic computations involved in these results are worked out, in order to explicitly classify $O(n)$-actions modeled on $3\rho_n$ on homology $(3n-1)$-spheres. When studying G_n-actions on homology spheres, the requirement that the action be modeled on $k\rho_n$ is not unreasonable; for the Hsiangs have proved that any G_n-action on an integral homology sphere (or acyclic manifold), with principal orbits equal to Stiefel manifolds of type G_n/G_{n-k}, is modeled on $k\rho_n$ (see section 12).

The point of view taken in this paper is straightforward. If G is a compact Lie group and if M is a smooth G-manifold, then $M^{(H)}$, the union of orbits of type G/H, is a smooth submanifold of M. The image of $M^{(H)}$ in the orbit space M/G is also a smooth manifold. Thus, both M and M/G are stratified by the orbit types. This stratification of M/G is sometimes called the orbit structure. If $\pi: M \longrightarrow M/G$ is the natural projection, then $\pi|_{M^{(H)}}: M^{(H)} \longrightarrow M^{(H)}/G$ is the projection map of a smooth fibre bundle with fibre G/H and with structure group $N(H)/H$, the normalizer of H mod H. Hence, the theory of smooth G-actions with one orbit type is identical with the theory of principal $N(H)/H$-bundles over smooth manifolds.

When there is more than one orbit type the situation can be formulated in a similar fashion. Just as one classifies bundles over a fixed base space, one can try to classify G-actions over a fixed orbit space. Two questions arise:

(1) Which spaces occur as orbit spaces?

(2) Over a given orbit space, how many G-manifolds are there up to equivariant diffeomorphism (or alternately, up to an equivariant diffeomorphism which covers the identity on the orbit space)?

It turns out that, at least for G_n-actions modeled on $k\rho_n$, the analogy with fibre bundles is much closer than one might suspect.

The first step towards answering question (1) is to analyze the local structure of an orbit space. There is a natural functional structure on M/G which essentially is obtained by identifying "smooth" real-valued functions on M/G with $C^\infty(M)^G$, the G-invariant smooth functions on M. By a slight abuse of language, this functional structure on M/G is called the quotient differential structure (M/G is in general not a smooth manifold). Theorem 1.1, which is a recent result of Schwarz [Sc], reduces the study of $C^\infty(M)^G$ to the classical theory of G-invariant polynomials. In particular, it says that M/G is locally semialgebraic. Thus, we show in section 1 that the orbit space of a G_n-action modeled on $k\rho_n$ is locally isomorphic to the direct product of Euclidean space with a space of positive semidefinite hermitian matrices.

Notice that questions (1) and (2) deal with three distinct types of objects: G-manifolds, orbit spaces and G-manifolds over a fixed orbit space. Thus, in studying modeled G_n-actions, we shall be discussing objects in each of the following three categories.

$\mathcal{G}_{n,k}$: the category of G_n-manifolds which are modeled on $k\rho_n$. The morphisms are smooth equivariant *strata preserving* maps (i.e., isovariant maps).

$\mathcal{B}_k$: the category of functionally structured spaces which are locally isomorphic to the orbit space of a G_n-manifold modeled on $k\rho_n$. The morphisms are strata preserving morphisms of functionally structured spaces.

$\mathcal{S}_{n,k}(B)$: a category for each $\mathcal{B}_k$-space B. Its objects are pairs (M, φ) where M is a $\mathcal{S}_{n,k}$-object and where $\varphi: M/G_n \longrightarrow B$ is a $\mathcal{B}_k$-isomorphism. The morphisms are those equivariant diffeomorphisms which cover maps commuting with the φ's.

Our best results are in this last category, which isolates the bundle theoretic aspect of this type of transformation group.

The main result of Chapter I is a smooth version of Palais' Covering Homotopy Theorem for G_n-manifolds modeled on $k\rho_n$. This is one of the technical underpinnings of our theory, as it plays a role analogous to that of the theorem of the same name in fibre bundle theory. The hardest part of the theorem is to prove that a $\mathcal{B}_k$-isotopy on the orbit spaces can be covered by an equivariant isotopy, if it can be covered on one end. This is proved in section 3 using calculus and a trick from classical invariant theory. Suppose that M is a $\mathcal{S}_{n,k}$-manifold, that B is a $\mathcal{B}_k$-space and that $f: B \longrightarrow M/G_n$ is a $\mathcal{B}_k$-morphism. In section 4 we define a $\mathcal{S}_{n,k}(B)$-manifold f^*M, called the "pullback of M via f," and use this notion to prove 4.5, the Covering Homotopy Theorem.

Chapter II is devoted to answering question (2). We develop the concept of a "normal system" for an object in each of the three categories. Roughly, a normal system consists of fixed tubular neighborhoods of each stratum which "match up" the same way as a standard system of tubular neighborhoods for the linear model. (This is a fairly standard technique for dealing with stratified spaces.) A morphism of normal systems is a map which is linear on each of the prescribed tubular neighborhoods. The two key properties of this theory of normal systems are proved in section 7. Basically, in each of the three categories we can choose a normal system for any compact object. Furthermore, any morphism is homotopic to a morphism of normal systems (and every isomorphism is isotopic to an isomorphism of normal systems). The proof of these properties is an easy consequence of the Equivariant Tubular Neighborhood Theorem and the Equivariant Isotopy Extension Lemma (or of the corresponding theorems in $\mathcal{B}_k$). An interesting corollary of these results is Theorem 7.17. It says that for any $n \geq k$, $\mathcal{J}_{n,k}$ and $\mathcal{J}_{k,k}$ are equivalent categories (similarly, $\mathcal{J}_{n,k}(B)$ and $\mathcal{J}_{k,k}(B)$ are equivalent).

In section 8 we use the pullback construction and the theory of normal systems to classify $\mathcal{J}_{n,k}(B)$-manifolds, as mentioned above. This answers question (2) modulo our ability to compute homotopy classes of $\mathcal{B}_k$-morphisms.

In Chapter III we study the structure of various spaces of normal system morphisms. One reason for doing this is that, in view of the Classification Theorem, we want to compute the set of homotopy classes of morphisms between two $\mathcal{B}_k$-spaces. As remarked above, this set is isomorphic to a set of homotopy classes of normal system morphisms. Secondly, we are interested in the topological group of equivariant self-diffeomorphisms of a $\mathcal{S}_{n,k}$-manifold M. Particularly susceptible to attack is the subgroup $\mathrm{Diff}_1^{G^n}(M)$ consisting of those equivariant diffeomorphisms which cover the identity on the orbit space. In Theorem 7.15 it is shown that $\mathrm{Diff}_1^{G^n}(M)$ deformation retracts to the space of automorphisms of a normal system for M (a different proof of this fact is given in Remark 10.9). In section 10 we show that this space of normal system isomorphisms can be regarded as a space of cross sections of certain bundles. Thus, the homotopy of $\mathrm{Diff}_1^{G^n}(M)$ can be computed by methods which are completely bundle theoretic. In section 11 we take a slightly different viewpoint on the problem of classifying $\mathcal{S}_{n,k}(B)$-manifolds. This viewpoint exploits the fact that we can compute $\pi_0 \mathrm{Diff}_1^{G^n}(M)$ and therefore sheds new light on the Classification Theorem.

It is a pleasure to acknowledge the help I received from several people. Many discussions I had with Glen Bredon were important in clarifying some of the main problems of this paper. His work, and in particular his fine book [B1], has had considerable influence on this thesis. I would also like to thank Bill Massey for explaining some of his results to me before they were published in [M2]. My thanks go also to Bill Allard, Bill Browder and Ronnie Lee for many enlightening conversations. I have benefited from numerous conversations with my fellow graduate students in topology and from their patience in listening to some extremely vague ideas. The enthusiasm, criticism and ideas of my advisor, Wu-Chung Hsiang, have been invaluable. Much can be learned of how to be a mathematician from the splendid example he sets.

List of Symbols

Symbol	Description	Page
$\mathbb{F}^d$	the division ring of real, complex or quaternionic numbers, as d=1, 2 or 4	10
$M^d(n,k)$ or $M(n,k)$	the set of $n \times k$ matrices with coefficients in $\mathbb{F}^d$	10, 38
$\Sigma^d(n,k)$ or $\Sigma(n,k)$	the unit sphere in $M^d(n,k)$	22
$GL^d(k)$ or GL_k	the general linear group with coefficients in $\mathbb{F}^d$	13, 38
$G^d(n)$ or G_n	the orthogonal, unitary or symplectic group, as d = 1, 2 or 4	10, 38
$Z^d(k)$ or Z_k	the scalar matrices in $G^d(k)$	33
$PG^d(k)$ or PG_k	the group $G^d(k)/Z^d(k)$	
$H_+^d(k)$ or $H_+(k)$	the space of positive semidefinite hermitian matrices over $\mathbb{F}^d$	12
$W_+^d(k)$ or $W_+(k)$	the trace one matrices of $H_+^d(k)$	22
$k\rho_n$	k times the standard linear representation of G_n	10
$M(n,k)^{(i)}$	the $n \times k$ matrices of rank i (the i^{th} stratum)	10
$H_+(k)^{(i)}$	the matrices in $H_+(k)$ of rank i (the i^{th} stratum)	12
B_k^i	the closed i^{th} stratum of $H_+(k)$	48
W_k^i	the closed i^{th} stratum of $W_+(k)$	138
M	a typical G_n-manifold, modeled on $k\rho_n$	10
$f^*(M)$	the pullback of M via f	34
$M_\#$ or M/G_n	the orbit space of M	9
$M^{(i)}$	the union of orbits of type G_n/G_{n-i} in M	10
B	a typical space, modeled on $H_+(k)$	17
$B^{(i)}$	the i^{th} stratum of B	17
B^i	the closed i^{th} stratum	62, 67
N^i	the normal bundle of $M^{(i)}$ in M	38
C^i	the normal cone bundle of B^i in B	18
C_k^i	the normal cone bundle of B_k^i in $H_+(k)$	51
η^i	the principal PG_{k-i}-bundle associated to $C^i \to B^i$	64
$\xi^{(i)}$ or ξ^i	the principal $G_i \times G_{k-i}$-bundle associated to the normal bundle of $M^{(i)}$ or M^i in M	38, 62, 67

Symbol	Description	Page
$\mathscr{S}^d_{n,k}$ or $\mathscr{S}_{n,k}$	the category of $G^d(n)$-manifolds, modeled on $k\rho_n$	3, 10
$\mathscr{B}^d_k$ or $\mathscr{B}_k$	the category of functionally structured spaces, modeled on $H^d_+(k)$	4, 17
$\mathscr{S}^d_{n,k}(B)$ or $\mathscr{S}_{n,k}(B)$	the category of $G^d(n)$-manifolds, modeled on $k\rho_n$ over B	4, 36
$\widetilde{\mathscr{S}}_{n,k}$	the category of $\mathscr{S}_{n,k}$-normal systems	64
$\widetilde{\mathscr{B}}_k$	the category of $\mathscr{B}_k$-normal systems	64
$\widetilde{\mathscr{S}}_{n,k}(\eta_B)$	the category of $\mathscr{S}_{n,k}(B)$ normal systems rel η_B	74
$C_{n,k}(B)$	the set of isomorphism classes of $\mathscr{S}_{n,k}(B)$-objects	93
A	the assemble functor	66
ξ_k	the standard $\mathscr{S}_{n,k}$ normal system for $k\rho_n$	82
η_k	the standard $\mathscr{B}_k$ normal system for $H_+(k)$	82
$\xi, \eta, \mathcal{E}$	typical normal systems	62, 64, 74
$[B;H_+(k)]$	$\mathscr{B}_k$-homotopy classes of $\mathscr{B}_k$-morphisms from B to $H_+(k)$	81
$EG_k \longrightarrow BG_k$	the universal G_k-bundle	93
$\mathrm{Diff}^{G_n}(M)$	the group of equivariant self-diffeomorphisms of M	32
$\mathrm{Diff}^{G_n}_1(M)$	the subgroup of $\mathrm{Diff}^{G_n}(M)$ consisting of those maps which cover the identity on $M_\#$	32
Diff (B)	the automorphisms of a $\mathscr{B}_k$-space B	32
α^*	the associated automorphic bundle to α	106
ζ	a typical automorphic normal system	108
$\Gamma(\zeta)$	the sections of ζ	109
$\mathcal{H}^*(B;\pi_q)$	the obstruction theory for computing sections of an automorphic normal system	118
$\mathbb{Z}_m$	the integers mod m	137
$\mathbb{Z}_{(2)}$	the integers localized at 2	137
$\mathbb{Z}^t$	a system of twisted integer coefficients	151

I. THE COVERING HOMOTOPY THEOREM

1. Smooth Invariant Theory

Preliminary Definitions: Let G be a compact Lie group and let M be a smooth G-manifold. The orbit space of M will usually be denoted by $M_{\#}$. The natural projection $\pi: M \longrightarrow M_{\#}$ is called the orbit map. For any subgroup $H \subset G$ we can define submanifolds, $M^{(H)}$, M^{H} and $M^{\langle H \rangle}$ as follows:

$$\begin{aligned} M^{(H)} &= \text{the union of orbits of type } G/H, \\ M^{H} &= \text{the fixed point set of } H, \text{ and} \\ M^{\langle H \rangle} &= \{M^{(H)}\}^{H}. \end{aligned}$$

$M^{(H)} \longrightarrow M_{\#}^{(H)}$ is a G/H-bundle with structure group $N(H)/H$. Its principal bundle can be identified with

$$\pi|_{M^{\langle H \rangle}}: M^{\langle H \rangle} \longrightarrow M_{\#}^{(H)},$$

which is called the orbit bundle.

Let $U \subset M_{\#}$ be an open subset. A function $f: U \longrightarrow \mathbb{R}$ is said to be smooth if and only if $f \circ \pi: \pi^{-1}(U) \longrightarrow \mathbb{R}$ is smooth. This defines a functional structure on $M_{\#}$ called the quotient differential structure. We shall refer to morphisms of functionally structured spaces as smooth maps and to isomorphisms as diffeomorphisms. The quotient differential structure is natural in that a smooth equivariant map will cover a smooth map of orbit spaces. It is the "smallest" functional structure on $M_{\#}$ such that π will be smootn.

For the remainder of Chapter I we shall use the notation $G^d(n)$ for $O(n)$, $U(n)$ or $Sp(n)$ as $d=1,2$, or 4. Let $\mathbb{F}^d$ denote the division ring of real, complex or quaternion numbers as $d=1,2$, or 4, and let $M^d(n,k)$ be the real vector space of $n \times k$ matrices with coefficients in $\mathbb{F}^d$. The representation

$$k\rho_n : G^d(n) \times M^d(n,k) \longrightarrow M^d(n,k)$$

is defined by left matrix multiplication. If $x \in M^d(n,k)$ is a matrix of rank i, then the isotropy subgroup at x is conjugate to $G^d(n-i)$. To simplify notation we shall write $M^d(n,k)^{(i)}$ rather than $M^d(n,k)^{(G^d(n-i))}$. Thus, $M^d(n,k)^{(i)}$ is the submanifold consisting of those matrices with rank i (over $\mathbb{F}^d$). The isotropy subgroup at $x \in M^d(n,k)^{(i)}$ acts on the fibre at x of the normal bundle to $M^d(n,k)^{(i)}$. This representation, called the <u>slice representation</u>, is easily seen to be equivalent to $(k-i)\rho_{n-i}$.

<u>Definition</u>: A smooth action $G^d(n) \times M \to M$ is <u>modeled on</u> $k\rho_n$ if it has the same orbit types and slice representations as does $k\rho_n$. The <u>i^{th}-stratum</u> $M^{(i)}$ is the union of orbits of type $G^d(n)/G^d(n-i)$, i.e., $M^{(i)} = M^{(G^d(n-i))}$. Thus, the slice representation at $x \in M^{(i)}$ is equivalent to $(k-i)\rho_{n-i}$. It is not required that $M^{(i)}$ be nonempty, i.e., the "smallest" orbits might be of type $G^d(n)/G^d(n-p)$, for some $p > 0$, rather than fixed points. (Actions modeled on $k\rho_n$ were called "regular actions" by W. Y. Hsiang, and this terminology is the most common in the literature.)

Let $\mathcal{S}^d_{n,k}$ be the category whose objects are $G^d(n)$-actions modeled on $k\rho_n$ and whose morphisms are smooth isovariant maps. The expression

$\mathcal{J}^{d}_{n,k}$-manifold will often be used to mean a $G^d(n)$-action which is modeled on $k\rho_n$.

The quotient differential structure of the orbit space: Let G be a compact Lie group. By the Equivariant Tubular Neighborhood Theorem every orbit of a smooth G-manifold has a G-invariant neighborhood of the form $G \times_H \mathbb{R}^m$, where the isotropy subgroup H acts on $\mathbb{R}^m$ via some orthogonal representation (the slice representation plus a trivial representation of H). By the naturality of the twisted product, a function $f\colon G \times_H \mathbb{R}^m \longrightarrow \mathbb{R}$ is G-invariant if and only if its restriction to $H \times_H \mathbb{R}^m = \mathbb{R}^m$ is H-invariant, i.e., $(G \times_H \mathbb{R}^m)/G$ is diffeomorphic to $\mathbb{R}^m/H$. Hence, the quotient differential structure of an orbit space is known once we have determined the invariant C^∞ functions on the various slice representations.

Classical invariant theory deals with the related problem of computing the invariant polynomials of a representation. Its so-called First Main Theorem states that if $G \times V \longrightarrow V$ is a real representation, then the ring of invariant polynomials $\mathbb{R}[V]^G$ is finitely generated (see [W; p. 275]). A natural conjecture is that any $f \in C^\infty(V)^G$ can be written in the form $f = g \circ q$, where $q \in \mathbb{R}[V]^G$ and $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ is smooth. This has recently been proved by Schwarz [Sc]. The result can be reformulated as a statement about the quotient differential structure of $V_\# = V/G$. Suppose that $\{\varphi_1, \ldots, \varphi_s\}$ generates $\mathbb{R}[V]^G$. Define $\varphi\colon V \longrightarrow \mathbb{R}^s$ by $\varphi(v) = (\varphi_1(v), \ldots, \varphi_s(v))$. Since φ is equivariant with respect to the trivial action on $\mathbb{R}^s$, it induces $\varphi_\#\colon V_\# \longrightarrow \mathbb{R}^s$. Since φ is smooth, so is $\varphi_\#$.

Since invariant polynomials separate the orbits of V, $\varphi_{\#}$ is a topological embedding. Hence, Schwarz's result amounts to the

1.1 <u>Theorem</u> (Schwarz): $\varphi_{\#}: V_{\#} \longrightarrow \mathbb{R}^s$ is a smooth embedding, that is, $\varphi_{\#}^{-1}: \varphi_{\#}(V_{\#}) \longrightarrow V_{\#}$ is smooth, where the differential structure (i.e., functional structure) of $\varphi_{\#}(V_{\#})$ is induced from the inclusion $\varphi_{\#}(V_{\#}) \subset \mathbb{R}^s$.

When the representation is $k\rho_n$, $n \geq k$, the above theorem follows from a result of Glaser [G] (see also [Bi]). Let $H^d(k)$ denote the real vector space of $(k \times k)$ $\mathbb{F}^d$-hermitian matrices (the entries are in $\mathbb{F}^d$), and let $H_+^d(k)$ be the semialgebraic subset of positive semidefinite matrices. Since we explicitly know the invariant polynomials of $k\rho_n$ (by the First Main Theorem for $O(n)$, $U(n)$ or $Sp(n)$ in [W]), we have the

1.2 <u>Corollary</u>: Let $\pi: M^d(n,k) \longrightarrow H_+^d(k)$ be defined by $\pi(a) = a^* \cdot a$. where a^* means the conjugate transpose of a. If $n \geq k$, then

$$\pi_{\#}: M^d(n,k)_{\#} \longrightarrow H_+^d(k)$$

is a diffeomorphism. (If $n \leq k$, $\pi_{\#}$ is a diffeomorphism onto its image which consists of those matrices of rank $\leq n$.)

Henceforth, we will identify $M^d(n,k)_{\#}$ with $H_+^d(k)$ and the orbit map with π (compare [B2]).

The image of the orbits of type $G^d(n)/G^d(n-i)$ is the submanifold $H_+^d(k)^{(i)}$ consisting of those hermitian matrices of rank i. Notice that this stratification (the orbit structure) and the intrinsic stratification of $H_+^d(k)$ as a semianalytic set are the same.

One well known application of Corollary 1.2 is the

1.3 Corollary: If X is a $G^d(n)$-manifold which is modeled on kp_n, $n \geq k$, then $X_\#$ is a topological manifold with boundary. $X_\#$ is a smooth manifold with boundary (in the ordinary sense) only at those orbits where the slice representation is trivial or equivalent to ρ_{n-1}.

Proof: Each orbit has an invariant neighborhood U of the form

$$U = \{G^d(n) \times_{G^d(n-i)} M^d(n-i, k-i)\} \times \mathbb{R}^q,$$

for some i, $0 \leq i \leq k$. Notice that $U_\# = H_+^d(k-i) \times \mathbb{R}^q$. The convex cone $H_+^d(k-i)$ is homeomorphic to the half space $\mathbb{R}_+ \times \mathbb{R}^{p-1}$, where $p = \dim_{\mathbb{R}} H^d(k-i)$. This proves the first statement. However, only $H_+^d(1) = \mathbb{R}_+$ is diffeomorphic to a half space, which proves the second. $\square$

The action of the centralizer of kp_n: The centralizer of kp_n is $GL^d(k)$, the general linear group with coefficients in $\mathbb{F}^d$. It acts by right matrix multiplication on $M^d(n, k)$. This induces a linear action on the orbit space

$$\beta_d: GL^d(k) \times H_+^d(k) \longrightarrow H_+^d(k)$$

defined by $\beta_d(g, z) = g^* \cdot z \cdot g$ (change of bases for hermitian forms). Conversely, we shall soon show that if $\varphi: H^d(k) \longrightarrow H^d(k)$ is a linear isomorphism such that

$$\varphi(H_+^d(k)) = H_+^d(k),$$

then φ can be written in the form $\varphi(z) = g^* \cdot z \cdot g$, for some $g \in GL^d(k)$.

Let $\Delta_d: H^d(k) \longrightarrow \mathbb{R}$ be defined by

$$\Delta_d(z) = \prod_{i=1}^{k} \text{(eigenvalues of } z\text{)}.$$

In the real and complex cases, Δ_d is just the determinant.

1.4 <u>Lemma</u>: $\Delta_d : H^d(k) \longrightarrow \mathbb{R}$ is an irreducible polynomial (over the reals).

<u>Proof</u>: We prove it only for the quaternionic case $d=4$; the cases $d=1$ or 2 being easier and well known (for example, see [W; p.248]). When $d=4$ it is not even clear why Δ_4 is a polynomial.

Let $b_k^i(z) = \sigma_i$ (eigenvalues of z), where σ_i is the i^{th} symmetric polynomial. Let $A: H^4(k) \longrightarrow H^2(2k)$ be the monomorphism which sends $z = z_1 + jz_2$ to

$$A(z) = \begin{bmatrix} z_1 & \bar{z}_2 \\ -z_2 & z_1 \end{bmatrix}.$$

Let $\Omega(\lambda; A(z))$ be the characteristic polynomial of $A(z)$. Clearly, $\Omega(\lambda, A(z)) = (\lambda^k - b_k^1 \lambda^{k-1} + \ldots + (-1)^k b_k^k)^2$.

The coefficient of λ^{2k-i} in $\Omega(\lambda, A(z))$ is a polynomial in z. Since this coefficient is $2(\ldots\ldots \pm b_k^1 b_k^{i-1} \pm b_k^i)$ and since $b_k^1 = \text{trace }(z)$, it follows by induction that b_k^i is a polynomial in z. In particular, $b_k^k = \Delta_d$ is a polynomial. Δ_d is irreducible since its restriction to the subspace $\mathbb{R}^k \subset H^d(k)$ of diagonal matrices is already irreducible.

□

The Zariski closure of

$$\partial H_+^d(k) = \bigcup_{i<k} H_+^d(k)^{(i)}$$

is just the irreducible real algebraic hypersurface $\Delta_d^{-1}(0)$. Thus, if

$$\varphi\colon H^d(k) \longrightarrow H^d(k)$$

is any polynomial mapping which preserves $\partial H_+^d(k)$, then

$$\Delta_d \circ \varphi(z) = \alpha(z) \cdot \Delta_d(z), \tag{1.5}$$

where $\alpha(z)$ is some polynomial.

We specialize this observation to the case where φ is a linear isomorphism, as follows. Let $GL(p)$ be the general linear group of the real vector space $H^d(k)$. There is an action $GL(p) \times \mathbb{R}[H^d(k)] \longrightarrow \mathbb{R}[H^d(k)]$, defined by $g \cdot f(z) = f(g^{-1}z)$. Let $J^d(k) \subset GL(p)$ be the subgroup consisting of those g in $GL(p)$ such that $g(H_+^d(k)) = H_+^d(k)$. In particular, such a g will preserve the boundary; so by (1.5),

$$g \cdot \Delta_d(z) = \alpha_g(z) \cdot \Delta_d(z).$$

Since both sides of this equation have the same degree, α_g is a positive constant (it is positive since $\Delta_d(z)$ is positive for z positive definite). This gives us a 1-dimensional representation $\alpha\colon J^d(k) \longrightarrow GL(1, \mathbb{R})$, defined by $\alpha(g) = \alpha_g$. We summarize the above remarks in the following

1.6 Lemma: For any $g \in J^d(k)$

$$g \cdot \Delta_d = \alpha(g) \cdot \Delta_d,$$

where α is a character with values in the positive component of $GL(1, \mathbb{R})$.

Recall the representation $\beta_d\colon GL^d(k) \longrightarrow GL(p)$, defined by $\beta_d(a) \cdot z = a^* z a$, $z \in H^d(k)$. Notice that the kernel of β_d is isomorphic to $G^d(1)$.

Clearly, $\beta_d(GL^d(k)) \subset J^d(k)$.

1.7 Theorem (Frobenius): $\beta_d(GL^d(k)) = J^d(k)$. In other words, any linear automorphism g of $H^d(k)$ which maps the positive semidefinite part onto itself can be written in the form

$$gz = a^* z a,$$

for some $a \in GL^d(k)$.

Our proof is based on an argument of Dynkin in the introduction of [D].

Proof: The following diagram commutes,

$$\begin{tikzcd} GL^d(k) \arrow[rr, "\beta_d"] \arrow[dr, "(D_d)^{r(d)}"'] & & GL(p) \arrow[dl, "\det"] \\ & GL(1, \mathbb{R}) & \end{tikzcd}$$

where $D_1 = \det_{\mathbb{R}}$, $D_2 = |\det_{\mathbb{C}}|$ and where the quaternionic determinant D_4 is defined to be the composition

$$GL^4(k) \xrightarrow{A} GL^2(2k) \xrightarrow{\det_{\mathbb{C}}} GL(1, \mathbb{C}),$$

and where

$$r(d) = \begin{cases} k+1 & ; \quad d=1 \\ 2k & ; \quad d=2 \\ 2k-1 & ; \quad d=4 \end{cases} .$$

The diagram shows that $\beta_d(GL^d(k)) \cap SL(p) \cong PSL^d(k)$, the projective special linear group.

Let $G(\Delta_d) \subset GL(p)$ be the subgroup which leaves Δ_d invariant, and let $SJ^d(k) = J^d(k) \cap G(\Delta_d)$. By the previous lemma we have the inclusions

$$\beta_d(GL^d(k)) \cap SL(p) \subset SJ^d(k) \subset G(\Delta_d).$$

Since $PSL^d(k)$ is a simple Lie group, the arguments in [D] show that $G(\Delta_d)$ is a $\mathbb{Z}_2$-extension of $PSL^d(k)$, i.e., that

$$G(\Delta_d) = \pm 1 \; \{\beta_d(GL^d(k)) \cap SL(p)\},$$

where $-1 \in GL(p)$. Since elements of $J^d(k)$ preserve positive definiteness, $-1 \notin J^d(k)$. Hence, $\beta_d(GL^d(k)) \cap SL(p) = SJ^d(k)$, and the result follows. $\square$

2. Normal Cone Bundles

2.1 Definition: A functionally structured space B is modeled on $H^d_+(k)$ if it is locally isomorphic to the orbit space of some $\mathcal{S}^d_{n,k}$-manifold, $n \geq k$. This means that every point $b \in B$ has a neighborhood U which is diffeomorphic to $H^d_+(k-i) \times \mathbb{R}^{m_i}$, $0 \leq i \leq k$, with b corresponding to a point of the form $(0, x)$. The i^{th} stratum $B^{(i)}$ is the manifold consisting of those points b with a neighborhood of the form $H^d_+(k-i) \times \mathbb{R}^{m_i}$, where $b = (0, x)$ and where $m_i = \dim B^{(i)}$. Let $\mathcal{B}^d_k$ denote the category of functionally structured spaces which are modeled on $H^d_+(k)$. Morphisms are smooth strata preserving maps (that is, maps which take the i^{th} stratum to the i^{th} stratum).

In the usual fashion, we can define the tangent space T_b at $b \in B$ to be the vector space of derivations $C^\infty(B)_{(b)} \longrightarrow \mathbb{R}$, where $C^\infty(B)_{(b)}$ denotes the local ring of functions at b. The tangent cone $T^c_b \subset T_b$ is defined to be the subset consisting of those derivations which come from differentiating along smooth curves starting at b. Notice that

$$(2.2)\qquad T_0(H^d_+(k)) \cong H^d(k)$$

$$T^c_0(H^d_+(k)) \cong H^d_+(k).$$

The tangent bundle of B is defined by

$$T(B) = \bigcup_{b\in B} T_b(B).$$

From the fact that the tangent bundle of $H^d_+(k) \times \mathbb{R}^m$ is a trivial vector bundle, we can prove the

2.3 Proposition: If B is modeled on $H^d_+(k)$, then $T(B) \longrightarrow B$ is a vector bundle.

Since the i^{th} stratum $B^{(i)}$ is a smooth manifold, the ordinary tangent bundle $T(B^{(i)})$ is a subbundle of $T(B)|_{B^{(i)}}$. The normal bundle of $B^{(i)}$, denoted by $N(B^{(i)})$, is the quotient bundle $T(B)|_{B^{(i)}}/T(B^{(i)})$. Let

$$E = \bigcup_{b\in B^{(i)}} T^c_b.$$

The image of E in $N(B^{(i)})$ is called the normal cone bundle of $B^{(i)}$, and denoted by C^i. $C^i \longrightarrow B^{(i)}$ is a fibre bundle with fibre $H^d_+(k-i)$ and structure group $GL^d(k-i)$ (or $PGL^d(k-i)$ if we require the action of the structure group to be effective). Any automorphism of the fibre of $N(B^{(i)})$ must leave the fibre of C^i invariant; so by Throerm 1.7, the structure group of $N(B^{(i)})$ is also $PGL^d(k-i)$.

By choosing a metric on $T(B)|_{B^{(i)}}$, we can define an exponential map

$$\exp\colon C^i \longrightarrow B,$$

which is an embedding on a neighborhood of the zero section $B^{(i)}$. This is all that is needed to prove the Tubular Neighborhood Theorem.

2.4 Proposition (The $\mathcal{B}_k^d$ Tubular Neighborhood Theorem): Suppose B is modeled on $H_+^d(k)$. For each stratum $B^{(i)}$ there is a fibre bundle $C^i \longrightarrow B^{(i)}$ with fibre $H_+^d(k-i)$ and structure group $PGL^d(k-i)$ and an embedding

$$\tau: C^i \longrightarrow B$$

such that $\tau|_{B^{(i)}}$ is the inclusion. Moreover, any such τ is unique up to a $\mathcal{B}_k^d$-isotopy on C^i and a fibre bundle equivalence.

2.5 Remark: Let $\varphi: B \longrightarrow D$ be a morphism of $\mathcal{B}_k^d$-spaces. Since the only linear endomorphisms of $H_+^d(j)$ which preserve the stratification are isomorphisms, it follows that each of the maps

$$d\varphi|_{B^{(i)}}: C^i(B) \longrightarrow C^i(D)$$

of normal cone bundles is transverse. For a similar reason, $\mathcal{S}_{n,k}^d$ and $\mathcal{S}_{n,k}^d(B)$ morphisms are automatically transverse on the normal bundle of each stratum.

2.6 Remark: It has been customary to give the orbit space of a $\mathcal{S}_{n,k}^d$-manifold X the differential structure of a smooth manifold with boundary. This can be done as follows. Let $H_0 \subset H^d(k)$ be the codimension one subspace defined by trace $(z) = 0$. We can define smooth homeomorphisms $\lambda_k: H_+^d(k) \longrightarrow H_0 \times \mathbb{R}_+$ (for example, by the equation $\lambda_k(z) = (p(z), \Delta_d(z))$, where p is the orthogonal projection onto H_0). By choosing a suitable invariant tubular neighborhood of each stratum in X, the maps induced by the λ_k's can be patched together to give $X/G^d(n)$ the functional structure of a smooth manifold with boundary. This functional structure is well-defined up to the choice of tubular neighborhoods. (See the appendix of [B2] for a further discussion.)

This approach has two major disadvantages. First of all smooth equivariant maps will not necessarily cover smooth maps. For example, the induced action of the centralizer $GL^d(k) \times (H_0 \times \mathbb{R}_+) \longrightarrow H_0 \times \mathbb{R}_+$ will no longer be smooth, yet alone linear. Secondly, it may well happen two orbit spaces $X_\#$ and $Y_\#$ are not isomorphic when given their quotient differential structure, but become diffeomorphic (through a strata preserving diffeomorphism), when considered as smooth manifolds.

The most convincing argument, however, for the use of the quotient differential structure is that it allows us to prove the next two theorems.

3. Covering Isotopies

Theorem 1.7 shows that a linear automorphism $\varphi: H_+^d(k) \longrightarrow H_+^d(k)$ can be covered by an equivariant linear automorphism $\tilde{\varphi}: M^d(n,k) \longrightarrow M^d(n,k)$ The corresponding result for diffeomorphisms is also true.

3.1 Theorem: If $\varphi: H_+^d(k) \times \mathbb{R}^m \longrightarrow H_+^d(k) \times \mathbb{R}^m$ is a diffeomorphism, then for any $n \geq k$ there is a $G^d(n)$-equivariant diffeomorphism

$\tilde{\varphi}: M^d(n,k) \times \mathbb{R}^m \longrightarrow M^d(n,k) \times \mathbb{R}^m$ covering φ (equivariant with respect to the linear action $k\rho_n$ + trivial).

The following global version is more useful.

3.2 Covering Isotopy Theorem: Let X be a $\mathscr{S}^d_{n,k}$-manifold, $n \geq k$, and let $f: X_\# \times I \longrightarrow X_\#$ be a smooth isotopy (i.e., $f_t = f|_{X_\# \times \{t\}}$ is a diffeomorphism for each t). If $F_0: X \times \{0\} \longrightarrow X$ is an equivariant diffeomorphism covering f_0, then there is an equivariant isotopy $F: X \times I \longrightarrow X$ extending F_0 and covering f.

The two theorems are proved simultaneously by an inductive argument, which is carried out as follows. Let $A(k)$ denote the statement of Theorem 3.1. For a given $\mathscr{S}^d_{n,k}$-manifold X there is a "smallest orbit type," say, $G^d(n)/G^d(n-k+r)$, where r is maximal. The slice representation at such an orbit is equivalent to $r\rho_{n-k+r}$. Let $CI(r)$ mean that the Covering Isotopy Theorem is true for all $\mathscr{S}^d_{n,k}$-manifolds X with $G^d(n)/G^d(n-k+r)$ as smallest orbit type (in other words, $X^{(k-r)}$ is the lowest stratum). The theorems will be proved according to the inductive scheme

$$A(r) \text{ and } CI(r-1) \Rightarrow CI(r) \Rightarrow A(r+1).$$

Since $A(0)$ is trivial and $CI(-1)$ is vacuous, both theorems will then follow. (Also, $A(1)$ and $CI(1)$ are proved in [Bl; pp. 325, 328]).

The first implication is straightforward. First, notice that $A(r)$ implies the Covering Isotopy Theorem for $X = M^d(n,r) \times \mathbb{R}^m$. Since an arbitrary X, satisfying the hypothesis of $CI(r)$, may be decomposed as

$X = N^{k-r} \cup X - N^{k-r}$, where N^{k-r} is an invariant tubular neightorhood of the lowest stratum, the implication now follows from a partition of unity argument (similar to an argument in [B1; p. 328]). The details may safely be left to the reader.

The second implication is subtler. Assuming CI(r), we want to show that a diffeomorphism $\varphi: H^d(r+1) \times \mathbb{R}^m \longrightarrow H^d(r+1) \times \mathbb{R}^m$ can be covered by an equivariant diffeomorphism $\tilde{\varphi}: M^d(n, r+1) \times \mathbb{R}^m \longrightarrow M^d(n, r+1) \times \mathbb{R}^m$. The $\mathbb{R}^m$ coordinates play no role and will be suppressed for the remainder of the proof.

Define a smooth isotopy $f: H_+^d(r+1) \times I \longrightarrow H_+^d(r+1)$ by

$$f(z,t) = \begin{cases} t^{-1}\varphi(tz) & ; \quad \text{for } t \neq 0 \\ d\varphi_0(z) & ; \quad \text{for } t = 0 \end{cases}$$

where $d\varphi_0(z) = \lim_{t \to 0} t^{-1}\varphi(tz)$ is the differential of φ at the origin. By Theorem 1.7, there is an equivariant linear isomorphism $F_0: M^d(n, r+1) \longrightarrow M^d(n, r+1)$ covering $f_0 = d\varphi_0$. The $\mathcal{S}^d_{n,r+1}$-manifold $M^d(n, r+1) - \{0\}$ satisfies the hypothesis of CI(r), so we obtain a smooth isotopy

$$F: M^d(n, r+1) - \{0\} \times I \longrightarrow M^d(n, r+1) - \{0\}$$

extending F_0 and covering f.

Next, we use F to construct $\tilde{\varphi}$. It suffices to construct $\tilde{\varphi}$ on the unit disk $D^d(n, r+1) \subset M^d(n, r+1)$. Let $\Sigma = \Sigma^d(n, r+1)$ be the unit sphere. The orbit space of Σ is the space $W_+^d(r+1) \subset H_+^d(r+1)$ consisting of those positive semidefinite matrices with trace $= 1$. We use the same letter to denote the restriction of F to Σ, $F: \Sigma \times I \longrightarrow M^d(n, r+1)$. Notice that we can recover φ from the restriction of f to $\Sigma_{\#} \times I$, by the formula

$$\varphi(z) = \begin{cases} T(z) \cdot f(z/T(z), T(z)) & ; \text{for } z \neq 0 \\ 0 & ; \text{for } z = 0 \end{cases}$$

where $T(z)$ is the trace of $z \in H_+^d(r+1)$. In an analogous fashion, we can define $\tilde{\varphi}$: $D^d(n, r+1) \longrightarrow M^d(n, r+1)$ by the formula

$$\tilde{\varphi}(x) = \begin{cases} |x| F(x/|x|, |x|^2) & ; \text{for } x \neq 0 \\ 0 & ; \text{for } x = 0 \end{cases} \tag{3.3}$$

where $|x| = T(x^* \cdot x)^{\frac{1}{2}}$, $x \in D^d(n, r+1)$.

$\tilde{\varphi}$ is clearly a continuous equivariant map and it is simple to check that it covers φ. Since φ is a homeomorphism, so is $\tilde{\varphi}$. From the formula it is clear that φ is a diffeomorphism except possibly at the origin, where it does not appear to be smooth.

In fact, $\tilde{\varphi}$ is smooth at the origin. The remainder of the proof is devoted to showing this. We might first remark that our condition on the initial lift, that $F(x, 0) = x \cdot a$, for some $a \in GL^d(r+1)$, is precisely the condition that $\tilde{\varphi}$ be differentiable at 0; for,

$$\begin{aligned} d\tilde{\varphi}_0(x) &= \lim_{t \to 0} t^{-1} \tilde{\varphi}(tx) \\ &= \lim_{t \to 0} x F(x/|x|, t^2|x|^2) \\ &= |x| F(x/|x|, 0) \\ &= x \cdot a . \end{aligned}$$

which is linear in x. Since $d\tilde{\varphi}_0$ is nonsingular, $\tilde{\varphi}$ will automatically be a diffeomorphism once we have shown that it is C^∞ at the origin. We proceed by a somewhat indirect route.

3.4 <u>Lemma</u>: Let $F: \Sigma^d(n, r+1) \times I \longrightarrow M^d(n, r+1)$ be as above. Then there is a unique smooth map

$$\eta: W_+^d(r+1) \times I \longrightarrow GL^d(r+1)$$

such that $F(x,t) = x \cdot \eta(\pi(x), t)$, where $\cdot$ means matrix multiplication and where $\pi: \Sigma^d(n, r+1) \longrightarrow W^d(r+1)$ is the orbit map.

<u>Proof</u>: For each j, $1 \leq j \leq r+1$, let $p_j: M^d(n, r+1) \longrightarrow M^d(n,1) = (\mathbb{F}^d)^n$ be projection on the j^{th} column. Define $\psi_j: \Sigma \times I \times M^d(n,1) \longrightarrow \mathbb{F}^d \cong \mathbb{R}^d$ by $\psi_j(x,t,u) = \langle p_j F(x,t), u\rangle$, where $\langle\ ,\ \rangle$ is the standard hermitian inner product in $(\mathbb{F}^d)^n$. Clearly, ψ_j is a smooth invariant $\mathbb{R}^d$-valued function (invariant with respect to the product action $(r+1)\rho_n +$ trivial $+ \rho_n$). Moreover, it is $\mathbb{F}^d$-conjugate linear in u. From classical invariant theory, we know that the smooth invariants which are $\mathbb{F}^d$-conjugate linear in u are all linear combinations of expressions of the form

$$\langle x_i, u\rangle \cdot Q(x,t),$$

where $x_i = p_i(x)$ and where $Q: \Sigma \times I \longrightarrow \mathbb{F}^d \cong \mathbb{R}^d$ is some smooth invariant function. Hence,

$$\psi_j(x,t,u) = \sum_{i=1}^{i=r+1} \langle x_i, u\rangle\, \hat{\eta}_{ij}(x,t), \tag{1}$$

where $\hat{\eta}_{ij}: \Sigma \times I \longrightarrow \mathbb{F}^d$ is smooth and invariant. (As long as $n \geq r+1$, this representation of ψ_j is unique.) Since $\hat{\eta}_{ij}$ is invariant, it factors through the orbit space, i.e., $\hat{\eta}_{ij}(x,t) = \eta_{ij}(\pi(x),t)$. The η_{ij} arrange themselves neatly into a $(r+1) \times (r+1)$ matrix

$$\eta = (\eta_{ij}) \colon W_{+}^{d}(r+1) \longrightarrow \mathbb{F}^{d}.$$

From (1) and the definition of ψ_j, we see that

$$\begin{aligned} F(x,t) &= \sum_{j=1}^{j=r+1} p_j F(x,t) \\ &= \sum_{i,j} x_i \cdot \eta_{ij}(\pi(x),t) \\ &= x \cdot \eta\,(\pi(x),t). \end{aligned}$$

To complete the proof it remains to observe that the image of η actually lies in $GL^{d}(r+1)$. This is obvious when $\pi(x)$ is nonsingular. When the rank of $\pi(x)$ is less than $r+1$, we can reach the same conclusion by considering the differential of $F_{\#} = f$ at $\pi(x)$.

$\square$

We can restate the above lemma in a more comprehensible form.

3.5 <u>Corollary</u>: For $n \geq k$, let $\lambda \colon M^{d}(n,k) \longrightarrow M^{d}(n,k)$ be an equivariant diffeomorphism; then there is a smooth $\theta \colon H_{+}^{d}(k) \longrightarrow GL^{d}(k)$ such that $\lambda(x) = x \cdot \theta(\pi(x))$.

From 3.3 and 3.4, we have that

$$\begin{aligned} \tilde{\varphi}(x) &= |x| F(x/|x|,\ |x|^2) \\ &= x \cdot \eta(\pi(x/|x|),\ |x|^2) \\ &= x \cdot \sigma(\pi(x)) \end{aligned} \tag{3.6}$$

where $\sigma\colon H^d_+(r+1) \longrightarrow GL^d(r+1)$ is defined by

$$\sigma(z) = \begin{cases} \eta(z/T(z),\ T(z)) & ;\ z \neq 0 \\ \eta(*, 0) = a & ;\ z = 0 \end{cases} \tag{3.7}$$

where $*$ is an arbitrary point in $H^d_+(r+1)$. To show that $\widetilde{\varphi}$ is smooth at $0 \in M^d(n, r+1)$, it clearly suffices to show that σ is smooth at $0 \in H^d_+(r+1)$.

Let $\{e_1, \ldots, e_p\}$ be a basis for $H^d(r+1)$, chosen so that each e_i lies in $H^d_+(r+1)$. Let $z_1, \ldots, z_p$ be the corresponding coordinate functions. We use the standard multi-index notation $z^\ell = (z_1)^{\ell_1} \cdot \ldots \cdot (z_p)^{\ell_p}$, etc., where $\ell = (\ell_1, \ldots, \ell_p)$ is a p-tuple of nonnegative integers. Also let $|\ell| = \ell_1 + \ldots + \ell_p$. Since η is defined by F, we may extend η to a function (also denoted by η) defined on $H^d_+(r+1) - \{0\} \times I$ (we do this in order to differentiate η with respect to the z_i's). The notation $\frac{\partial \eta}{\partial y}$ will be used for the matrix

$$\left(\frac{\partial \eta_{ij}}{\partial y} \right)$$

where each of the d component functions of $\eta_{ij}\colon H^d_+(r+1) - \{0\} \times I \longrightarrow \mathbb{R}^d$ is differentiated separately.

The next lemma characterizes the differentiability of σ at the origin in terms of the partial derivatives of η.

3.8 <u>Lemma</u>: For σ to be n times continuously differentiable at the origin it is necessary and sufficient that for all $q \leq n$ and for all p-tuples ℓ with $|\ell| = q+1$, the partial derivatives

$$\frac{\partial^{2q+1} \eta}{\partial z^\ell \partial t^q}(w, 0) \equiv 0,$$

for all $w \in W_+^d(r+1)$.

Proof: We prove only sufficiency, since this is all that is needed in the proof of Theorems 3.1 and 3.2. Once 3.1 has been proved, necessity will follow immediately from 3.5. A direct proof of necessity, similar to the proof of the next lemma, could also be given.

Suppose that for all $q \leq n$ and for all ℓ, with $|\ell| = q+1$,

$$\frac{\partial^{2q+1}\eta}{\partial z^{\ell}\partial t^{q}}(w,0) \equiv 0, \quad w \in W^d(r+1).$$

Expanding η about 0 and using (3.7), we get

$$(1) \qquad \eta(z,t) = \left\{\eta(w,0) + \sum_{i=1}^{i=n} \sum_{|\ell| \leq i} \frac{t^i}{(i+|\ell|)!} \cdot \frac{\partial^{i+|\ell|}\eta}{\partial z^{\ell}\partial t^{i}}(w,0) \cdot (z-w)^{\ell}\right\}$$

$$+ t^{n+1} R(z,t)$$

$$(2) \qquad \sigma(z) = \left\{a + \ldots + T^n \sum_{|\ell| \leq n} \frac{1}{(n+|\ell|)!} \cdot \frac{\partial^{n+|\ell|}\eta}{\partial z^{\ell}\partial t^{n}}(w,0) \cdot (z/T - w)^{\ell}\right\}$$

$$+ Q_n(z)$$

where $a = \eta(w,0)$, $T = T(z)$ is the trace, and where

$$Q_n(z) = \begin{cases} T^{n+1} R(z/T, T) & ; \ z \neq 0 \\ 0 & ; \ z = 0. \end{cases}$$

The expression in brackets in (2) is clearly a degree n polynomial in z. Define a norm on $H^d(r+1)$ by $|z| = T(z \cdot z)^{\frac{1}{2}}$. For a positive semidefinite z,

$$\lim_{|z| \to 0} T(z)^{n+1}/|z|^n = 0.$$

It follows that

$$\lim_{|z|\to 0} Q_n(z)/|z|^n = 0.$$

Hence, (2) is the n^{th} Taylor's series expansion of σ at the origin. This proves the lemma. (Implicit use of the Whitney Extension Lemma is made in this proof and in the proof of the following lemma.)

□

The next lemma completes the proof that $CI(r) \Longrightarrow A(r+1)$ and hence the proof of Theorems 3.1 and 3.2.

3.9 Lemma: σ is C^∞.

The idea is this. We know that σ is C^∞ except possibly at the origin. We also know that $\varphi(z) = \sigma(z)^* \cdot z \cdot \sigma(z)$ defines a diffeomorphism of $H_+^d(r+1)$. We will show that this condition insures that the partial derivatives of η satisfy the condition of the previous lemma.

Proof: Recall that $\sigma(0) = \eta(z,0) = a \in GL^d(r+1)$. We may as well assume that $a = 1$, for clearly σ is smooth if and only if $a^{-1} \cdot \sigma$ is smooth.

Let n be the smallest integer such that for some $w \in W_+^d(r+1)$

$$\frac{\partial^{2n+1} \eta}{\partial z^\ell \partial t^n}(w,0) \neq 0, \quad |\ell| = n+1.$$

Let $d^{n+1}\varphi_0$ denote the $(n+1)^{th}$ differential at the origin of the C^∞-diffeomorphism $\varphi\colon H_+^d(r+1) \longrightarrow H_+^d(r+1)$. We can regard $d^{n+1}\varphi_0$ as a $(r+1)\times(r+1)$ hermitian matrix, the entries of which are d-tuples of degree $n+1$ homogeneous real-valued polynomials in z. From the matrix equation

$$\varphi(z) = \eta(z/T,T)^* \cdot z \cdot \eta(z/T,T),$$

it follows easily that

$$\text{(1)}\quad d^{n+1}\varphi_0(z) = T^n \left\{ z \cdot \frac{\partial^n \eta}{\partial t^n}(z/T,0) + \frac{\partial^n \eta}{\partial t^n}(z/T,0)^* \cdot z \right\}$$

+ terms the entries of which are obviously degree $n+1$ polynomials,

where $T = T(z)$.

Expanding $\frac{\partial^n \eta}{\partial t^n}(z/T,0)$ about $(w,0)$,

$$\text{(2)}\quad T^n \frac{\partial^n \eta}{\partial t^n}(z/T,0) = \left\{ T^n \sum_{i=0}^{i=n} \sum_{|\ell|=i} \frac{1}{(n+|\ell|)!} \cdot \frac{\partial^{n+|\ell|}\eta}{\partial z^\ell \partial t^n}(w,0)\cdot(z/T-w)^\ell \right\} + T^n R$$

$$\text{(3)}\quad R = \sum_{|\ell|=n+1} \left[\frac{1}{n!}(z/T-w)^\ell \cdot \int_0^1 (1-s)^n \frac{\partial^{n+|\ell|}\eta}{\partial z^\ell \partial t^n}(w+s(z/T-w))\,ds \right].$$

In (2), the entries of the expression in brackets are homogeneous degree n polynomials in z. Hence, the entries of $T^n(z\cdot R + R^*\cdot z)$ must be homogeneous degree $n+1$ polynomials. In other words, the entries of

$$\text{(4)}\quad T^n \sum_{|\ell|=n+1} (z/T-w)^\ell \cdot [z \cdot A_\ell(z) + A_\ell(z)^* \cdot z]$$

must be homogeneous polynomials, where $A_\ell(z)$ denotes the integral in (3). Rewriting (4) as

$$T^{-1}\sum(z - Tw)^\ell \cdot [z \cdot A_\ell + A_\ell^* \cdot z]$$

and expanding each of the terms $(z - Tw)^\ell$, we see that only those terms involving the highest powers of z can fail to be polynomials. So (4) can again be rewritten as

(5) $$T^{-1}\sum_{\ell} z^{\ell}\cdot[z\cdot A_{\ell} + A_{\ell}^{*}\cdot z]$$

+ terms whose entries are homogeneous degree $n+1$ polynomials.

In the case $r=0$, we have $H_{+}^{d}(1) = \mathbb{R}_{+}$, $z = T(z)$, and so the integral $A_{\ell}(z) = 0$, by inspection. Since (5) must itself have homogeneous polynomial entries, we see that the nonzero entries of the matrix $(z\cdot A_{\ell}(z) + A_{\ell}(z)^{*}\cdot z)$ must be homogeneous linear polynomials which are multiples of $T(z)$. For $r \neq 0$, this implies that the matrix equation

(6) $$z\cdot A_{\ell}(z) + A_{\ell}(z)^{*}\cdot z \equiv 0$$

holds identically in z.

For any $A \in M^{d}(r+1, r+1)$, define a subspace $P_A \subset H^{d}(r+1)$, by $P_A = \{z \in H^{d}(r+1) \mid z\cdot A + A^{*}\cdot z = 0\}$. It is easy to see that $P_A \neq H^{d}(r+1)$ unless $A = 0$.

By assumption, for some ℓ with $|\ell| = n+1$, we have that

$$A_{\ell}(w) = \frac{\partial^{2n+1}\eta}{\partial z^{\ell}\partial t^{n}}(w,0) \neq 0.$$

Since $P_{A_{\ell}(w)}$, therefore, has positive codimension in $H_{+}^{d}(r+1)$, we can find a z such that z/T is arbitrarily close to w and such that

$$z\cdot A_{\ell}(w) + A_{\ell}(w)^{*}\cdot z \neq 0.$$

In fact, replacing z by Kz for some large scalar K, we may assume that the nonzero entries of $z\cdot A_{\ell}(w) + A_{\ell}(w)^{*}\cdot z$ are arbitrarily large while z/T remains close to w. Recall that

$$A_{\ell}(z) = \frac{1}{n!}\int_0^1 (1-s)^n \frac{\partial^{2n+1}\eta}{\partial z^{\ell}\partial t^{n}}(w + s(z/T - w)\,ds.$$

For a z chosen as above, we see that the nonzero entries of $z \cdot A_\ell(w) + A_\ell(w)^* \cdot z$ will not change sign during the integration to $z \cdot A_\ell(z) + A_\ell(z)^* \cdot z$. Thus, $z \cdot A_\ell(z) + A_\ell(z)^* \cdot z \neq 0$, contradicting (6). Hence, we must have had, from the outset, that

$$A_\ell(w) = \frac{\partial^{2n+1} \eta}{\partial z^\ell \partial t^n}(w, 0) = 0$$

for all $w \in W_+^d(r+1)$ and for all n and ℓ, with $|\ell| = n+1$. So by Lemma 3.8, σ is smooth.

$\square$

Before going on we mention two results of peripheral interest. The first is a mild generalization of Corollary 3.5, which we shall need in the proof of Theorem 7.17. The second is an analog of (1.5).

3.10 <u>Corollary</u>: For $n \geq k$, let

$$U = G^d(n+i) \times_{G^d(n)} M^d(n,k).$$

If $\lambda: U \times \mathbb{R}^m \longrightarrow U$ is an m-parameter family of $G^d(n+i)$-equivariant diffeomorphisms, then there is a unique smooth map $\theta = (\theta_1, \theta_2): U_\# \times \mathbb{R}^m \longrightarrow G^d(i) \times GL^d(k)$ such that

$$\lambda(u,y) = [g \cdot \theta_1(\pi(u),y),\ x \cdot \theta_2(\pi(u),y)],$$

where $u = [g,x] \in G^d(n+i) \times_{G^d(n)} M^d(n,k)$, $y \in \mathbb{R}^m$, and $\pi: U \longrightarrow U_\# = H_+^d(k)$ is the orbit map.

3.11 <u>Corollary</u>: If $\varphi: H_+^d(k) \longrightarrow H_+^d(k)$ is a diffeomorphism, then there is a smooth function $\alpha: H_+^d(k) \longrightarrow (0,\infty)$ such that $\Delta_d\varphi(z) = \alpha(z) \cdot \Delta_d(z)$.

Proof: By 3.1, φ can be covered by $\tilde{\varphi}: M^d(n,k) \longrightarrow M^d(n,k)$. By 3.5, there is a $\theta: H_+^d(k) \longrightarrow GL^d(k)$ with $\tilde{\varphi}(x) = x \cdot \theta(\pi(x))$. Let α be the composition

$$H_+^d(k) \xrightarrow{\theta} GL^d(k) \xrightarrow{|\det|^2} (0,\infty).$$

□

For any smooth G-manifold X, let Diff $(X_\#)$, $\mathrm{Diff}^G(X)$ and $\mathrm{Diff}_1^G(X)$ be the topological groups of, respectively, self-diffeomorphisms of $X_\#$, equivariant self-diffeomorphisms of X, and equivariant self-diffeomorphisms of X which cover the identity on $X_\#$. Each of these function spaces is given the coarse C^∞-topology. Notice that $\mathrm{Diff}_1^G(X) \subset \mathrm{Diff}^G(X)$ and that there is a natural projection $p: \mathrm{Diff}^G(X) \longrightarrow \mathrm{Diff}(X_\#)$, defined by $p(\varphi) = \varphi_\#$.

The Covering Isotopy Theorem shows that p has the Covering Homotopy Property. This important reformulation of 3.2 can be stated as follows.

3.12 Theorem: If X is a $\mathscr{S}^d_{n,k}$-manifold, $n \geq k$, then $p: \mathrm{Diff}^G(X) \longrightarrow \mathrm{Diff}(X_\#)$ is a principal fibration, with fibre $\mathrm{Diff}_1^G(X)$, (here $G = G^d(n)$).

3.13 Example: Suppose $X = M^d(n,k)$. Let $\mathrm{Diff}^G(M^d(n,k)) \longrightarrow \mathrm{Diff}^G(M^d(n,k))$ be defined by

$$h_t(\varphi)(x) = \begin{cases} t^{-1}\varphi(tx) & ; \; t \neq 0 \\ d\varphi_0(x) & ; \; t = 0 . \end{cases}$$

Clearly, h_0 is a deformation retraction onto the linear self-diffeomorphism of $M^d(n,k)$, that is, onto $GL^d(k)$. Moreover, h_t preserves the subgroup $\mathrm{Diff}_1^G(M^d(n,k))$ and h_0 maps it onto $Z^d(k)$, the kernel of $\beta_d\colon GL^d(k) \to GL(p)$ ($Z^d(k) \cong G^d(1)$, the scalar matrices in $G^d(k)$). Similarly, we can define a homotopy $h_t'\colon \mathrm{Diff}(H_+^d(k)) \to \mathrm{Diff}(H_+^d(k))$, such that $p \circ h_t = h_t' \circ p$. By Theorem 1.7, the image of h_0' is just $PGL^d(k) = GL^d(k)/Z^d(k)$. We conclude that the commutative diagram

$$\begin{array}{ccccc} \mathrm{Diff}_1^G(M^d(n,k)) & \subset & \mathrm{Diff}^G(M^d(n,k)) & \longrightarrow & \mathrm{Diff}(H_+^d(k)) \\ \Big\downarrow h_0 & & \Big\downarrow h_0 & & \Big\downarrow h_0' \\ Z^d(k) & \subset & GL^d(k) & \longrightarrow & PGL^d(k) \end{array}$$

is an equivalence of fibrations (i.e., the vertical maps are homotopy equivalences).

4. Pullbacks and the Covering Homotopy Theorem

One consequence of the Covering Isotopy Theorem is the following

4.1 Theorem: Let B be a compact $\mathcal{B}_k^d$-space. Suppose that W is a $\mathcal{B}_{n,k}^d$-manifold with orbit space $B \times I$ (where $B \times I$ has the product orbit structure and functional structure). Let $W_0 = \pi^{-1}(B \times \{0\})$, where π is the orbit map. Then there is an equivariant diffeomorphism $F\colon W_0 \times I \to W$ such that

(i) the restriction of F to $W_0 \times \{0\}$ is the inclusion, and

(ii) $F_\#\colon B \times I \to B \times I$ is the identity.

Proof: Let $p: B \times I \longrightarrow I$ be projection on the second factor. Since $p \circ \pi: W \longrightarrow I$ has no critical points, there is a gradient like vector field ∇ on W such that $dp \circ d\pi(\nabla) = \frac{d}{dt}$. We can define an invariant vector field ∇^* by integrating over $G^d(n)$,

$$\nabla^*_w = \int_{G^d(n)} dg\, \nabla_{g^{-1}w}\, dg.$$

We can define an equivariant diffeomorphism $\psi: W_0 \times I \longrightarrow W$ by following the trajectories of ∇^*. Notice that $\psi|W_0 \times \{0\}$ is the inclusion and that $\psi_\#: B \times I \longrightarrow B \times I$ preserves the I-coordinate. However, $\psi_\#$ need not be the identity.

By 3.2, $\psi_\#^{-1}$ cen be covered by an equivariant diffeomorphism $H: W_0 \times I \longrightarrow W_0 \times I$. Setting $F = \psi \circ H$, the result follows.

$\square$

4.2 Definition: Let M be a $\mathcal{S}^d_{n,k}$-manifold, $n \geq k$, and let $\pi: M \longrightarrow M_\#$ be the orbit map. Suppose that B is modeled on $H^d_+(k)$ and that $f: B \longrightarrow M_\#$ is a $\mathcal{B}^d_k$-morphism (i.e., a smooth, strata preserving map). Define the pullback of M via f to be

$$f^*(M) = \{(b,m) \in B \times M \mid f(b) = \pi(m)\}.$$

This terminology is justified by the following proposition, which is essentially due to Bredon [B2].

4.3 Proposition: $f^*(M)$ is a $\mathcal{S}^d_{n,k}$-manifold and the diagram

$$\begin{array}{ccc} f^*(M) & \longrightarrow & M \\ \downarrow & & \downarrow \pi \\ B & \xrightarrow{f} & M_\# \end{array}$$

is a pullback square.

Proof: Give B the trivial $G^d(n)$-action. The product action on $B \times M$ is smooth (with respect to the product functional structure), and $f^*(M)$ is clearly an invariant subspace.

We claim that $f^*(M)$ is a "smooth submanifold" of $B \times M$. By this, we mean the following. Every point in B has a neighborhood of the form $H^d_+(k-j) \times \mathbb{R}^q$. For each such neighborhood, we require that

$$f^*(M) \cap H^d(k-j) \times \mathbb{R}^q \times M$$

is a smooth submanifold of $H^d(k-j) \times \mathbb{R}^q \times M$. In particular, this will imply that $f^*(M)$ is a smooth $G^d(n)$-manifold.

Since the question is local, we may assume that $M = \{G^d(n) \times_{G^d(n-i)} M^d(n-i, k-i)\} \times \mathbb{R}^m$ and hence, that $M_\# = H^d_+(k-i) \times \mathbb{R}^m$. Define $\theta: B \times M \longrightarrow H^d_+(k-i) \times \mathbb{R}^m$ by $\theta(b,m) = f(b) - \pi(m)$. By definition, $f^*(M) = \theta^{-1}(0)$. Let $N^j(B)$ and $N^j(M_\#)$ be the normal bundles of $B^{(j)} \subset B$ and of $M_\#^{(j)} \subset M_\#$, respectively. By Remark 2.5,

(1) $$df_b: N^j_b(B) \longrightarrow N^j_{f(b)}(M_\#)$$

is an isomorphism, for any $b \in B^{(j)}$. Also, $d\pi|_{T(M^{(j)})}: T(M^{(j)}) \longrightarrow T(M_\#^{(j)})$ is a surjection. It follows that $d\theta = (df, -d\pi)$ is a surjection at any $(b,m) \in \theta^{-1}(0)$. Therefore, by the Implicit Function Theorem, $f^*(M)$ is a smooth submanifold.

Let $\gamma: f^*(M) \longrightarrow M$ be projection on the second factor. It follows from (1) that $d\gamma_x$ maps the slice representation at $x \in f^*(M)^{(j)}$ isomorphically to the slice representation at $f(x)$. Hence, $f^*(M)$ is a $\mathscr{S}^d_{n,k}$-manifold. It is obvious that all four maps in the diagram are smooth morphisms and that the diagram satisfies the universal property for pullbacks.

□

4.4 Remark: Projection on the first factor $f^*(M) \longrightarrow B$ induces a diffeomorphism $\varphi\colon f^*(M)_\# \longrightarrow B$. The pair $(f^*(M), \varphi)$ is a $\mathscr{S}^d_{n,k}(B)$-manifold. Thus, the pullback of M via $f\colon B \longrightarrow M_\#$ has the natural structure of a $\mathscr{S}^d_{n,k}(B)$-manifold; it will often be convenient to regard it as such.

The main result of this chapter is the following smooth version of a theorem of Palais (compare [B1; p. 97]).

4.5 The Covering Homotopy Theorem: Let X and Y be $\mathscr{S}^d_{n,k}$-manifolds, $n \geq k$. Suppose that $h\colon X_\# \times I \longrightarrow Y_\#$ is a $\mathscr{B}^d_k$-homotopy and that $H_0\colon X \longrightarrow Y$ is a smooth isovariant map covering $h_0 = h|_{X_\# \times \{0\}}$. Then there is an isovariant homotopy $H\colon X \times I \longrightarrow Y$ covering h and extending H_0.

Proof: We have the pullback square

$$\begin{tikzcd} h^*(Y) \arrow[r, "\tilde{h}"] \arrow[d] & Y \arrow[d, "\pi"] \\ X_\# \times I \arrow[r, "h"'] & Y_\# \end{tikzcd} .$$

By 4.1, there is an equivariant diffeomorphism $F\colon h_0^*(Y) \times I \longrightarrow h^*(Y)$, covering the identity on $X_\# \times I$. By the universal property of pullbacks, there is a smooth isovariant map Φ, making the following diagram commute

$$\begin{tikzcd}
h_0^*(Y) \arrow[rr, "\widetilde{h}_0"] \arrow[d] & & Y \arrow[d] \\
& X \arrow[ul, dashed, "\Phi"] \arrow[ur, "H_0"] \arrow[dl] & \\
X_\# \arrow[rr, "h_0"] & & Y_\#
\end{tikzcd}$$

(In fact, Φ is an equivariant diffeomorphism, since it covers the identity on $X_\#$.) Set $H = \widetilde{h} \circ F \circ (\Phi \times 1)$, where $\Phi \times 1 : X \times I \longrightarrow h_0^*(Y) \times I$ is defined by $(\Phi \times 1)(x,t) = (\Phi(x), t)$. Since $F \circ (\Phi \times 1)$ covers the identity, H covers h.

$\square$

II. THE CLASSIFICATION OF ACTIONS MODELED ON $k\rho_n$

5. Equivariant Normal Bundles

In this section, we describe the normal bundles of the strata of a $\mathscr{S}^d_{n,k}$-manifold X. This description will be one of the necessary ingredients to be used in the definition of a "normal system" for X.

Notation: From now on, we shall omit the "d" in our notation, whenever no confusion arises. Rather than $G^d(n)$, $GL^d(n)$, $M^d(n,k)$ and $H^d_+(k)$, we shall write, instead, G_n, GL_n, $M(n,k)$, $H_+(k)$, etc.

5.1 Definition: Let $X^{\langle i \rangle} = \{X^{(i)}\}^{G_{n-i}}$; $\pi|_{X^{\langle i \rangle}}: X^{\langle i \rangle} \longrightarrow X^{(i)}_{\#}$ is the projection map of a smooth principal bundle called the i^{th} orbit bundle. Its fibre is $G_i \cong N(G_{n-i})/G_{n-i}$, where $N(G_{n-i}) = G_i \times G_{n-i}$ means the normalizer of G_{n-i}. There is an action $N(G_{n-i})/G_{n-i} \times G_n/G_{n-i} \longrightarrow G_n/G_{n-i}$, defined by $(n, g[G_{n-i}]) \longrightarrow g n^{-1}[G_{n-i}]$. In this way, G_i is identified with $\mathrm{Diff}^{G_n}(G_n/G_{n-i})$, the group of equivariant diffeomorphisms of an orbit (see [B1]). It is easy to see that the orbit bundle is just the principal bundle associated to $G_n/G_{n-i} \longrightarrow X^{(i)} \longrightarrow X^{(i)}_{\#}$, i.e., $X^{(i)} \cong G_n/G_{n-i} \times_{G_i} X^{\langle i \rangle}$.

As usual, let X be a G_n-manifold which is modeled on $k\rho_n$. Let $N^i \longrightarrow X^{(i)}$ be the equivariant normal bundle of the i^{th} stratum and let $N^i_{\#} \longrightarrow X^{(i)}_{\#}$ be the corresponding normal cone bundle (sometimes denoted by $C^i \longrightarrow X^{(i)}_{\#}$). We are going to define principal $G_i \times G_{k-i}$-bundles $\xi^{(i)} \longrightarrow X^{(i)}_{\#}$, which will classify N^i. (The same symbol $\xi^{(i)}$ will be used to denote both the bundle and its total space.) The G_i part of $\xi^{(i)}$ will correspond to the orbit bundle $X^{\langle i\rangle} \longrightarrow X^{(i)}_{\#}$; while the G_{k-i} part will determine the normal cone bundle. In particular since $N^k \longrightarrow X^{(k)}$ is a 0-disk bundle, $\xi^{(k)} \longrightarrow X^{(k)}_{\#}$ will just be the principal orbit bundle. We will also define $G_i \times G_{k-i}$-equivariant embeddings $F^{ik}: \xi^{(i)} \longrightarrow \xi^{(k)}$, which will, in some sense, represent tubular neighborhood maps.

5.2 Definition: Let Q be a smooth manifold and let $n \geq k$. A modeled G_n-vector bundle of type (k, i) over Q consists of a pair (E, Y) such that

(i) Y is a G_n-manifold with all its orbits of type G_n/G_{n-i}.

(ii) $E \longrightarrow Y$ is a G_n-vector bundle with fibre $M(n-i, k-i)$ and isotropy representation equivalent to $(k-i)\rho_{n-i}$.

Two such pairs (E_1, Y_1) and (E_2, Y_2) are equivalent if there is a G_n-vector bundle equivalence $\psi: E_1 \longrightarrow E_2$ such that $(\psi/Y_1)_{\#}$ is the identity on Q.

Of course, what we have in mind is where $Q = X^{(i)}_{\#}$, $Y = X^{(i)}$ and $E = N^i$.

The composition $E \longrightarrow Y \longrightarrow Y_{\#} = Q$ is the projection map of a smooth bundle with fibre $G_n \times_{G_{n-i}} M(n-i, k-i)$, (a normal tube about each G_n/G_{n-i}-orbit). The idea is to classify E as a bundle over Q. The structure group of $E \longrightarrow Q$ is $\mathrm{Diff}^{G_n}(G_n \times_{G_{n-i}} M(n-i, k-i))$. As in 3.13, this space obviously deformation retracts onto the space consisting of those equivariant diffeomorphisms which are linear on each fibre, that is, onto $G_i \times GL_{k-i}$. In fact, we could further deformation retract it to $G_i \times G_{k-i} \cong N(G_{n-k}) \cap N(G_{n-i})/G_{n-k}$, the group of equivariant diffeomorphisms of $G_n \times_{G_{n-i}} M(n-i, k-i)$ which are orthogonal linear maps on each fibre. Thus, the structure group of $E \longrightarrow Q$ can be reduced to $G_i \times G_{k-i}$. Let $\xi \longrightarrow Q$ be the associated principal bundle.

Next, we describe ξ from a slightly different angle. Consider the $G_i \times G_{n-i}$-vector bundle $E|_{Y\langle i\rangle} \longrightarrow Y^{\langle i\rangle}$. Set

(5.3) $$\tilde{\xi} = \{E|_{Y\langle i\rangle}\}^{\langle G_{n-k}\rangle}.$$

In other words, $\tilde{\xi}$ is the total space of the principal orbit bundle of the $G_i \times G_{n-i}$-manifold $E|_{Y\langle i\rangle}$.

5.4 Theorem: The composition

$$\tilde{\xi} \subset E|_{Y\langle i\rangle} \longrightarrow Y^{\langle i\rangle} \longrightarrow Q$$

is the projection map of a smooth principal $G_i \times GL_{k-i}$-bundle. $\tilde{\xi}$ is the principal bundle associated to $E \longrightarrow Q$, i.e., E is G_n-vector bundle equivalent to

$$\{G_n \times_{G_{n-i}} M(n-i, k-i)\} \times_{G_i \times GL_{k-i}} \tilde{\xi},$$

where $G_i \times GL_{k-i}$ acts on the expression in brackets as follows. Regard G_i as $N(G_{n-i})/G_{n-i}$. Let $[g, x] \in \{G_n \times_{G_{n-i}} M(n-i, k-i)\}$ and let $(h, a) \in G_i \times GL_{k-i}$, then $(h, a) \cdot [g, x] = [gh^{-1}, xa^{-1}]$. This sets up a natural 1-1 correspondence between equivalence classes of modeled G_n-vector bundles of type (k, i) over Q and principal $G_i \times GL_{k-i}$-bundles over Q.

The theorem is virtually self evident and we omit its proof. Similar results are proved in [B1], [EH] and [J].

By an <u>invariant metric</u> for a G_n-vector bundle we mean a reduction of the structure group to an orthogonal group. Since the maximal compact subgroup of GL_{k-i} is G_{k-i}, an invariant metric for the modeled G_n-vector bundle $E \longrightarrow Y$ corresponds to a reduction of the structure group of $\tilde{\xi}$ from $G_i \times GL_{k-i}$ to $G_i \times G_{k-i}$, i.e., to a section

$$s: Q \longrightarrow \tilde{\xi}/G_i \times G_{k-i}.$$

(Notice that the total space of $\tilde{\xi}/G_i \times G_{k-i}$ can be identified with the interior of $E_{\#}$.)

Once we have chosen a metric, a principal $G_i \times G_{k-i}$-bundle $\xi \longrightarrow Q$ is defined by the pullback diagram

$$(5.5) \qquad \begin{array}{ccc} \xi & \longrightarrow & \tilde{\xi} \\ \downarrow & & \downarrow \\ Q & \xrightarrow{s} & \tilde{\xi}/G_i \times G_{k-i} \end{array}.$$

A <u>modeled</u> G_n<u>-disk bundle of type</u> (k-i) <u>over Q</u> is just a modeled G_n-vector bundle together with a choice of invariant metric.

5.6 <u>Corollary</u>: Equivalence classes of modeled G_n-disk bundles of type (k-i) over Q are in 1-1 correspondence with principal $G_i \times G_{k-i}$-bundles $\xi \longrightarrow Q$. In other words, if $\overline{E} \longrightarrow Y$ is a modeled G_n-disk bundle (i.e., with fibre the unit disk $D(n-i,k-i) \subset M(n-i,k-i)$) and if $\xi \longrightarrow Q$ is defined by (5.5), then there is an orthogonal G_n-vector bundle equivalence

$$\overline{E} \cong \{G_n \times_{G_{n-i}} D(n-i,k-i)\} \times_{G_i \times G_{k-i}} \xi .$$

Thus, if X is a $\mathcal{J}_{n,k}$-manifold, we can define principal $G_i \times G_{k-i}$-bundles

(5.7) $$G_i \times G_{k-i} \longrightarrow \xi^{(i)}(X) \longrightarrow X_{\#}^{(i)}$$

such that

(5.8) $$N^i \cong \{G_n \times_{G_{n-i}} M(n-i,k-i)\} \times_{G_i \times G_{k-i}} \xi^{(i)}(X).$$

Let $Z_{k-i} \subset G_{k-i}$ be the subgroup defined in 3.13 $(Z_{k-i} \cong G_1)$. We can also define principal PG_{k-i}-bundles $\eta^{(i)}(X_{\#}) \longrightarrow X_{\#}^{(i)}$ by

(5.9) $$\eta^{(i)}(X_{\#}) = \xi^{(i)}(X)/G_i \times Z_{k-i},$$

which is the principal bundle of the normal cone bundle of $X_{\#}^{(i)}$, i.e.,

(5.10) $$N_{\#}^i \cong H_+(k-i) \times_{PG_{k-i}} \eta^{(i)}(X_{\#}).$$

We also have that

(5.11) $$X^{\langle i \rangle} \cong \xi^{(i)}(X)/G_{k-i}.$$

$\xi^{(i)}(X)$ and $\eta^{(i)}(X_{\#})$ will be abbreviated as $\xi^{(i)}$ and $\eta^{(i)}$, when no confusion arises.

The last two equations exhibit an important general principal; namely, that the equivariant normal bundle is almost determined by the orbit bundle and the normal cone bundle. We say "almost" because $\eta^{(i)}$ does not necessarily determine $\xi^{(i)}/G_i$. The difficulty is the lifting problem explained in the following diagram

$$\begin{tikzcd} & BZ_{k-i} \arrow[d] \\ & BG_{k-i} \arrow[d] \\ X_{\#}^{(i)} \arrow[r] \arrow[ur, dashed] & BPG_{k-i} \end{tikzcd}$$

In the orthogonal and unitary cases, $BZ_{k-i} = BG_1$ is an Eilenberg-Mac Lane space; so, there is a single obstruction to lifting $\eta^{(i)}$ to a G_{k-i}-bundle and also a single obstruction to uniqueness. The symplectic case is more complicated. In practice, these obstructions will not cause any problems.

Suppose that $\sigma^i: N^i \rightarrow X$ is an equivariant tubular neighborhood map. The restriction of σ^i to $\{N^i|_{X^{\langle i \rangle}}\}^{\langle G_{n-k} \rangle} = \tilde{\xi}^{(i)}$ is a G_k-equivariant embedding, $\bar{\sigma}^i: \tilde{\xi}^{(i)} \rightarrow \xi^{(k)}$. We can then define a $G_i \times G_{k-i}$-equivariant embedding

(5.12) $$F^{ik}: \xi^{(i)} \longrightarrow \xi^{(k)}$$

to be the composition

$$\xi^{(i)} \xrightarrow{\tilde{s}} \tilde{\xi}^{(i)} \xrightarrow{\bar{\sigma}^i} \xi^{(k)},$$

where $\tilde{s}$ covers the metric.

The map F^{ik} is an extremely strong relationship between $\xi^{(i)}$ and the principal orbit bundle ($F^{k-1,k}$ is sometimes called the "characteristic reduction"). Let n^i = interior of $\sigma_{\#}^i (N_{\#}^i)$ and let $p_i: n^i \longrightarrow X_{\#}^{(i)}$ be the projection. Since $p_i: n^i \longrightarrow X_{\#}^{(i)}$ is a vector bundle with fibre $GL_i/G_i \cong H(i)$, p_i is, in particular, a homotopy equivalence. Hence, from (5.12), (5.13) $$\xi^{(k)}\big|_{n^i} \cong p_i^*\,(G_k \times_{G_i \times G_{k-i}} \xi^{(i)}).$$

At the fixed point set, this equation becomes

(5.14) $$\xi^{(k)}\big|_{n^0} \cong p_0^*\ \xi^{(0)}\ ,$$

which is a theorem of [EH; Thm. 2.3].

When the principal orbit bundle is trivial, (5.12) is particularly interesting. Let $L: \xi^{(k)} \longrightarrow G_k$ be a trivialization. Notice that $L \circ F^{ik}: \xi^{(i)} \longrightarrow G_k$ is $G_i \times G_{k-i}$-equivariant and therefore covers a map of orbit spaces $\theta^i: X^i_\# \longrightarrow G_k/G_i \times G_{k-i}$. Let $\alpha^i_k = G_k \longrightarrow G_k/G_i \times G_{k-i}$ denote the canonical bundle. We have proved the

5.15 <u>Proposition</u>: Let X be a $\mathscr{S}_{n,k}$-manifold. Let $L: \xi^{(k)} \longrightarrow G_k$ be a trivialization, and let $\theta^i = (L \circ F^{ik})_\#: X^{(i)}_\# \longrightarrow G_k/G_i \times G_k$ be as above. Then

$$\xi^{(i)} \cong \theta^{i*}\,(\alpha^i_k)\ ,$$

that is, $\xi^{(i)}$ is the pullback of the canonical bundle over a Grassmannian. If the fixed point set is nonempty, this says that $\xi^{(0)} \cong X^{(0)}_\# \times G_k$, i.e., the equivariant normal bundle of the fixed point set is trivial if the principal orbit bundle is trivial.

In the next two sections, we sharpen these notions by considering a "compatible system" of tubular neighborhood maps $\{\sigma^i: N^i \longrightarrow X\}$. For each pair of integers (i,j), $0 \leq i < k$, we obtain maps f^{ij}, which generalize the F^{ik}'s and which fit together in a commutative diagram. Although these refinements are necessary to prove the classification results of section 8, the basic idea remains that of the above proposition.

6. Cutting Apart the Linear Action

In this section we discuss how to cut apart the linear model, $k\rho_n\colon G_n \times M(n,k) \longrightarrow M(n,k)$, into closed tubular neighborhoods of each orbit type. In the previous section we showed how the normal bundles to the strata were classified by principal $G_i \times G_{k-i}$-bundles. The first step is to compute these bundles for $k\rho_n$; we show how such a bundle can be described in terms of canonical bundles over a Grassmannian. As we proceed it is important to keep track of the action of G_k, the orthogonal centralizer, which acts by right matrix multiplication on $M(n,k)$. Next we explicitly cut apart the orbit space $H_+(k)$ into closed tubular neighborhoods of each stratum. This allows us to describe, in a fairly natural way, how these tubular neighborhoods are pasted together (in terms of maps between certain principal bundles). Unfortunately, the details are somewhat complicated.

Let $N^i \longrightarrow M(n,k)^{(i)}$ be the equivariant normal bundle of $M(n,k)^{(i)}$ in $M(n,k)$. Since $M(n,k)^{(i)}$ is also G_k-invariant, N^i is, in fact, a $G_n \times G_k$-vector bundle. Let $C^i \longrightarrow H_+(k)^{(i)}$ be the corresponding normal cone bundle. G_k acts by conjugation on $H_+(k)$, and this induces a G_k-action on C^i. As in 5.6, let $\xi_k^{(i)} = \xi^{(i)}(M(n,k))$ denote the $G_i \times G_{k-i}$-bundle associated to the modeled G_n-vector bundle $N^i \longrightarrow M(n,k)^{(i)}$. In the next lemma we compute these bundles.

6.1 Lemma:

(1) $H_+(k)^{(i)}$ is G_k-diffeomorphic to $H_+(i)^{(i)} \times_{G_i \times G_{k-i}} G_k$, where G_i acts on $H_+(i)^{(i)}$ by conjugation and G_{k-i} acts trivially on $H_+(i)^{(i)}$. In other words, $H_+(k)^{(i)}$ is a canonical G_k-vector bundle over the Grassmannian $G_k/G_i \times G_{k-i}$.

(2) $\xi_k^{(i)} \cong H_+(i)^{(i)} \times G_k$, which is a $G_i \times G_{k-i}$-bundle over $H_+(k)^{(i)}$ (by (1)). The natural right G_k-action on $\xi_k^{(i)}$ corresponds to the action of the centralizer on N^i.

Proof: Notice that the total space of the i^{th} orbit bundle is $M(n,k)^{\langle i \rangle} = M(i,k)^{\langle i \rangle}$. Here $M(i,k)^{\langle i \rangle}$ denotes those matrices of rank i with zeroes in the last $n-i$ rows. G_k acts by right multiplication on $M(i,k)^{\langle i \rangle}$ and the orbits are all of type G_k/G_{k-i}. Hence,

$$
\begin{aligned}
M(i,k)^{\langle i \rangle} &= \left\{M(i,k)^{\langle i \rangle}\right\}^{G_{k-i}} \times_{G_i} G_k/G_{k-i} \\
&= GL_i \times_{G_i} G_k/G_{k-i} \\
&\cong H_+(i)^{(i)} \times G_k/G_{k-i}
\end{aligned}
$$

and so,

$$
\begin{aligned}
H_+(k)^{(i)} &= M(i,k)^{\langle i \rangle}/G_i \\
&\cong H_+(i)^{(i)} \times_{G_i \times G_{k-i}} G_k,
\end{aligned}
$$

which proves (1).

Now, recall the definition of $\xi_k^{(i)}$. First, we must consider

$N^i|_{M(i,k)^{\langle i \rangle}}$. This is a (left) $G_i \times G_{n-i}$-vector bundle and also a (right) G_k-vector bundle. Since $M(i,k)^{\langle i \rangle}/G_k \cong H_+(i)^{(i)}$ is contractible, it follows that as a G_k-vector bundle $N^i|_{M(i,k)^{\langle i \rangle}}$ is determined by its restriction to a single orbit, for example, the orbit of

$$E_i = \begin{bmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ 0 & 0 \end{bmatrix} \in M(i,k)^{\langle i \rangle}.$$

(E_i is the identity matrix in the upper left-hand corner and zero elsewhere.) The fibre of N^i at E_i is clearly $M(n-i,k-i)$, on which the isotropy subgroup, $(G_k)_{E^i} = G_{k-i}$, acts by right multiplication. Thus, there is a G_k-vector bundle isomorphism

(a) $$N^i|_{M(i,k)^{\langle i \rangle}} \cong M(n-i,k-i) \times_{G_{k-i}} (H_+(i)^{(i)} \times G_k).$$

It is easy to see that $G_i \times G_{n-i}$ acts on the right-hand expression as follows. G_i acts trivially on $M(n-i,k-i)$, by conjugation on $H_+(i)^{(i)}$ and by left multiplication on G_k; while G_{n-i} acts by left multiplication on $M(n-i,k-i)$ and trivially on the other two factors. From 5.6, we know that as a $G_i \times G_{n-i}$-vector bundle.

(b) $$N^i|_{M(i,k)^{\langle i \rangle}} \cong M(n-i,k-i) \times_{G_{k-i}} \xi_k^{(i)}.$$

Comparing (a) and (b) we see that

$$\xi_k^{(i)} = H_+(i)^{(i)} \times G_k,$$

which proves (2). $\square$

Let $\rho = \rho_0 > \rho_1 \cdots > \rho_{k-1} > \rho_k = 0$ be a decreasing sequence of real numbers such that $\rho_i / \rho_{i+1} < 1/2$. Given such a sequence, we will first define manifolds with corners $B^i_k(\rho) \subset H_+(k)^{(i)}$, which will be the i^{th} stratum minus tubular neighborhoods of the lower strata. We will then cut $H_+(k)$ (and hence $M(n,k)$) into G_k-invariant pieces denoted by $C^i_k(\rho)$,

$$H_+(k) = \bigcup_i C^i_k(\rho) .$$

$C^i_k(\rho)$ will be a closed tubular neighborhood of $B^i_k(\rho)$.

6.2 <u>Definition:</u> Let $x \in H_+(k)$ and let

$$e_1(x) \geq e_2(x) \ldots\ldots \geq e_k(x) \geq 0$$

be the eigenvalues of x arranged in decreasing order. (The eigenvalues determine the G_k-orbit of x.) For each i, $0 \leq i \leq k$, let $B^i_k(\rho)$ be the G_k-invariant subset of $H_+(k)^{(i)}$ defined by the conditions

$$(6.3) \quad \sum_{j=q+1}^{j=i} e_j(x) \geq \rho_q - \rho_i, \quad \text{for each } q,\ 0 \leq q \leq i-1.$$

For each $q < i$, $B^i_k(\rho)$ has a <u>codimension one face</u> $\partial_q B^i_k(\rho)$ defined by

$$(6.4) \qquad \sum_{j=q+1}^{j=i} e_j(x) = \rho_q - \rho_i .$$

(Also, by convention, set $\partial_i B^i_k(\rho) = B^i_k(\rho)$). Similarly, if $\omega = (\omega_1, \ldots, \omega_\ell)$ is a sequence of nonnegative integers such that $|\omega| = \sum \omega_j \leq i$ and such that $\omega_j > 0$ for $j > 0$, we can define a codimension ℓ-face $\partial_\omega B^i_k(\rho)$ to be the intersection

$$\partial_{\omega_1} B^i_k \cap \partial_{\omega_1+\omega_2} B^i_k \cdots \cap \partial_{\omega_1+\omega_2+\ldots\omega_\ell} B^i_k .$$

Next we define a G_k-invariant neighborhood of the origin in $H_+(k)$, which will be denoted by $F_k(\rho)$. $F_k(\rho)$ is the set of $x \in H_+(k)$ satisfying the conditions

(6.5) $$\sum_{j=1}^{j=q} e_j(x) \leq \rho_0 - \rho_q; \text{ for each } q,\ 1 \leq q \leq k.$$

For each q, $F_k(\rho)$ has a codimension one face $\partial_q F_k(\rho)$ defined by

(6.6) $$\sum_{j=1}^{j=q} e_j(x) = \rho_0 - \rho_q$$

(and by convention, $\partial_0 F_k(\rho) = F_k(\rho)$).

Notice that there is a natural, face preserving G_k-diffeomorphism

(6.7) $$B_i^i(\rho') \times_{G_i} G_k/G_{k-i} \cong B_k^i(\rho),$$

$\rho' = (\rho_0 - \rho_i, \rho_1 - \rho_i, \ldots, \rho_{i-1} - \rho_i)$, defined by

$$[b, g] \longrightarrow g^{-1}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} g \cdot \in B_k^i(\rho).$$

Let $\xi_k^{(i)} \longrightarrow H_+(k)^{(i)}$ be as in 5.1, and let

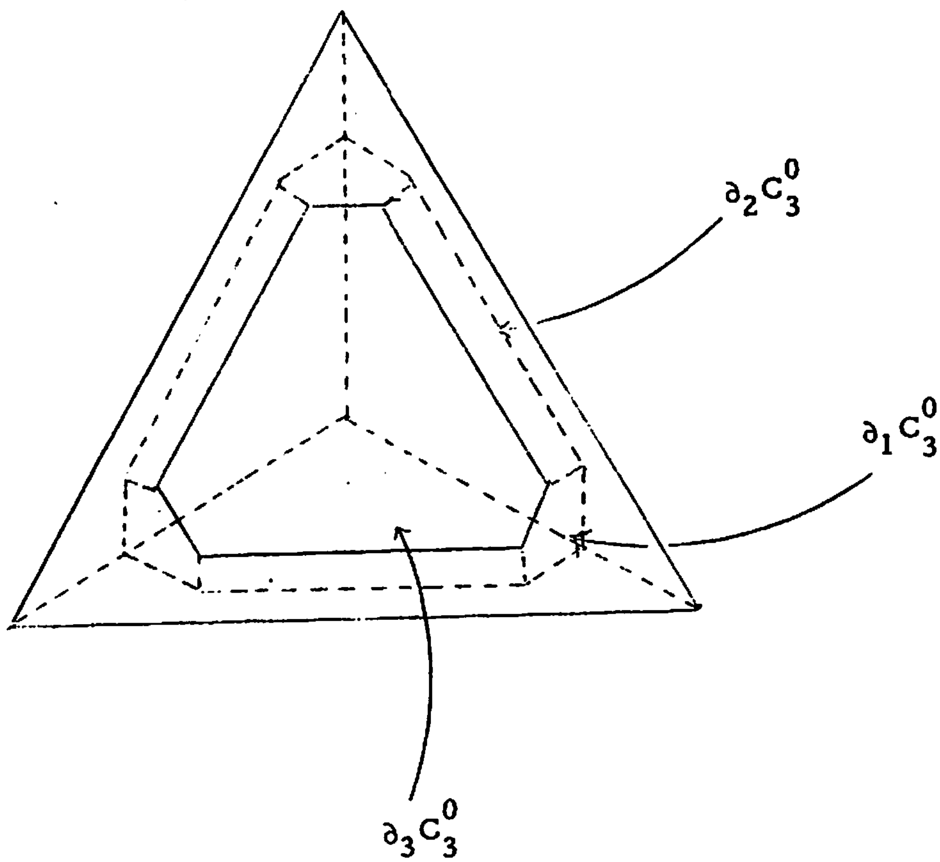
(6.8) $$\xi_k^i(\rho) = \xi_k^{(i)} \,\Big|_{B_k^i(\rho)}.$$

(Also, let $n_k^i(\rho) = \xi_k^i(\rho)/G_i \times Z_{k-i}$).

From Lemma 6.1 and (6.7), it follows that

(6.9) $$\xi_k^i(\rho) = B_i^i(\rho') \times G_k,$$

where $G_i \times G_{k-i}$ and G_k act on the right-hand side of the equation as in 6.1. As before, for a sequence of nonnegative integers $\omega = (\omega_1, \ldots, \omega_\ell)$, $|\omega| < i$, we can define



$C_3^0(\rho) \cap$ diagonal matrices in $H_+^1(3)$

$C_3^0(\rho) = F_3(\rho)$

$$(6.10) \qquad \partial_\omega \xi_k^i(\rho) = \xi_k^{(i)}\Big|_{\partial_\omega B_k^i(\rho)}.$$

6.11 <u>Definition:</u> Let $C_k^i(\rho) \longrightarrow B_k^i(\rho)$ be the bundle defined by

$$(6.12) \qquad C_k^i(\rho) = F_{k-i}(\rho'') \times_{G_{k-i}} \xi_k^i(\rho)/G_i$$

$$= F_{k-i}(\rho'') \times_{G_{k-i}} (B_i^i(\rho') \times_{G_i} G_k),$$

where $\rho'' = (\rho_i, \rho_{i+1}, \ldots, \rho_k)$ and $\rho' = (\rho_0 - \rho_i, \ldots, \rho_{i-1} - \rho_i, 0)$. There is a natural embedding $C_k^i(\rho) \longrightarrow H_+(k)$ defined by

$$(6.13) \qquad [f, b, g] \longrightarrow g^{-1}\begin{pmatrix} b & 0 \\ 0 & f \end{pmatrix} g \in H_+(k),$$

where $[f, b, g] \in F_{k-i}(\rho'') \times_{G_{k-i}} B_i^i(\rho') \times_{G_i} G_k$. We consider this map an inclusion. As usual, for each q, $0 \leq q < k$, we can define a <u>codimension one face</u> $\partial_q C_k^i(\rho)$ by

$$(6.14) \qquad \partial_q C_k^i(\rho) = \begin{cases} F_{k-i}(\rho'') \times_{G_{k-i}} \partial_q \xi_k^i/G_i; & \text{for } q \leq i \\ \partial_{q-i} F_{k-i}(\rho'') \times_{G_{k-i}} \xi_k^i/G_i; & \text{for } q > i. \end{cases}$$

6.15 <u>Remark:</u> Let $b \in B_i^i(\rho')$, $f \in F_{k-i}(\rho'')$ and let $e_1(b) \geq \ldots \geq e_i(b) \geq 0$ and $e_1(f) \geq \cdots \geq e_{k-i}(f) \geq 0$ be the eigenvalues of b and f, respectively. By (6.3) and (6.5),

$$e_i(b) \geq \rho_{i-1} - \rho_i \quad \text{and}$$

$$e_1(b) \leq \rho_i - \rho_{i+1}.$$

Since $\rho_{i-1} - 2\rho_i > 0 > -\rho_{i+1}$, it follows that $\rho_{i-1} - \rho_i > \rho_i - \rho_{i+1}$; hence

$$e_i(b) > e_1(f).$$

Therefore, using (6.3), (6.5) and (6.13), we see that $C_k^i(\rho)$ can be characterized as the set of x in $H_+(k)$ satisfying the conditions

$$(6.16) \quad (a) \quad \sum_{\ell=q+1}^{\ell=i} e_\ell(x) \geq \rho_q - \rho_i \; ; \qquad 0 \leq q \leq i-1$$

$$(b) \quad \sum_{\ell=i+1}^{\ell=q} e_\ell(x) \leq \rho_i - \rho_q \; ; \qquad i+1 \leq q \leq k.$$

At this point we have actually succeeded in cutting $H_+(k)$ into pieces which intersect along codimension one faces, as is shown by the following

6.17 Lemma:

(A) $C_k^j(\rho) \cap C_k^i(\rho) = \partial_i C_k^j(\rho) = \partial_j C_k^i(\rho)$

(B) $H_+(k) = \bigcup_i C_k^i(\rho)$.

Proof: If $j=i$, (A) is true by convention. Now suppose that $j < i$ and that $x \in \partial_j C_k^i(\rho)$, then

$$(1) \qquad \sum_{\ell=j+1}^{\ell=k} e_\ell(x) = \rho_j - \rho_i .$$

Subtracting (1) from 6.16 (a) and adding it to 6.16 (b), we get

$$(2) \quad (a) \quad \sum_{\ell=q+1}^{\ell=j} e_\ell(x) \geq \rho_q - \rho_j \; ; \qquad 0 \leq q \leq j-1$$

$$(b) \quad \sum_{\ell=j+1}^{\ell=q} e_\ell(x) \leq \rho_j - \rho_q \; ; \qquad j+1 \leq q \leq k.$$

Hence, $x \in \partial_i C_k^j(\rho)$. A similar argument shows that $\partial_j C_k^i(\rho) \subset \partial_i C_k^j(\rho)$, for $j > i$. By symmetry, $\partial_i C_k^j(\rho) = \partial_j C_k^i(\rho) \subset C_k^i(\rho) \cap C_k^j(\rho)$. On the other hand, if $x \in C_k^i(\rho) \cap C_k^j(\rho)$ and if $j < i$, then

$$\sum_{\ell=j+1}^{\ell=i} e_\ell(x) \geq \rho_j - \rho_i, \quad \text{since } x \in C_k^i(\rho), \text{ and}$$

$$\sum_{\ell=j+1}^{\ell=i} e_\ell(x) \leq \rho_j - \rho_i, \quad \text{since } x \in C_k^j(\rho).$$

Hence, (1) is true, from which it follows that $x \in \partial_j C^i(\rho)$. This proves (A). Using (A) and the fact $C_k^i(\rho)$ is closed in $H_+(k)$, it is obvious that $\bigcup_i C_k^i(\rho)$ is both open and closed in $H_+(k)$; hence, they are equal.

$\square$

6.18 <u>Lemma:</u> Let $\rho = (\rho_0, \rho_1, \ldots, \rho_{k-1}, \rho_k = 0)$ and $\epsilon = (\epsilon_0, \ldots, \epsilon_{k-1}, \epsilon_k = 0)$ be two sequences such that $\rho_i > 2\rho_{i+1}$ and $\epsilon_i > 2\epsilon_{i+1}$. Then $B_k^i(\rho)$ and $C_k^i(\rho)$ are G_k-invariant semialgebraic spaces (as are the faces). Furthermore, $B_k^i(\rho) \cong B_k^i(\epsilon)$ and $C_k^i(\rho) \cong C_k^i(\epsilon)$, where $\cong$ means that there is a face preserving diffeomorphism (i.e., a face preserving isomorphism of functionally structured spaces).

<u>Proof:</u> The first statement is clear, since the conditions (6.3) and (6.18) defining $B_k^i(\rho)$ and $C_k^i(\rho)$ can be written in terms of inequalities involving the symmetric polynomials in the eigenvalues of x, and since these polynomials are just the G_k-invariant polynomials on $H(k)$. To prove the last sentence, we may assume, without loss of generality, that for each i, $\rho_i > \epsilon_i$. One then shows that $B_k^i(\epsilon_0, \ldots, \epsilon_k) \cong B_k^i(\rho_0, \epsilon_1, \ldots, \epsilon_k)$ $\cong B_k^i(\rho_0, \rho_1, \epsilon_2 \ldots \epsilon_k) \cong \ldots \cong B_k^i(\rho_0, \rho_1 \ldots \rho_k)$. In a similar fashion,

$F_k(\epsilon) \cong F_k(\rho)$, and hence $C_k^i(\epsilon) \cong C_k^i(\rho)$. The details are somewhat messy and we omit them.

□

Notation: In view of the above lemma, we can omit the "ρ" in our notation and write simply B_k^i, C_k^i, ξ_k^i, etc.

6.19 Definition ([J; p. 54]): A smooth manifold with corners X is a functionally structured space which is differentiably modeled on the quadrant $\{x \in \mathbb{R}^m | x_1 \geq 0, \ldots, x_m \geq 0\}$. If $x \in X$ is represented by local coordinates $(x_1, \ldots, x_m)$, we denote by $c(x)$ the number of zeroes in this m-tuple. X is called a manifold with faces if every $x \in X$ belongs to exactly $c(x)$ different connected components of $\{p \in X | c(p) = 1\}$. Any disjoint union of these connected components is called a codimension one face.

Following Jänich, we define a $\langle j \rangle$-manifold to be a manifold with faces X together with a j-tuple $(\partial_0 X, \ldots, \partial_{j-1} X)$ of codimension one faces such that

(i) $\partial_0 X \cup \cdots \cup \partial_{j-1} X = \partial X$

(ii) $\partial_i X \cap \partial_j X$ is a codimension one face of $\partial_i X$ and of $\partial_j X$ for $i \neq j$.

6.20 Proposition

(i) B_k^j is a $\langle j \rangle$-manifold. $\partial_0 B_j^j$ is diffeomorphic to a disk D^{p-1} (with faces) and so $B_j^j = \partial_0 B_j^j \times [\rho_0, \infty)$ is diffeomorphic to the half space $\mathbb{R}_+^p$, where $p = \frac{1}{2} k(d(k-1) + 2)$ and where $\mathbb{R}_+^p$ is given the structure of a $\langle j \rangle$-manifold and D^{p-1} is given the structure of a $\langle j-1 \rangle$-manifold.

(ii) $B^j_k \cong B^j_j \times_{G_j} G_k/G_{k-j}$. Furthermore, if $\omega = (\omega_1, \ldots, \omega_\ell)$ is a sequence of nonnegative integers with $|\omega| < j$ and with $\omega_i > 0$ for $i > 0$, then

$$\partial_\omega B^j_k \cong \left\{\partial_0 B^{j-|\omega|}_{j-|\omega|} \times \partial_0 B^{\omega_\ell}_{\omega_\ell} \times \ldots \partial_0 B^{\omega_2}_{\omega_2} \times B^{\omega_1}_{\omega_1}\right\} \times_{G_{j-|\omega|} \times G_\omega} G_k/G_{k-j},$$

where $G_\omega = G_{\omega_1} \times \ldots \times G_{\omega_\ell}$. Here $G_{j-|\omega|} \times G_\omega$ acts by left multiplication on G_k/G_{k-j} and by conjugation on each factor of the expression in brackets. In other words, $B^j_k \longrightarrow G_k/G_j \times G_{k-j}$ is a bundle with contractible fibre, as is $\partial_\omega B^j_k \longrightarrow G_k/G_{j-|\omega|} \times G_\omega \times G_{k-j}$.

Proof: It is easily checked that B^j_j and hence B^j_k are $\langle j \rangle$-manifolds. B^j_j is clearly G_k-diffeomorphic to {$H_+(j)$-collared neighborhood of the boundary}. $H_+(j)^{(j)}$ is G_k-diffeomorphic to Euclidean space $H(j) = \mathbb{R}^p$ (by the exponential map). Similarly $\partial_0 B^j_j$ is G_k-diffeomorphic to {$W_+(j)$-collared neighborhood of the boundary} which is D^{p-1}. Also, the natural map $\partial_0 B^j_j \times [\rho_0, \infty) \longrightarrow B^j_j$, sending (x,t) to tx is obviously a face preserving G_k-diffeomorphism. This proves (i).

According to equation (6.7),

$$B^j_k \cong B^j_j \times_{G_j} G_k/G_{k-j} \text{ and so}$$

(1) $\quad \partial_i B^j_k \cong \partial_i B^j_j \times_{G_j} G_k/G_{k-j}$, for $i < j$.

Using Lemma 6.17 twice,

$$\partial_i B^j_j = \partial_i C^j_j$$

$$= \partial_j C^i_j$$

$$= \partial_{j-i} F_{j-i} \times_{G_{j-i}} \xi^i_j$$

$$= \{\partial_0 B^{j-i}_{j-i} \times B^i_i\} \times_{G_{j-i} \times G_i} G_j .$$

Combining this with (1),

$$\text{(2)} \qquad \partial_i B^j_k \cong \{\partial_0 B^{j-i}_{j-i} \times B^i_i\} \times_{G_{j-i} \times G_i} G_k / G_{k-j} .$$

Repeating this process proves (ii).

□

We can cut apart $M(n,k)$ the same way we cut apart its orbit space. Thus, if $\pi: M(n,k) \longrightarrow H_+(k)$ is the orbit map, define

$$\text{(6.21)} \qquad D(n,k) = \pi^{-1}(F_k)$$

$$\text{(6.22)} \qquad M^j_k = G_n / G_{n-j} \times_{G_j} \xi^j_k / G_{k-j}$$

$$\text{(6.23)} \qquad N^j_k = \{G_n \times_{G_{n-j}} D(n-j, k-j)\} \times_{G_j \times G_{k-j}} \xi^j_k .$$

There is a natural isomorphism $M^j_k \cong \pi^{-1}(B^j_k) \subset M(n,k)^{(j)}$, defined by

$$\text{(6.24)} \qquad [h,b,g] \longrightarrow h \cdot \begin{pmatrix} b^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \cdot g$$

where $[h,b,g] \in M^j_k = G_n \times_{G_{n-j} \times G_j \times G_{k-j}} \{B^j_j \times G_k\}$, and where $b^{\frac{1}{2}}$ is the unique positive definite square root of $b \in B^j_j$. Similarly, there is an isomorphism $N^j_k \cong \pi^{-1}(C^j_k) \subset M(n,k)$, covering $C^j_k \subset H_+(k)$, defined by

$$(6.25) \qquad [h,d,b,g] \longrightarrow h \cdot \begin{pmatrix} b^{\frac{1}{2}} & 0 \\ 0 & d \end{pmatrix} \cdot g,$$

where $[h,d,b,g] \in N_k^j = \{G_n \times_{G_{n-j}} D(n-j,k-j)\} \times_{G_j \times G_{k-j}} \{B_j^j \times G_k\}$. We will regard the maps defined by (6.24) and (6.25) as inclusions. Thus, $M_k^j \subset M(n,k)^{(j)}$ is just the j^{th} stratum with tubular neighborhoods of the orbits of smaller type removed, and N_k^j is a normal disk bundle about M_k^j. Of course, we can also define

$$(6.26) \qquad \partial_\omega M_k^j = \pi^{-1}(\partial_\omega B_k^j)$$

$$\partial_\omega N_k^j = \pi^{-1}(\partial_\omega C_k^j), \quad \text{etc.}$$

From Lemma 6.17 and (6.25) we have the natural identifications $\partial_{i+j} N_k^i = \partial_i N_k^{i+j} = N_k^i \cap N_k^{i+j}$, which we now consider.

6.27 <u>Definition</u>: Let ξ_k^{ij} be defined by

$$\xi_k^{ij} = \partial_0 \xi_{k-i}^j \times_{G_{k-i}} \xi_k^i$$

$$= \partial_0 B_j^j \times B_i^i \times G_k.$$

This is naturally a principal $G_i \times G_j \times G_{k-i-j}$-bundle over

$$\{\partial_0 B_j^j \times B_i^i\} \times_{G_i \times G_j} G_k / G_{k-i-j} \cong \partial_i B_k^{i+j}.$$

By using 6.17, notice that

$$(6.28) \qquad \partial_{i+j} N_k^i = \partial_{i+j}(\{G_n \times_{G_{n-i}} D(n-i,k-i)\} \times_{G_i \times G_{k-i}} \xi_k^i)$$

$$= \{G_n \times_{G_{n-i}} \partial_j D(n-i,k-i)\} \times_{G_i \times G_{k-i}} \xi_k^i$$

$$= \{G_n \times_{G_{n-i-j}} D(n-i-j, k-i-j)\} \times_{G_i \times G_j \times G_{k-i-j}} (\partial_0 \xi^j_{k-i} \times_{G_{k-i}} \xi^i_k)$$

$$= \{G_n \times_{G_{n-i-j}} D(n-i-j, k-i-j)\} \times_{G_i \times G_j \times G_{k-i-j}} \xi^{ij}_k .$$

The last equation expresses the fact that $\partial_{i+j} N^i_k$ is the total space of a normal $D(n-i-j, k-i-j)$-bundle over $\{\partial_{i+j} N^i_k\}^{(i+j)} =$ $G_n/G_{n-i-j} \times_{G_i \times G_j} \xi^{ij}_k / G_{k-i-j}$.

On the other hand,

$$(6.29) \qquad \partial_i N^{i+j}_k = \{G_n \times_{G_{n-i-j}} D(n-i-j, k-i-j)\} \times_{G_{i-j} \times G_{k-i-j}} \partial_i \xi^{i+j}_k .$$

Using (6.25), there is an isomorphism $\partial_{i+j} N^i_k \longrightarrow \partial_i N^{i+j}_k$ of $D(n-i-j, k-i-j)$-bundles. To be more explicit, we can define $G_i \times G_j \times G_{k-i-j}$-equivariant embeddings $f^{ij}_k \colon \xi^{ij}_k \longrightarrow \partial_i \xi^{i+j}_k$ by

$$(6.30) \qquad f^{ij}_k (b_j, b_i, g) = \left(\begin{bmatrix} b_i & 0 \\ 0 & b_j \end{bmatrix}, g \right),$$

where $(b_j, b_i, g) \in \partial_0 B^j_j \times B^i_i \times G_k = \xi^{ij}_k$, so that the following diagram commutes

$$(6.31) \qquad \begin{array}{ccc} \partial_{i+j} N^i_k & \xrightarrow{\times f^{ij}_k} & \partial_i N^{i+j}_k \\ \cap & & \cap \\ N^i_k & \subset M(n,k) \supset & N^j_k \end{array} .$$

Slightly more generally, if $\omega = (i_1, \ldots, i_p)$ is a sequence of non-negative integers, $|\omega| \leq k$, and if $G_\omega = G_{i_1} \times \ldots \times G_{i_p}$, then we can define a principal $G_\omega \times G_{k-|\omega|}$-bundle, $\xi_k^\omega \longrightarrow \partial_\omega B_k^{|\omega|}$ by

$$(6.32)\qquad \xi_k^\omega = \partial_0 \xi_{k-(i_1+\dots i_{p-1})}^{i_p} \times_{G_{k-(i_1+\dots+i_{p-1})}} \partial_0 \xi_{k-(i_1+\dots i_{p-2})}^{i_{p-1}} \times \dots$$

$$\times \partial_0 \xi_{k-i_1}^{i_2} \times_{G_{k-i_1}} \xi_k^{i_1}$$

$$= \partial_0 B_{i_p}^{i_p} \times \dots \times \partial_0 B_{i_2}^{i_2} \times B_{i_1}^{i_1} \times G_k .$$

It follows from part (ii) of Proposition 6.20 that the base space of ξ_k^ω is isomorphic to $\partial_\omega B_k^{|\omega|}$ (here the usual boundary conventions $\partial_i B_k^i = B_k^i$ and $\partial_0 C_k^0 = C_k^0$ are enforced). Our construction is such that,

$$(6.33)\qquad \partial_\omega N_k^{i_1} = \pi^{-1}(\partial_\omega C_k^{i_1}) = \{G_n \times G_{n-|\omega|} D(n-|\omega|, k-|\omega|) \times_{G_\omega \times G_{k-|\omega|}} \xi_k^\omega .$$

Using (6.25) again, there is a natural identification of $\partial_\omega N_k^{i_1}$ with, say,

$$\partial_\omega N_k^{|\omega|} = \pi^{-1}(\partial_\omega C_k^{|\omega|})$$

$$= \{G_n \times_{G_{n-|\omega|}} D(n-|\omega|, k-|\omega|)\} \times_{G_{|\omega|} \times G_{k-|\omega|}} \partial_\omega \xi_k^{|\omega|}$$

which is defined by a $G_\omega \times G_{k-|\omega|}$-equivariant embedding $f_k(\omega, |\omega|)\colon \xi_k^\omega \longrightarrow \partial_\omega \xi_k^{|\omega|}$. If $(b_{i_p}, \dots, b_{i_1}, g) \in \xi_k^\omega =$ $\partial_0 B_{i_p}^{i_p} \times \dots \times \partial_0 B_{i_2}^{i_2} \times B_{i_1}^{i_1} \times G_k$, then checking through the definitions we see that

$$(6.34)\quad f_k(\omega, |\omega|)(b_{i_p}, \ldots, b_{i_1}, g) = \left(\begin{bmatrix} b_{i_p} & & & & 0 \\ & \ddots & & & \\ & & \ddots & & \\ & & & b_{i_2} & \\ 0 & & & & b_{i_1} \end{bmatrix}, g \right).$$

In other words, $\times f_k(\omega, |\omega|): \partial_\omega N_k^{i_1} \cong \partial_\omega N_k^{|\omega|}$ is the natural identification.

The maps $f_k(\omega, |\omega|)$ also satisfy a rather obvious commutativity condition. To formulate it properly we need one further definition.

6.35 <u>Definition:</u> Let $\omega' = (j_1, \ldots, j_s)$ be a sequence of nonnegative integers. A partition of ω' is a sequence

$$\begin{aligned} \omega &= (i_1, \ldots, i_{p_1}, i_{p_1+1}, \ldots, i_{p_2}, \ldots, i_{p_s}) \\ &= (\omega_1, \omega_2, \ldots, \omega_s) \end{aligned}$$

where $\omega_\ell = (i_{p_{\ell-1}+1}, \ldots, i_{p_\ell})$ and $|\omega_\ell| = j_\ell$. This situation is denoted by $\omega \subset \omega'$. Define $\partial_\omega \xi_k^{\omega'}$ by

$$\partial_\omega \xi_k^{\omega'} = \partial_0 \partial_{\omega_p} \xi^{j_p}_{k-(i_1+\ldots i_{p_{s-1}})} \times_{G_{k-(i_1+\ldots i_{p_{s-1}})}} \cdots$$

$$\cdots \times \partial_0 \partial_{\omega_2} \xi^{j_2}_{k-j_1} \times_{G_{k-j_1}} \partial_{\omega_1} \xi_k^{j_1}.$$

Similarly, we can define $\partial_\omega N_k^{\omega'}$, $\partial_\omega \eta_k^{\omega'}$, $\partial_\omega C_k^{\omega'}$, etc. As in (6.34), we get a $G_\omega \times G_{k-|\omega|}$-embedding, $f_k(\omega, \omega'): \xi_k^\omega \longrightarrow \partial_\omega \xi_k^{\omega'}$, which exhibits the isomorphism $\partial_\omega N_k^{i_1} \cong \partial_\omega N_k^{\omega'}$.

The following lemma is obvious.

6.36 Lemma: Suppose that $\omega \subset \omega' \subset \Omega$ and that $\omega \subset \omega'' \subset \Omega$. Then the following diagram commutes

$$\begin{tikzcd}
\xi_k^{\omega} \arrow[rr, "f_k(\omega,\omega')"] \arrow[d, dashed, "f_k(\omega,\omega'')"'] \arrow[drr, "f_k(\omega,\Omega)"] & & \partial_\omega \xi_k^{\omega'} \arrow[d, "f_k(\omega',\Omega)|_{\partial_\omega \xi_k^{\omega'}}"] \\
\partial_\omega \xi_k^{\omega''} \arrow[rr, "f_k(\omega'',\Omega)|_{\partial_\omega \xi_k^{\omega''}}"] & & \partial_\omega \xi_k^{\Omega}
\end{tikzcd}$$

7. Normal Systems

In section 5 we showed that the normal bundle of the i^{th}-stratum of a $\mathcal{S}_{n,k}$-manifold X is determined by a principal $G_i \times G_{k-i}$-bundle $\xi^{(i)} \longrightarrow X_{\#}^{(i)}$. When X was the linear model $M(n,k)$, we described, in section 6, how these normal bundles are pasted together via principal bundle maps. Similarly, we can cut apart an arbitrary X, modeled on $k\rho_n$, by choosing a system of tubular neighborhoods of the strata which match up the same way as do the tubular neighborhoods in our linear model. Formalizing this notion leads to the definition of a "normal system." Corresponding to each of tht three categories $\mathcal{S}_{n,k}$, $\mathcal{B}_k$ and $\mathcal{S}_{n,k}(B)$ we will define categories of normal systems denoted by $\widetilde{\mathscr{D}}_{n,k}$, $\widetilde{\mathscr{B}}_k$ and $\widetilde{\mathscr{D}}_{n,k}(\eta_B)$, respectively. An object in $\widetilde{\mathscr{B}}_k$, for example, should be thought of as a space B, modeled on $H_+(k)$, together with a "compatible" system of tubular neighborhoods of the strata. A $\widetilde{\mathscr{B}}_k$-morphism will correspond to a smooth strata preserving map $B \longrightarrow B'$ which is linear on the prescribed tubular neighborhoods.

Although they are not the same, the following definition of a normal system bears some similarity to Jänich's notion of a "(n, r)-complete system" in [J].

7.1 __Definition:__ A $\mathscr{D}_{n,k}$-__normal system__, $\xi = \{\xi^i \longrightarrow B^i, f^{ij}: \xi^{ij} \longrightarrow \partial_i \xi^{i+j}\}$ consists of the following data:

(i) For each i, $0 \leq i \leq k$, an $\langle i \rangle$-manifold B^i (possibly the empty manifold).

(ii) Principal $G_i \times G_{k-i}$-bundles $\xi^i \longrightarrow B^i$.

(iii) If $\partial_i \xi^{i+j} = \xi^{i+j}|_{\partial_i B^{i+j}}$ and if

$$\xi^{ij} = \partial_0 \xi^j_{k-i} \times_{G_{k-i}} \xi^i,$$

then for each pair of nonnegative integers (i, j), $0 < i\ j \leq k$, there is a $G_i \times G_j \times G_{k-i-j}$-equivariant embedding

$$f^{ij}: \xi^{ij} \longrightarrow \partial_i \xi^{i+j}$$

such that

$$\times f^{ij}: G_{i+j} \times_{G_i \times G_j} \xi^{ij} \longrightarrow \partial_i \xi^{i+j}$$

is an isomorphism of principal $G_{i+j} \times G_{k-i-j}$-bundles (covering a face preserving diffeomorphism of the base spaces). Finally the f^{ij}'s are subject to the following compatibility condition.

(iv) For each triple of nonnegative integers i, j, ℓ, $0 \leq i + j + \ell \leq k$, the following diagram commutes

$$\begin{tikzcd} \xi^{ij\ell} \arrow[r, "\times f^{j\ell}_{k-i}"] \arrow[d, "\times f^{ij}"'] & \partial_{i+j}\, \xi^{i, j+\ell} \arrow[d, "f^{i, j+\ell}"] \\ \partial_i\, \xi^{i+j, \ell} \arrow[r] & \partial_{ij}\xi^{i+j+\ell} \end{tikzcd}$$

where

$$\xi^{ij\ell} = \partial_0\, \xi^{\ell}_{k-i-j} \times_{G_{k-i-j}} \partial_0\, \xi^{j}_{k-i} \times_{G_{k-i}} \xi^{i}$$

is a principal $G_i \times G_j \times G_\ell \times G_{k-i-j-\ell}$-bundle. (Once the above diagram commutes it is easy to see that a diagram similar to that of Lemma 6.36 also commutes.) The difference in notation between B^i and $B^{(i)}$ is intentional. Eventually it will be clear that B^i is the closed i^{th}-stratum of a $\mathcal{B}_k$-space B, that is, B^i will be $B^{(i)}$ with open tubular neighborhoods of the lower strata removed.

If $\xi = \{\xi^i \longrightarrow B^i,\ f^{ij}: \xi^{ij} \longrightarrow \partial_i \xi^{i+j}\}$ and $\gamma = \{\gamma^i \longrightarrow D^i, \bar{f}^{ij}: \gamma^{ij} \longrightarrow \partial_i \gamma^{i+j}\}$ are $\mathscr{D}_{n,k}$-normal systems, then a morphism $\theta: \xi \longrightarrow \gamma$ of normal systems means a collection $\{\theta^i;\ 0 \leq i \leq k\}$ of $G_i \times G_{k-i}$-bundle maps $\theta^i: \xi^i \longrightarrow \gamma^i$ such that

(1) θ^i covers a smooth map of $\langle i \rangle$-manifolds (preserving faces), $\theta^i_{\#}: B^i \longrightarrow D^i$, and

(2) θ commutes with the f^{ij}'s, i.e., the following diagram commutes

$$\begin{tikzcd} \xi^{ij} \arrow[r, "\times\theta^i"] \arrow[d, "f^{ij}"'] & \gamma^{ij} \arrow[d] \\ \partial_i \xi^{i+j} \arrow[r, "\theta^{i+j}"] & \partial_i \gamma^{i+j} \end{tikzcd} .$$

Notice that θ will automatically map the diagram of condition (iv) to the corresponding diagram for γ. Let $\widetilde{\mathscr{S}}_{n,k}$ denote the category with objects $\mathscr{S}_{n,k}$-normal systems and with morphisms defined as above.

In a similar fashion, we can define $\widetilde{\mathcal{B}}_k$, the <u>category of $\mathcal{B}_k$-normal systems</u>. A $\mathcal{B}_k$-normal system $\eta = \{\eta^i \longrightarrow B^i,\ g^{ij}: \eta^{ij} \longrightarrow \partial^{ij} \eta^{i+j}\}$ consists of

(i)′ $\langle i \rangle$-manifolds, B^i,

(ii)′ Principal PG_{k-i}-bundles $\eta^i \longrightarrow B^i$,

(iii)′ Isomorphisms of PG_{k-i-j}-bundles $g^{ij}: \eta^{ij} \longrightarrow \partial_i \eta^{i+j}$, where

$$\eta^{ij} = \partial_0 \eta^j_{k-i} \times_{PG_{k-i}} \eta^i .$$

(iv)′ The g^{ij}'s satisfy a compatibility condition analogous to (iv).

Notice that naturally associated to a $\mathscr{S}_{n,k}$-normal

$\xi = \{\xi^i \longrightarrow B^i,\ f^{ij}: \xi^{ij} \longrightarrow \partial_i \xi^{i+j}\}$, there is a $\mathcal{B}_k$-normal system

$\eta = \{\eta^i \longrightarrow B^i,\ g^{ij}: \eta^{ij} \longrightarrow \partial_i \eta^{i+j}\}$, which is defined as follows:

$$\eta^i = \xi^i / G_i \times Z_{k-i},$$

and the g^{ij}'s are defined by the following diagram:

$$\begin{tikzcd}
\xi^{ij} \arrow[r, "f^{ij}"] \arrow[d] & \partial_i \xi^{i+j} \arrow[d] \\
\xi^{ij}/G_i \times G_j \times Z_{k-i-j} \arrow[r] \arrow[d, equal] & \partial_i \xi^{i+j}/G_i \times G_j \times Z_{k-i-j} \arrow[d] \\
\eta^{ij} \arrow[ddr, dashed, "g^{ij}"] & \partial_i \xi^{i+j}/G_{i+j} \times Z_{k-i-j} \arrow[d, equal] \\
 & \partial_i \eta^{i+j} .
\end{tikzcd}$$

In this situation we say that ξ covers η.

Next we want to see that our intuitive description of $\widetilde{\mathscr{S}}_{n,k}$ is correct, i.e., that $\widetilde{\mathscr{S}}_{n,k}$ is the category an object of which is a $\mathscr{S}_{n,k}$-manifold together with a prescribed "compatible" system of tubular neighborhoods of the strata. Let $\xi = \{\xi^i \longrightarrow B^i,\ f^{ij}: \xi^{ij} \longrightarrow \partial_i \xi^{i+j}\}$ be a $\mathscr{S}_{n,k}$-normal system. Define modeled G_n-disk bundles of type (k, i) by

$$(7.2) \qquad N^i = \{G_n \times_{G_{n-i}} D(n-i, k-i)\} \times_{G_i \times G_{k-i}} \xi^i,$$

where $D(n-i, k-i) = \pi^{-1}(F_k) \subset M(n-i, k-i)$ is as in (6.21). For $0 < j \leq k-i$, let $\partial_{i+j} N^i \subset N^i$ be the modeled G_n-disk bundle of type $(k, i+j)$ defined by

$$(7.3) \qquad \partial_{i+j} N^i = \{G_n \times_{G_{n-i-j}} D(n-i-j, k-i-j)\} \times_{G_i \times G_j \times G_{k-i-j}} \xi^{ij}.$$

Note that $\partial_i N^{i+j} = N^{i+j}|_{\partial_i B^{i+j}} \subset N^{i+j}$ is defined by

$$\partial_i N^{i+j} = \{G_n \times_{G_{n-i-j}} D(n-i-j, k-i-j)\} \times_{G_{i+j} \times G_{k-i-j}} \partial_i \xi^{i+j} .$$

Using $f^{ij}: \xi^{ij} \longrightarrow \partial_i \xi^{i+j}$, we get a G_n-disk bundle isomorphism

$$(7.4) \qquad F^{ij} = \times f^{ij}: \partial_{i+j} N^i \cong \partial_i N^{i+j}.$$

We can now define an <u>assemble functor</u> $A: \widetilde{\mathscr{S}}_{n,k} \longrightarrow \mathscr{S}_{n,k}$ by setting

$$(7.5) \qquad A(\xi) = \coprod_i N^i / \sim F^{ij}, \text{ i.e.,}$$

$A(\xi)$ is the space formed from the disjoint union of the N^i by identifying $\partial_{i+j} N^i$ with $\partial_i N^{i+j}$ via F^{ij}, for each i, j with $0 < i + j \leq k$. Condition (iv) of Definition 7.1 insures that $A(\xi)$ is well defined. $A(\xi)$ is readily seen to be a smooth G_n-manifold modeled on $k\rho_n$. Let $X^i \subset A(\xi)^{(i)}$ denote the zero-section of N^i ($X^i = \xi^i / G_{k-i}$ is the i^{th} stratum minus tubular neighborhoods of the orbits of smaller type). $A(\xi)$ comes naturally equipped with equivariant tubular neighborhood maps for $X^i \subset A(\xi)$, namely the inclusion maps $\tau^i: N^i \longrightarrow A(\xi)$. If $\theta: \xi \longrightarrow \gamma$ is a morphism, then we can define an isovariant map $A(\theta): A(\xi) \longrightarrow A(\gamma)$ in the obvious way, by

$$(7.6) \qquad A(\theta)|_{N^i(\xi)} = \times \theta^i: N^i(\xi) \longrightarrow N^i(\gamma).$$

Thus, $A(\theta)$ is a linear map on the tubular neighborhood of each stratum.

In a similar manner we can define an <u>assemble functor</u> $A: \widetilde{\mathscr{B}}_k \longrightarrow \mathscr{B}_k$ (also denoted by A). If $\eta = \{\eta^i \longrightarrow B^i; g^{ij}: \eta^{ij} \longrightarrow \partial_i \eta^{i+j}\}$ is a $\mathscr{B}_k$-normal system, then set

$$(7.7) \qquad C^i = F_{k-i} \times_{PG_{k-i}} \eta^i,$$

where F_{k-i} is as in (6.5), and define an isomorphism G^{ij} by

(7.8) $$G^{ij} = \times g^{ij} : \partial_{i+j} C^i \longrightarrow \partial_i C^{i+j} .$$

Just as in the $\widetilde{\mathscr{F}}_{n,k}$ case, we can define

(7.9) $$A(\eta) = \coprod_i C^i / \sim G^{ij} ,$$

which is a space modeled on $H_+(k)$.

The next proposition is a very careful version of the existence part of the Equivariant Tubular Neighborhood Theorem.

7.10 Proposition

(i) If X is a $\mathscr{S}_{n,k}$-manifold, then there is a $\mathscr{S}_{n,k}$-normal system ξ such that $A(\xi)$ is equivariantly diffeomorphic to X.

(ii) If B is a $\mathscr{B}_k$-space, then there is a $\mathscr{B}_k$-normal system η such that $A(\eta) \cong B$.

Proof: We only prove (i), since the proof of (ii) is essentially the same. We are going to inductively define a decreasing sequence of $\mathscr{S}_{n,k}$-manifolds with faces

$$X = Y(0) \supset Y(1) \supset Y(2) \ldots\ldots \supset Y(k).$$

Assuming, for the moment, that $Y(i)$ has been defined, let

(1) $B^i = Y(i)^{(i)}_{\#}$ and let

(2) $\xi^i \longrightarrow B^i$

be the principal $G_i \times G_{k-i}$-bundle which classifies the equivariant normal bundle of $Y(i)^{(i)} \subset Y(i)$. Also, let

(3) $$N^i = \{G_n \times_{G_{n-i}} D(n-i, k-i)\} \times_{G_i \times G_{k-i}} \xi^i$$

be this normal disk-bundle.

If $X^{(p)}$ is the lowest stratum of X, then set $Y(0) = Y(1) = \dots = Y(p) = X$. By induction, suppose that for each $i < r$, we have defined $Y(i)$ and tubular neighborhood maps $\tau_i : N^i \longrightarrow Y(i)$ with the following two properties:

(A) For each pair (i, j), with $i + j < r$, there is a $G_i \times G_j \times G_{k-i-j}$-equivariant embedding $f^{ij}: \xi^{ij} \longrightarrow \partial_i \xi^{i+j}$ making the following diagram commute:

$$\begin{tikzcd}
\partial_{i+j} N^i \arrow[r, "\tau_i"] \arrow[d, dashed, "\times f^{ij}"] & Y(i) \arrow[d, phantom, "\cup"] \\
\partial_i N^{i+j} \arrow[r, "\tau_{i+j}"] & Y(i+j)
\end{tikzcd}$$

where ξ^{ij} is defined as in 7.1.

(B) For each triple (i, j, ℓ) with $i+j+\ell < r$, the following diagram commutes

$$\begin{tikzcd}
\partial_{i+j,\ell} N^i \arrow[rr, "\times f^{i,j+\ell}"] \arrow[dr, "\times f^{i,j}"'] & & \partial_{i,j} N^{i+j+\ell} \arrow[dl, "\times f^{i+j,\ell}"] \\
& \partial_{i,j+\ell} N^{i+j} &
\end{tikzcd}$$

where $\partial_{i,j}(*) = \partial_i(*) \cap \partial_{i+j}(*)$.

We can now define $Y(r)$ by

$$(4) \qquad Y(r) = \text{closure of } \{Y(r-1) - \tau_{r-1}(N^{r-1})\}$$
$$= \text{closure of } \{X - \bigcup_{i=p}^{i=r-1} \tau_i(N^i)\}.$$

The definition of $\tau_r: N^r \longrightarrow Y(r)$ is a little more complicated.

By the Equivariant Tubular Neighborhood Theorem, we can find some tubular neighborhood map $\overline{\tau}_r: N^r \longrightarrow Y(r)$. We must alter $\overline{\tau}_r$ so that diagrams (A) and (B) are satisfied. Recall that the lowest stratum of X is $X^{(p)}$. First notice that we may alter $\overline{\tau}_r$ by an isotopy so that $\overline{\tau}_r(\partial_p N^r) = \overline{\tau}_p(\partial_r N^p)$, and so that $(\overline{\tau}_r)^{-1} \circ \overline{\tau}_p: \partial_r N^p \longrightarrow \partial_p N^r$ is an orthogonal isomorphism of modeled G_n-disk bundles of type (k, r). By 5.6, $(\overline{\tau}_r)^{-1} \circ \overline{\tau}_p$ defines a $G_p \times G_{r-p} \times G_{k-r}$-equivariant embedding $f^{p,r-p}: \xi^{p,r-p} \longrightarrow \partial_p \xi^r$ (as in (6.30)). Therefore, diagram (A) holds with $i=p$ and $j=r-p$.

Suppose, by a further induction, that we have altered $\overline{\tau}_r$ by isotopies so that for some $s \leq r$, diagram (A) holds whenever $i < s$ and $j=r-i$ and that diagram (B) holds whenever $i+j < s$ and $i+j+\ell = r$. Consider the problem of deforming $\overline{\tau}_r$ so that it "matches up" with τ_s on $\partial_r N^s \cong \partial_s N^r$. Notice that $\overline{\tau}_r$ already "matches up" with τ_s on the codimension two faces, i.e., $(\tau_r)^{-1} \circ \tau_s: \partial_{i,r-i} N^s \longrightarrow \partial_{i,s-i} N^r$ is an orthogonal isomorphism of G_n-disk bundles. This is because the other two maps in the commutative diagram

$$\begin{tikzcd}[column sep=small] \partial_{s,r-s}N^i \arrow[rr, "(\tau_r)^{-1}\circ\tau_i"] \arrow[dr, "(\tau_s)^{-1}\circ\tau_i"'] & & \partial_{i,s-i}N^r \arrow[dl, "(\tau_r)^{-1}\circ\tau_s"] \\ & \partial_{i,r-i}N^s & \end{tikzcd}$$

are already disk-bundle isomorphisms. Finally, by a standard argument, we can alter $\overline{\tau}_r$ by an isotopy rel $\bigcup_{i<s} \partial_{i,s-i}N^r$ to a new tubular neighborhood map τ_r so that $(\tau_r)^{-1}\circ\tau_s\colon \partial_r N^s \longrightarrow \partial_s N^r$ is an orthogonal isomorphism. As before $(\tau_r)^{-1}\circ\tau_s$ defines a $G_s \times G_{r-s} \times G_{k-r}$-equivariant embedding $f^{s,r-s}\colon \xi^{s,r-s} \longrightarrow \partial_s \xi^r$ making diagrams (A) and (B) commute.

As this completes the induction, we have, therefore, defined the $\mathscr{S}_{n,k}$-manifolds $Y(i)$ for all $i \leq k$, together with tubular neighborhood maps $\tau_i\colon N^i \longrightarrow Y(i)$. The definition of the normal system $\xi = \{\xi^i \longrightarrow B^i;\, f^{ij}\colon \xi^{ij} \longrightarrow \partial_i \xi^{i+j}\}$ is transparent. The bundles $\xi^i \longrightarrow B^i$ are defined by (1) and (2). The maps f^{ij} are defined by $(\tau_{i+j})^{-1}\circ\tau_i\colon \partial_{i+j}N^i \longrightarrow \partial_i N^{i+j}$. That the f^{ij}'s satisfy the compatibility condition (iv) of Definition 7.1 is insured by the commutativity of diagram (B). Finally, there is an obvious equivariant diffeomorphism

$$T\colon A(\xi) = \coprod_i N^i/\sim \;\; F^{ij} \longrightarrow X$$

defined by

$$T|_{N^i} = \tau_i\colon N^i \longrightarrow X.$$

$\square$

7.11 Definition: Let X be a $\mathcal{S}_{n,k}$-manifold. A normal system for X is a pair (ξ, T) where ξ is a $\mathcal{S}_{n,k}$ normal system and $T: A(\xi) \longrightarrow X$ is an equivariant diffeomorphism. Two normal systems for X, (ξ, T) and (μ, S), are isotopic if there is a level preserving equivariant diffeomorphism $F: A(\xi) \times I \longrightarrow X \times I$ and an isomorphism of normal systems $\theta: \xi \longrightarrow \mu$ such that $F_0 = T$ and $F_1 = S \circ A(\theta)$.

If B is a $\mathcal{B}_k$-space, then the notions of a normal system for B and of isotopic normal systems for B can be defined in an analogous fashion.

The next proposition is essentially the uniqueness part of the Equivariant Tubular Neighborhood Theorem.

7.12 Proposition:

(i) If X is a compact $\mathcal{S}_{n,k}$-manifold, then any two normal systems for X are isotopic.

(ii) If B is a compact $\mathcal{B}_k$-space, then any two normal systems for B are isotopic.

Proof: As in 7.10, we need only prove (i). Let $\xi = \{\xi^i \longrightarrow B^i,\ f^{ij}: \xi^{ij} \longrightarrow \partial_i \xi^{i+j}\}$ and $\mu = \{\mu^i \longrightarrow D^i,\ h^{ij}: \mu^{ij} \longrightarrow \partial_i \mu^{i+j}\}$ be $\mathcal{S}_{n,k}$-normal systems and let $T: A(\xi) \longrightarrow X$ and $S: A(\mu) \longrightarrow X$ be equivariant diffeomorphisms. Let $N^i(\xi)$ and $N^i(\mu)$ denote the closed tubular neighborhoods in $A(\xi)$ and $A(\mu)$ corresponding to ξ^i and μ^i, respectively. Suppose that G_n/G_{n-p} is the smallest orbit type which occurs in X. The proposition is proved by constructing our isotopy

$F: A(\xi) \times [0, k-p] \longrightarrow X \times [0, k-p]$ one stratum at a time.

First, we can alter T by an isotopy so that $T(N^p(\xi)) \subset S(N^p(\mu))$. Let ψ_0 denote the restriction

$$(1) \qquad \psi_0 = S^{-1} \circ T|_{N^p(\xi)} : N^p(\xi) \longrightarrow N^p(\mu).$$

As in the proof of the uniqueness part of the Equivariant Tubular Neighborhood Theorem (see [B1, p. 310]), we can deform ψ_0 by an isotopy

$$(2) \qquad \psi: N^p(\xi) \times [0,1] \longrightarrow N^p(\mu) \times [0,1]$$

so that ψ_1 is a G_n-disk bundle isomorphism. (First, on a slightly larger open tubular neighborhood deform ψ_0 to a G_n-vector bundle isomorphism, then use the invariant metric to deform it to an orthogonal isomorphism.) Since the disk bundle $N^p(\xi)$ is compact, it follows from the Equivariant Isotopy Extension Lemma (see [B1, p. 313]), that we can extend

$$(S \times 1) \circ \psi: N^p(\xi) \times [0,1] \longrightarrow X \times [0,1]$$

to an isotopy

$$(3) \qquad F: A(\xi) \times [0,1] \longrightarrow X \times [0,1]$$

such that $F_0 = S \circ \psi_0 = T$ and such that $F_1|_{N^p(\xi)} = S \circ \psi_1$. Let $\theta^p: \xi^p \longrightarrow \mu^p$ be the $G_p \times G_{k-p}$-bundle isomorphism defined by ψ_1 (that is, $\psi_1 = \times\, \theta^p : N^p(\xi) \longrightarrow N^p(\mu)$).

Next, we fix up the isotopy on a neighborhood of the $(p+1)^{th}$-stratum. Notice that $S^{-1} \circ F_1$ is already linear on $N^p(\xi)$. As before, we can alter F_1 by an isotopy rel $N^p(\xi)$ so that $F_1(N^{p+1}(\xi)) \subset S(N^{p+1}(\mu))$. Proceeding as above, we get an isotopy of $S^{-1} \circ F_1 : N^{p+1}(\xi) \longrightarrow N^{p+1}(\mu)$ to a G_n-disk bundle equivalence. Moreover, this isotopy is fixed on $\partial_p N^{p+1}(\xi)$. Using the Equivariant Isotopy Extension Lemma again, we obtain an isotopy which can be patched together with the F defined in (3) to get a new F defined for $t \in [0,2]$,

(4) $$F: A(\xi) \times [0,2] \longrightarrow X \times [0,2].$$

F will have the following two properties:

(5) $$F_t|_{N^p(\xi)} = F_1|_{N^p(\xi)}, \quad \text{for all } t \in [1,2], \text{ and}$$

(6) $$S^{-1} \circ F_2|_{N^{p+1}(\xi)} : N^{p+1}(\xi) \longrightarrow N^{p+1}(\mu) \text{ will be an orthogonal } G_n\text{-disk}$$

bundle isomorphism. Let $\theta^{p+1} : \xi^{p+1} \longrightarrow \mu^{p+1}$ be the $G_{p+1} \times G_{k-p-1}$-principal bundle isomorphism defined by (6). By construction, the following diagram commutes

$$\begin{tikzcd} \xi^{p,1} \arrow[r, "\times\theta^p"] \arrow[d, "f^{p,1}"'] & \mu^{p,1} \arrow[d, "h^{p,1}"] \\ \partial_p \xi^{p+1} \arrow[r, "\theta^{p+1}"'] & \partial_p \mu^{p+1} \end{tikzcd}.$$

Continuing in this fashion, we end up with an isotopy $F: A(\xi) \times [0,k-p] \longrightarrow X \times [0,k-p]$ and an isomorphism of normal systems $\theta: \xi \longrightarrow \mu$, $\theta = \{\theta^i : \xi^i \longrightarrow \mu^i\}$, such that $F_0 = T$ and $F_{k-p} = S \circ A(\theta)$. □

To realize its full force, the theory of normal systems must be extended to the category $\mathscr{I}_{n,k}(B)$.

7.13 Definition: Recall that a $\mathscr{I}_{n,k}(B)$-manifold $\mathbb{X}$ is a pair $\mathbb{X} = (X, \varphi)$, where X is a G_n-manifold modeled on $k\rho_n$ and where $\varphi\colon X_{\#} \longrightarrow B$ is a diffeomorphism. Let η_B be a $\mathscr{B}_k$-normal system for B (i.e., there is a diffeomorphism $A(\eta_B) \cong B$, which we regard as the identity map). A $\mathscr{I}_{n,k}(B)$-normal system rel η_B is a triple $\mathcal{E} = (\xi, \eta, \psi)$, where ξ is a $\mathscr{I}_{n,k}$-normal system covering a $\mathscr{B}_k$-normal system η and where $\psi\colon \eta \longrightarrow \eta_B$ is an isomorphism of $\mathscr{B}_k$-normal systems. Let $\mathcal{E}' = (\xi', \eta', \psi')$ be another $\mathscr{I}_{n,k}(B)$ normal system rel η_B. A morphism $\theta\colon \mathcal{E} \longrightarrow \mathcal{E}'$ is just a $\widetilde{\mathscr{I}}_{n,k}$-morphism $\theta\colon \xi \longrightarrow \xi'$ which makes the following diagram commute

$$
\begin{tikzcd}
A(\xi) \arrow[rr, "A(\theta)"] \arrow[d, dashed] & & A(\xi') \arrow[d, dashed] \\
A(\eta) \arrow[rr, "A(\theta)_{\#}"] \arrow[dr, "A(\psi)"'] & & A(\eta') \arrow[dl, "A(\psi')"] \\
& B &
\end{tikzcd}
$$

Notice that any such morphism is automatically an isomorphism. Let $\widetilde{\mathscr{I}}_{n,k}(\eta_B)$ denote the category of $\mathscr{I}_{n,k}(B)$-normal systems rel η_B. Obviously, there is an assemble functor

$$A\colon \widetilde{\mathscr{I}}_{n,k}(\eta_B) \longrightarrow \mathscr{I}_{n,k}(B)$$

taking $\mathcal{E} = (\xi, \eta, \psi)$ to the $\mathscr{I}_{n,k}(B)$-manifold $A(\mathcal{E}) = (A(\xi), A(\psi))$, where $A(\psi)\colon A(\eta) = A(\xi)_{\#} \longrightarrow B$ is a $\mathscr{B}_k$-isomorphism.

Just as before, we can define a normal system rel η_B for a

$\mathscr{D}_{n,k}(B)$-manifold $\mathbb{X}$ to be a pair $(\mathcal{E}, T)$, where $T: A(\mathcal{E}) \longrightarrow \mathbb{X}$ is a $\mathscr{D}_{n,k}(B)$-diffeomorphism (that is, T is an equivariant diffeomorphism covering the identity on B).

The most interesting version of the Equivariant Tubular Neighborhood Theorem is in $\mathscr{D}_{n,k}(B)$.

7.14 Theorem: Let B be a compact $\mathcal{B}_k$-space and let η_B be a normal system for B. Let $\mathbb{X} = (X, \varphi)$ be a $\mathscr{D}_{n,k}(B)$-manifold, $n \geq k$. The following statements are true.

(i) There is a normal system rel η_B for $\mathbb{X}$.

(ii) Any two normal systems rel η_B for $\mathbb{X}$ are isotopic. In other words, if $\mathcal{E}$ and $\mathcal{E}'$ are $\mathscr{D}_{n,k}(B)$-normal systems rel η_B and if $F: A(\mathcal{E}) \longrightarrow A(\mathcal{E}')$ is a $\mathscr{D}_{n,k}(B)$-diffeomorphism, then F is $\mathscr{D}_{n,k}(B)$-isotopic to $A(\theta): A(\mathcal{E}) \longrightarrow A(\mathcal{E}')$ where $\theta: \mathcal{E} \longrightarrow \mathcal{E}'$ is a $\widetilde{\mathscr{D}}_{n,k}(\eta_B)$-isomorphism.

Proof: By 7.10, we can choose a $\mathscr{D}_{n,k}$-normal system (ξ, T) for X. Let $(\eta, T_\#)$ be the corresponding $\mathcal{B}_k$-normal system for $X_\#$. Notice that $T_\#: A(\eta) \longrightarrow X_\#$ and $\varphi^{-1}: B = A(\eta_B) \longrightarrow X_\#$ are two normal systems for $X_\#$. So, by part (ii) of Proposition 7.12, there is an isotopy $H: A(\eta) \times I \longrightarrow X_\# \times I$ such that $H_0 = T_\#$ and $H_1 = \varphi^{-1} \circ A(\psi)$, where $\psi: \eta \longrightarrow \eta_B$ is a $\widetilde{\mathcal{B}}_k$-isomorphism. By the Covering Isotopy Theorem 3.2, H can be covered by an equivariant isotopy $\widetilde{H}: A(\xi) \times I \longrightarrow X \times I$ such that $\widetilde{H}_0 = T$. The triple $\mathcal{E} = (\xi, \eta, \psi)$ is a $\mathscr{D}_{n,k}(B)$-normal system rel η_B, and $\widetilde{H}_1: A(\mathcal{E}) \longrightarrow \mathbb{X}$ is a $\mathscr{D}_{n,k}(B)$-diffeomorphism. This proves (i).

Let $\mathcal{E} = (\xi, \eta, \psi)$ and $\mathcal{E}' = (\xi' = (\xi', \eta', \psi')$ be a $\mathcal{J}_{n,k}(B)$-normal system rel η_B. Let $F: A(\xi) \longrightarrow A(\xi')$ be an equivariant diffeomorphism covering $A(\psi'^{-1} \circ \psi): A(\eta) \longrightarrow A(\eta')$. Since $\psi'^{-1} \circ \psi$ is a $\widetilde{\mathfrak{R}}_k$-isomorphism, it follows that

$$F(N^i(\xi)) = N^i(\xi') .$$

Observe that if $f: M(n,k) \longrightarrow M(n,k)$ is some equivariant diffeomorphism covering the linear diffeomorphism $\beta_g: H_+(k) \longrightarrow H_+(k)$ (where $\beta_g(z) = g^{-1} \cdot z \cdot g$ for some $g \in G_k$), then the isotopy f_t defined by

$$f_t(x) = \begin{cases} t^{-1} f(tx) & ; \ t \neq 0 \\ df_0(x) & ; \ t = 0 \end{cases}$$

will cover β_g at each time t. Also, note that df_0 will be the linear map $x \longrightarrow x \cdot h$, for some $h \in g \cdot [Z_k] \subset G_k$. Bearing the above observations in mind, we see that the isotopy of F constructed in the proof of Proposition 7.12 will automatically be a $\mathcal{J}_{n,k}(B)$-isotopy, i.e., it will cover $A(\psi'^{-1} \circ \psi) \times 1: A(\eta) \times I \longrightarrow A(\eta') \times I$. This proves (ii).

$\square$

Let $\mathcal{E}$ be a $\mathcal{J}_{n,k}(B)$-normal system rel η_B and let Aut $(\mathcal{E})$ denote the set of $\mathcal{J}_{n,k}(\eta_B)$-automorphism of $\mathcal{E}$. The assemble functor defines an inclusion

$$A: \text{Aut}\,(\mathcal{E}) \longrightarrow \text{Diff}_1^{G_n}(A(\mathcal{E})) .$$

Give Aut $(\mathcal{E})$ the induced topology. Let $F \in \text{Diff}_1^{G_n}(A(\mathcal{E}))$. As in the above proof, for $t \in [0, k-1]$, we can define an isotopy $F_t \in \text{Diff}_1^{G_n}(A(\mathcal{E}))$ such that $F_0 = F$ and $F_{k-1} = A(\theta)$ for some $\theta \in \text{Aut}\,(\mathcal{E})$. Thus,

$\pi_0 \operatorname{Diff}_1^{G^n}(A(\mathcal{E})) \cong \pi_0 \operatorname{Aut}(\mathcal{E})$. In fact, a much stronger result is true. Recall that F_t can be defined so that for $t \in [i, i+1]$, F_t is constant off a small open neighborhood of N^i and so that $F_t|_{N^i}$ is the isotopy which deforms $F|_{N^i}$ to its differential along the fibre. Since the definition of F_t can be made essentially independent of F, the map

$$D: \operatorname{Diff}_1^{G^n}(A(\mathcal{E})) \times [0, k-1] \longrightarrow \operatorname{Diff}_1^{G^n}(A(\mathcal{E})),$$

defined by $D(F, t) = F_t$, is a deformation retraction onto $\operatorname{Aut}(\mathcal{E})$. We restate this as the

7.15 <u>Theorem</u>: Let η_B be a normal system for a compact $\mathcal{B}_k$-space B and let $\mathcal{E}$ be a $\mathcal{J}_{n,k}(B)$-normal system rel η_B. Then, for any $n \geq k$, $\operatorname{Aut}(\mathcal{E})$ is a deformation retract of $\operatorname{Diff}_1^{G^n}(A(\mathcal{E}))$.

The value of this theorem is that the space $\operatorname{Aut}(\mathcal{E})$ can be given a completely bundle theoretic description. This description is the subject matter of section 10.

Our various versions of the Equivariant Tubular Neighborhood Theorems 7.10, 7.12 and 7.14 can be summarized as the following

7.16 <u>Theorem</u>: Let C denote one of the categories $\mathcal{J}_{n,k}$, $\mathcal{B}_k$ or $\mathcal{J}_{n,k}(B)$, $n \geq k$, and let $\widetilde{C}$ denote the corresponding category $\widetilde{\mathcal{J}}_{n,k}$, $\widetilde{\mathcal{B}}_k$ or $\widetilde{\mathcal{J}}_{n,k}(\eta_B)$ of normal systems, then the following statements are true.

(1) There is a faithful functor $A: \widetilde{C} \longrightarrow C$ (the assemble functor).

(2) If Y is a compact C-object, then Y is isomorphic to $A(\mu)$ for some $\widetilde{C}$-object μ.

(3) If $\mathrm{Iso}_C(A(\mu), A(\gamma))$ is given the coarse C^∞ topology and if $\mathrm{Iso}_{\widetilde{C}}(\mu,\gamma)$ is given the induced topology (via A), then the induced map

$$A_*: \pi_0 \mathrm{Iso}_{\widetilde{C}}(\mu,\gamma) \longrightarrow \pi_0 \mathrm{Iso}_C(A(\mu)), A(\gamma))$$

is an isomorphism.

Finally, we mention an interesting peripheral result suggested in [J]. Essentially it says that no information is lost by passing to the fixed point set of the principal isotropy subgroup.

7.17 __Theorem:__ Suppose that $n \geq k$ and that X denotes a $\mathscr{S}_{n,k}$-manifold. The correspondence $X \longrightarrow X^{G_{n-k}}$ defines functors

$$\mathscr{F}: \mathscr{S}_{n,k} \longrightarrow \mathscr{S}_{k,k} \quad \text{and}$$

$$\mathscr{F}_B: \mathscr{S}_{n,k}(B) \longrightarrow \mathscr{S}_{k,k}(B),$$

which are equivalences of categories.

__Proof:__ We only prove the statement about $\mathscr{F}$, the proof for $\mathscr{F}_B$ being identical. $\mathscr{F}$ is clearly a functor, since a G_n-isovariant morphism $X \longrightarrow M$ restricts to a G_k-isovariant morphism $X^{G_{n-k}} \longrightarrow M^{G_{n-k}}$. To show that $\mathscr{F}$ is an equivalence it suffices to show that

(1) it is a full faithful functor (i.e., it is an isomorphism on Hom's), and

(2) every $\mathscr{S}_{k,k}$-object is isomorphic to some $\mathscr{F}(X)$,

(see [Mc; p. 91]). By Theorem 7.16 the isomorphism class of X is determined by the isomorphism class of a normal system for X. Since the definition of a normal system is independent of n, condition (2) follows.

(This observation is essentially due to Jänich in [J].) Since the restriction of an isovariant $F: X \longrightarrow M$ to $X^{G_{n-k}}$ determines F on each orbit, it follows that $\mathcal{F}$ is injective on Hom's, i.e., $\mathcal{F}$ is faithful. To complete the proof we must show that $\mathcal{F}$ is surjective on Hom's. The heart of the proof is, therefore, the following lemma.

7.18 <u>Lemma</u>: For any $n \geq k$, $\mathcal{F}: \mathcal{S}_{n,k} \longrightarrow \mathcal{S}_{k,k}$ (respectively, $\mathcal{F}_B$) is a full functor. In other words, if X and M are $\mathcal{S}_{n,k}$-manifolds, then any smooth isovariant map $f: X^{G_{n-k}} \longrightarrow M^{G_{n-k}}$ has a (unique) smooth isovariant extension to $\tilde{f}: X \longrightarrow M$.

<u>Proof</u>: As in section 3, it suffices to prove this locally. Let

$$V = G_n \times_{G_{n-i}} M(n-i, k-i) \qquad \text{and let}$$

$$\begin{aligned} U &= V^{G_{n-k}} \\ &= G_k \times_{G_{k-i}} M(k-i, k-i). \end{aligned}$$

If $f: U \times \mathbb{R}^m \longrightarrow U \times \mathbb{R}^p$ is a smooth isovariant map, then we may decompose f into component functions $f = (f_1, f_2): U \times \mathbb{R}^m \longrightarrow U \times \mathbb{R}^p$. By Remark 2.5, for any $z \in \mathbb{R}^m$, the differential at $(0, z)$ of $f_1|_{U \times \{z\}}: U \times \{z\} \longrightarrow U$ is nonsingular. Using the Inverse Function Theorem, we may assume (by cutting $U \times \mathbb{R}^m$ down to a smaller neighborhood of $\{0\} \times \mathbb{R}^m$, if necessary) that

(1) $\qquad f_1: U \times \mathbb{R}^m \longrightarrow U$

is an m-parameter family of G_k-embeddings. Notice also that

(2) $f_2 \colon U \times \mathbb{R}^m \longrightarrow \mathbb{R}^p$

is G_k-invariant. Let $\pi \colon U \longrightarrow U_\# = H_+(k-i)$ and $\pi' \colon V \longrightarrow V_\# = H_+(k-i)$ be the orbit maps. Since f_2 is invariant

(3) $f_2(u, z) = f_{2\#}(\pi(u), z)$.

By Corollary 3.10, there is a smooth map

$$\theta = (\theta_1, \theta_2); H_+(k-i) \times \mathbb{R}^m \longrightarrow G_i \times GL_{k-i}$$

such that

(4) $f_1([g, x], z) = [g \cdot \theta_1(\pi(u), z),\ x \cdot \theta_2(\pi(u), z)]$,

where $u = [g, x] \in U$. We can define $\tilde{f}_1 \colon V \times \mathbb{R}^m \longrightarrow V$ by almost the same formula (4), that is, by

(5) $\tilde{f}_1([g', x], z) = [g' \cdot \theta_1(\pi'(v), z),\ x \cdot \theta_2(\pi'(v), z)]$,

where $v = [g', x] \in V$. Similarly, we can define $\tilde{f}_2 \colon V \times \mathbb{R}^m \longrightarrow \mathbb{R}^p$ by

(6) $\tilde{f}_2(v, z) = f_{2\#}(\pi'(v), z)$.

Clearly, $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) \colon V \times \mathbb{R}^m \longrightarrow V \times \mathbb{R}^p$ is the desired G_n-isovariant map. Using an equivariant partition of unity, the lemma can be proved globally. $\square$

7.19 <u>Remark:</u> The fact that $\mathcal{F} \colon \mathcal{S}_{n,k} \longrightarrow \mathcal{S}_{k,k}$ is an equivalence implies that there is an adjoint functor for $\mathcal{F}$, $\mathcal{S}_{k,k} \longrightarrow \mathcal{S}_{n,k}$. It could be extremely interesting if there were a simple geometric construction for such an adjoint (i.e., a construction not using normal systems); however, there seems to be no obvious candidate for this.

8. The Classification Theorem

In this section we make use of the theory of normal systems and the pullback construction to prove the following result.

8.1 Theorem: Suppose that B is a compact $\mathcal{B}_k$-space. Then, for any $n \geq k$, there is a one-to-one correspondence between isomorphism classes of $\mathcal{S}_{n,k}(B)$-manifolds with trivial principal orbit bundle and

$$[B; H_+(k)]/\mathfrak{A},$$

where $[B; H_+(k)]$ means the set of $\mathcal{B}_k$-homotopy classes of $\mathcal{B}_k$-morphisms $B \longrightarrow H_+(k)$, and where $\mathfrak{A} \cong [B; G_k]$ is the group of homotopy classes of trivializations of the principal orbit bundle.

8.2 Remarks: The action of $\mathfrak{A}$ on $[B; H_+(k)]$ will be described in the course of the proof. In many cases of interest it will be the trivial action.

This theorem means, among other things, that B is the orbit space of a $\mathcal{S}_{n,k}$-manifold with trivial principal orbit bundle if and only if there exists a smooth strata preserving map $B \longrightarrow H_+(k)$.

We should also mention that the requirement that the principal orbit bundle be trivial is only made to facilitate the proof; the general result is stated in 8.10.

When $k=2$, the above theorem essentially is the well-known result of the Hsiangs, Jänich, and Bredon. It was first formulated in the above form by Bredon in [B1, p. 258], when the action has no fixed points, and in [B2], in the knot manifold case.

To prove the theorem we must show how to produce a $\mathcal{S}_{n,k}(B)$-manifold from a map $f: B \longrightarrow H_+(k)$ and vice versa. In one direction this is easy. For, if $f: B \longrightarrow H_+(k)$ is a smooth strata preserving map, we have the pullback square

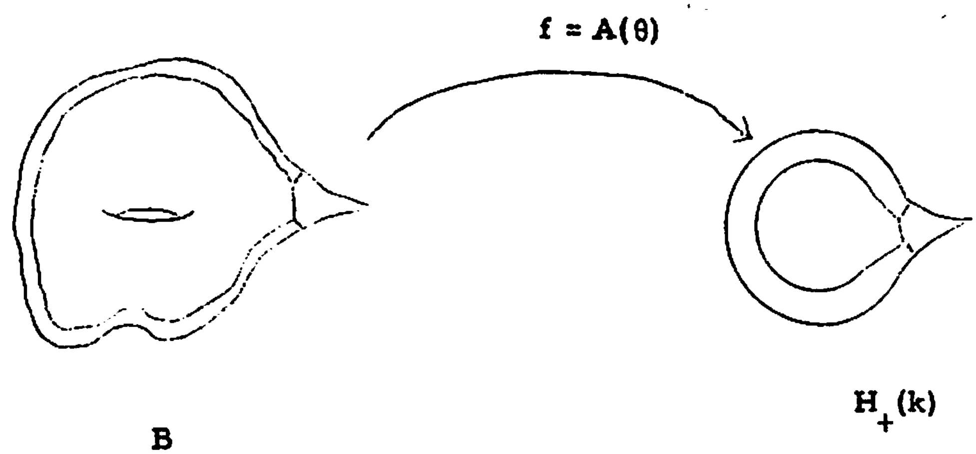
$$\begin{array}{ccc} f^*(M(n,k)) & \xrightarrow{\tilde{f}} & M(n,k) \\ \downarrow p & & \downarrow \pi \\ B & \xrightarrow{f} & H_+(k). \end{array}$$

As noted in Remark 4.4, the pair $\mathbb{X} = (f^*(M(n,k)), p_\#)$ is a $\mathcal{S}_{n,k}(B)$-manifold. If $h: B \times I \longrightarrow H_+(k)$ is a $\mathcal{B}_k$-homotopy of f, then by Theorem 4.1 there is a $\mathcal{S}_{n,k}(B \times I)$-diffeomorphism $\Phi: h^*(M(n,k)) \longrightarrow f^*(M(n,k)) \times I$. Hence, the isomorphism class of $\mathbb{X}$ depends only on the homotopy class of f.

In the other direction we must show how, given a $\mathcal{S}_{n,k}(B)$-manifold $\mathbb{X}$, to define a map $f: B \longrightarrow H_+(k)$ so that $\mathbb{X}$ is $\mathcal{S}_{n,k}(B)$-diffeomorphic to $(f^* M(n,k), p_\#)$. In order to do this, we will first choose a normal system η_B for B and a $\mathcal{S}_{n,k}(B)$ narmal system rel η_B for $\mathbb{X}$, $\mathcal{E} = (\xi, \eta, \psi)$, (where the isomorphisms $A(\eta_B) \cong B$ and $A(\mathcal{E}) \cong \mathbb{X}$ are suppressed). Let

$\xi_k = \{\xi_k^i \longrightarrow B_k^i \, ; f_k^{ij}: \xi_k^{ij} \longrightarrow \partial_i \, \xi_k^{i+j}\}$ and $\eta_k = \{\eta_k^i \rightarrow B_k^i ; g_k^{ij}: \eta_k^{ij} \rightarrow \partial_i \, \eta_k^{i+j}\}$

denote respectively the standard normal systems for $M(n,k)$ and $H_+(k)$, defined in section 6. Using a trivialization of the principal orbit bundle ξ^k, we will then construct a morphism of normal systems $\lambda: \xi \longrightarrow \xi_k$. If $\theta = \lambda_\# \circ \psi^{-1}: \eta_B \longrightarrow \eta_k$, then $A(\theta): B \longrightarrow H_+(k)$ will be the desired map.



8.3 Remark: Before proceeding with the details of this argument we should mention that the pullback construction can be reformulated in the language of normal systems. If $\theta: \eta_B \longrightarrow \eta_k$ is a $\widetilde{\mathcal{B}}_k$-morphism, then a $\mathcal{S}_{n,k}(B)$-normal system rel η_B, $\theta^*(\xi_k) = (\xi, \eta, \psi)$, can be defined. For example, if $\theta_i = \theta|_{B^i}$, then the bundles of ξ and η are defined by $\xi^i = \theta_i^*(\xi_k^i)$ and $\eta^i = \theta_i^*(\eta_k^i)$. The definitions of the remaining data of the normal system can safely be left to the reader. Clearly,

$$A(\theta^*(\xi_k)) \cong A(\theta)^* M(n,k).$$

Also notice that any $\mathcal{B}_k$-morphism $f: B \longrightarrow H_+(k)$ is $\mathcal{B}_k$-homotopic to some $A(\theta): B \longrightarrow H_+(k)$. (The proof of this is basically the same as the proof of 7.12.)

Next we describe the inverse of the above construction.

Proof of Theorem 8.1: Let $\mathbb{X} = (X, \varphi)$ be a $\mathcal{S}_{n,k}(B)$-manifold and let η_B be a normal system for B. By Theorem 7.14, we can choose a $\mathcal{S}_{n,k}(B)$ normal system rel η_B for $\mathbb{X}$, call it $\mathcal{E} = (\xi, \eta, \psi)$, where

$\xi = \{\xi^i \longrightarrow B^i,\ f^{ij}: \xi^{ij} \longrightarrow \partial_i \xi^{i+j}\}$ and $\eta = \{\eta^i \longrightarrow B^i,\ g^{ij}: \eta^{ij} \longrightarrow \partial_i \eta^{i+j}\}$.

By hypothesis, there is a trivialization

$$L^k: \xi^k \longrightarrow G_k$$

of the principal orbit bundle.

There are natural $G_i \times G_{k-i}$-equivariant embeddings $s_i: \xi^i \longrightarrow \xi^{i,k-i}$, which essentially are the invariant metrics. They are defined as follows. Recall that

$$\xi^{i,k-i} = \partial_0 \xi^{k-i}_{k-i} \times_{G_{k-i}} \xi^i$$

$$\cong \partial_0 B^{k-i}_{k-i} \times \xi^i .$$

If $c \in \partial_0 B^{k-i}_{k-i}$ is a scalar matrix, then define $s_i: \xi^i \longrightarrow \partial_0 B^{k-i}_{k-i} \times \xi^i$ by $s_i(x) = (c, x)$. We can also define a $G_i \times G_{k-i}$-equivariant map $L^i: \xi^i \longrightarrow G_k$ as the composition

$$(1) \qquad \xi^i \xrightarrow{s_i} \xi^{i,k-i} \xrightarrow{f^{i,k-i}} \partial_i \xi^k \xrightarrow{L^k} G_k .$$

Finally, let $t^i: G_k \longrightarrow \xi^i_k \cong B^i_i \times G_k$ be defined by $t^i(g) = (e, g)$, where e is the $(i \times i)$ identity matrix. Notice that $t^i \circ L^i: \xi^i \longrightarrow \xi^i_k$ is a bundle map. We are going to define inductively a morphism $\lambda: \xi \longrightarrow \xi_k$ such that $\lambda^i: \xi^i \longrightarrow \xi^i_k$ will equal $t^i \circ L^i$ off a collared neighborhood of $\partial \xi^i$.

Suppose that $X^{(p)}$ is the lowest stratum of X. Set

$$(2) \qquad \lambda^p = t^p \circ L^p: \xi^p \longrightarrow \xi^p_k .$$

By induction, suppose that our morphism λ is defined up to the i^{th}-

stratum. In other words, suppose that for all $j < i$ we have defined smooth face preserving bundle maps $\lambda^j \colon \xi^j \longrightarrow \xi^j_k$ such that for any j and ℓ with $j+\ell < i$, the following diagram commutes

(3)
$$\begin{tikzcd}
\xi^{j,\ell} \arrow[r, "\times\lambda^j"] \arrow[d, "f^{j\ell}"'] & \xi^{j\ell}_k \arrow[d, "f^{j\ell}_k"] \\
\partial_j\xi^{j+\ell} \arrow[r, "\lambda^{j+\ell}"'] & \partial_j\xi^{j+\ell}_k
\end{tikzcd}$$

Also suppose, by induction, that for all $j < i$ the following diagram commutes

$$\begin{tikzcd}
\xi^j \arrow[rr, "\lambda^j"] \arrow[dr, "L^j"'] & & \xi^j_k \arrow[dl, "p_j"] \\
& G_k &
\end{tikzcd}$$

where $p_j \colon \xi^j_k = B^j_j \times G_k \longrightarrow G_k$ is projection on the second factor. Essentially, λ^i is already defined on $\partial\xi^i = \bigcup_{j<i} \partial_j\xi^i$, for we are forced to make the following diagram commute

(5)
$$\begin{tikzcd}
\partial_j\xi^i \arrow[r, dashed, "\lambda^i|_{\partial_j\xi^i}"] & \partial_j\xi^i_k \\
G_i\times G_j\times G_{i-j}\xi^{j,i-j} \arrow[u, "\times f^{j,i-j}", "\cong"'] \arrow[r, "\times\lambda^j"'] & G_i\times G_j\times G_{i-j}\xi^{j,i-j}_k \arrow[u, "\cong"']
\end{tikzcd}$$

Therefore, the problem is to extend λ^i to the interior of ξ^i.

It is a routine matter of checking various definitions to verify that each piece of the following diagram commutes.

$$
\begin{tikzcd}
 & G_i \times G_j \times G_{i-j}\, \xi^{j,i-j} \arrow[r, "\times f^{ij}"] \arrow[d, "\text{projection}"] & \partial_j \xi^i \arrow[dd, "s^i"] \arrow[ddddd, bend left=60, "L^i"] \\
 & \xi^{j,i-j} \arrow[dl, "\text{projection}"'] \arrow[d, "\times s^j"] & \\
\xi^j \arrow[d, "s^j"] \arrow[dddr, bend right=60, "L^j"'] & \xi^{j,i-j,k-i} \arrow[r, "\times f^{j,i-j}"] \arrow[d, dashed] & \partial_j \xi^{i,k-i} \arrow[d] \\
\xi^{j,k-j} \arrow[dr] & \partial_{i-j}\, \xi^{j,k-j} \arrow[l] \arrow[r, "f^{j,k-j}"] & \partial_{j,i-j}\, \xi^k \arrow[ddl, "L^k"] \\
 & \partial_j \xi^k \arrow[d, "L"] & \\
 & G_k &
\end{tikzcd}
$$

In particular the outer part of the diagram commutes, or

(6)
$$
\begin{tikzcd}
G_i \times G_j \times G_{i-j}\, \xi^{j,i-j} \arrow[r, "\times f^{ij}", "\cong"'] \arrow[d, "\text{projection}"] & \partial_j \xi^i \arrow[d, "L^i"] \\
\xi^j \arrow[r, "L^j"'] & G_k
\end{tikzcd}
$$

If we stick diagrams (4), (5) and (6) together we get the following commutative diagram.

$$
\begin{tikzcd}
\partial_j \xi^i \arrow[rr, "\lambda^i"] \arrow[dddr, bend right=60, "L^i"'] & & \partial_j \xi^i_k \arrow[dddl, bend left=60, "p^i"] \\
G_i \times G_j \times G_{i-j} \xi^{j,i-j} \arrow[u] \arrow[rr, "\times\lambda^i"] \arrow[d, "\text{projection}"] & & G_i \times G_j \times G_{i-j} \xi_k^{j,i-j} \arrow[u] \arrow[d, "\text{projection}"'] \\
\xi^j \arrow[rr, "\lambda^j"] \arrow[dr, "L^j"'] & & \xi^j_k \arrow[dl, "p^j"] \\
& G_k &
\end{tikzcd}
$$

Again, the outer part commutes

(7)

$$
\begin{tikzcd}
\partial_j \xi^i \arrow[rr, "\lambda^i"] \arrow[dr, "L^i"'] & & \partial_j \xi^i_k \arrow[dl, "p^i"] \\
& G_k &
\end{tikzcd}
$$

This diagram suggests a way to extend λ^i to the interior of ξ^i. Recall that $\xi^i_k = B^i_i \times G_k$. Let $S\colon B^i_i \times I \longrightarrow B^i_i$ be a G_i-equivariant deformation retraction of B^i_i to a point $e \in B^i_i$, where e denotes the identity matrix. If $\exp\colon H(i) \longrightarrow H_+(i)^{(i)}$ denotes the exponential map, then S can be defined, for example, by considering the deformation which shrinks $\exp^{-1}(B^i_i)$ to $0 \in H(i)$ via scalar multiplication and then by conjugating this deformation by $\exp$. Let $\overline{S}\colon \xi^i_k \times I \longrightarrow \xi^i_k$ be the

$G_i \times G_{k-i}$-equivariant deformation defined by

$$\overline{S}(b, g, t) = (S(b, t), g).$$

For each $j < i$ define a homotopy $H^j: \partial_j \xi^i \times I \longrightarrow \xi^i_k$ by

(8) $$H^j(x, t) = \overline{S}(\lambda^i(x), t),$$

where $x \in \partial_j \xi^i$ and $\lambda^i|_{\partial_j \xi^i}$ is defined by (5). Notice that

$$H^j(x, 0) = \lambda^i(x) \qquad \text{and}$$
$$H^j(x, 1) = t^i \circ L^i(x).$$

Since the $\lambda^i|_{\partial_j \xi^i}$'s match up on the pairwise intersections, it follows immediately that the H^j's also agree on the intersections of the form $\partial_{\ell, j-\ell} \xi^i \times I$.

We can use the H^j's to extend λ^i to all of ξ^i as follows. Let

$$Q \cong \partial B^i \times I = \bigcup_{j<i} \partial_j B^i \times I$$

be a collared neighborhood of ∂B^i. There is no loss of generality in supposing that the map

$$L^i_{\#}|_Q : Q \cong \partial B^i \times I \longrightarrow G_k / G_i \times G_{k-i}$$

is constant in the I coordinate. Hence, there is an isomorphism $T: \partial \xi^i \times I \longrightarrow \xi^i|_Q$ such that

(9) $$L^i \circ T(x, t) = L^i(x).$$

The H^j's can be patched together to define a bundle map

(10) $$H: \xi^i|_Q \longrightarrow \xi^i_k$$

by setting $H(z) = H^j(x,t)$, where $T^{-1}(z) = (x,t) \in \partial_j \xi^i \times \{t\}$. By construction, $H = t^i \circ L^i$ on $\xi^i|_{\partial B \times \{1\}}$, so we can define $\lambda^i: \xi^i \longrightarrow \xi^i_k$ by the formula

$$(11) \qquad \lambda^i(z) = \begin{cases} H(z);\ z \in \xi^i|_Q \\ t^i \circ L^i(z);\ z \in \xi^i|_{B^i - Q} \end{cases}.$$

λ^i clearly satisfies the morphism part of the induction hypothesis (3), since on $\partial_j \xi^i$ it is defined by (5). We must also show that λ^i satisfies diagram (4), i.e., that $L^i = p_i \circ \lambda^i$. This is automatic except possibly at points in $\xi^i|_Q$. Using (7), (9), and the definition of $\overline{S}$, we see that for $T^{-1}(z) = (x,t) \in \partial_j \xi^i \times I$,

$$\begin{aligned} p_i \circ \lambda^i(z) &= p_i \circ H^j(x,t) \\ &= p_i \circ \overline{S}(\lambda^i(x), t) \\ &= p_i \lambda^i(x) \\ &= L^i(x) \\ &= L^i(z)\ . \end{aligned}$$

This completes the inductive definition of our $\widetilde{\mathscr{S}}_{n,k}$-morphism $\lambda: \xi \longrightarrow \xi_k$.

Recall that $\mathcal{E} = (\xi, \eta, \psi)$ is a normal system for $\mathbb{X}$. The $\widetilde{\mathscr{S}}_{n,k}$-morphism $\lambda: \xi \longrightarrow \xi_k$ covers a $\widetilde{\mathfrak{B}}_k$-morphism $\lambda_\#: \eta \longrightarrow \eta_k$. Set

$$\theta = \lambda_\# \circ \psi^{-1}: \eta_B \longrightarrow \eta_k\,.$$

By the universal property of pullbacks $\mathcal{E}$ and $\theta^*(\xi_k)$ are isomorphic $\widetilde{\mathscr{S}}_{n,k}(\eta_B)$-objects (see Remark 8.3). In other words, there are $\mathscr{S}_{n,k}(B)$-diffeomorphisms

$$\mathfrak{X} \cong A(\mathcal{E}) \cong A(\theta)^* M(n,k).$$

The only significant choice in the definition of λ is the original choice of trivialization $L^k: \xi^k \longrightarrow G_k$. If $\hat{L}^k: \xi^k \longrightarrow G_k$ is another such trivialization, then there is a map $\alpha: B^k \longrightarrow G_k$ defined by $\alpha(b) = \hat{L}^k(x) \cdot L^k(x)^{-1}$ where x is any point in the fibre $\pi^{-1}(b) \subset \xi^k$. Conversely, any $\alpha \in \mathrm{Map}\,(B^k, G_k)$ defines a new trivialization $\hat{L}^k$ by $\hat{L}^k(x) = \alpha(\pi(x)) \cdot L^k(x)$. Once we have $\hat{L}^k$ we can define $\hat{L}^i: \xi^i \longrightarrow G^k$, $\hat{\lambda}: \xi \longrightarrow \xi_k$ and $\hat{\theta}: \eta_B \longrightarrow \eta_k$ as above. This defines an action

$$\mathrm{Map}\,(B^k, G_k) \times \mathrm{Hom}_{\tilde{\mathfrak{B}}_k}(\eta_B, \eta_k) \longrightarrow \mathrm{Hom}_{\tilde{\mathfrak{B}}_k}(\eta_B, \eta_k).$$

If $\hat{L}^k$ and L^k are G_k-homotopic we check first that $\hat{L}^i$ and L^i are homotopic as maps of $G_i \times G_{k-i}$-bundles and consequently that $\hat{\lambda}$ and $\lambda: \xi \longrightarrow \xi_k$ are homotopic $\tilde{\mathcal{S}}_{n,k}$-morphisms. In particular this means that $A(\hat{\theta})$ and $A(\theta): B \longrightarrow H_+(k)$ will be $\mathfrak{B}_k$-homotopic. Hence, the action of $\alpha \in \mathrm{Map}\,(B^k, G_k)$ on $[B; H_+(k)]$ depends only on the homotopy class of α, i.e., the action of $\mathfrak{A} = [B^k, G_k]$ on $[B; H_+(k)]$ is well defined.

Suppose that $\overline{\mathfrak{X}} = (\overline{X}, \overline{\varphi})$ is another $\mathcal{S}_{n,k}(B)$-manifold and that $F: \mathfrak{X} \longrightarrow \overline{\mathfrak{X}}$ is a $\mathcal{S}_{n,k}(B)$-diffeomorphism. Let $\overline{\mathcal{E}} = (\overline{\xi}, \overline{\eta}, \overline{\phi})$ be a normal system for $\overline{\mathfrak{X}}$. By Theorem 7.14, F is isotopic to $A(\gamma): A(\mathcal{E}) \longrightarrow A(\overline{\mathcal{E}})$, where $\gamma: \mathcal{E} \longrightarrow \overline{\mathcal{E}}$ is a $\tilde{\mathcal{S}}_{n,k}(\eta_B)$-isomorphism, $\gamma = \{\gamma^i: \xi^i \longrightarrow \overline{\xi}^i\}$. Let $\overline{L}^k = L^k \circ (\gamma^k)^{-1}: \overline{\xi}^k \longrightarrow G_k$. Using $\overline{L}^k$ we can define a morphism $\overline{\lambda}: \overline{\xi} \longrightarrow \xi_k$. Notice that we will have $\overline{\lambda} = \lambda \circ (\gamma)^{-1}: \overline{\xi} \longrightarrow \xi_k$. Since γ is a $\mathcal{S}_{n,k}(\eta_B)$-morphism, $\gamma_{\#} = \overline{\phi}^{-1} \circ \phi: \eta \longrightarrow \overline{\eta}$. Hence,

$$\begin{aligned}\overline{\theta} &= \overline{\lambda}_{\#} \circ \overline{\psi}^{-1} \\ &= \lambda_{\#} \circ \gamma_{\#}^{-1} \circ \overline{\psi}^{-1} \\ &= \lambda_{\#} \circ \psi^{-1} \\ &= \theta .\end{aligned}$$

In other words, by making a judicious choice for $\overline{L}^k$, $\overline{\mathbb{X}}$ and $\mathbb{X}$ are defined as pullbacks via the same map $A(\theta) = A(\overline{\theta}): B \longrightarrow H_+(k)$. This completes the proof of Theorem 8.1.

□

8.4 Remark: Suppose that the lowest stratum of B is B^p. Then we may certainly assume that the image of a classifying map $B \longrightarrow H_+(k)$ is contained in $H_+(k) - \bigcup_{i<p} C_k^i$. In particular, if $B^0 = \phi$, then the image is contained in $H_+(k) - C_k^0$; but this space $\mathfrak{G}_k$-deformation retracts onto

$$W_+(k) = \{x \in H_+(k) \,|\, \text{trace}\,(x) = 1\}.$$

$(W_+(k) = \Sigma(n,k)_{\#})$. Thus, isomorphism classes of $\mathscr{G}_{n,k}(B)$-manifolds with trivial principal orbit bundle and empty fixed point set are in one-to-one correspondence with $[B; W_+(k)]/\mathfrak{A}$.

8.5 Remark: There is a simpler description of the action of $\mathfrak{A}$. Let $f: B \longrightarrow H_+(k)$ be a classifying map and let $\alpha \in \text{Map}\,(B^k; G_k)$. Using the normal system η_B, α can be extended to a map $\overline{\alpha}: B \longrightarrow G_k$. Define $\alpha \cdot f$ by using the linear action of G_k on $H_+(k)$ by conjugation, i.e.,

$$\alpha \cdot f(b) = \overline{\alpha}(b)^{-1} \cdot f(b) \cdot \overline{\alpha}(b).$$

This defines an action $\mathfrak{A}\times[B,H_+(k)] \longrightarrow [B,H_+(k)]$, and it is not difficult to see that this action is the same as the one defined in the above proof.

If B is contractible, then the principal orbit bundle is automatically trivial. Moreover, $\mathfrak{A} = [B^k; G_k] \cong \pi_0(G_k)$. So, if $G_n = U(n)$ or $Sp(n)$, $\mathfrak{A}$ is trivial; while if $G_n = O(n)$, then $\mathfrak{A} = \mathbb{Z}_2$. If k is odd, then the matrix $(-1) \in O(k) - SO(k)$. Since any map $B^k \longrightarrow O(k) - SO(k)$ is homotopic to the constant map $b \longrightarrow (-1)$ and since (-1) acts trivially on $H_+(k)$, it follows that, for k odd, $\mathfrak{A} = [B^k, O(k)]$ acts trivially on $[B; H_+(k)]$. On the other hand, if k is even, this action is in general nontrivial (for example, when $B = D^2 = W_+^1(2)$, see [B1, p. 258]).

The hypothesis that the principal orbit bundle be trivial is not really essential in the proof of Theorem 8.1. For suppose that $\mathbb{X}$ is an arbitrary $\mathscr{I}_{n,k}(B)$-manifold and that $\mathcal{E} = (\xi, \eta, \psi)$ is a normal system for $\mathbb{X}$ (rel η_B). Suppose further that $E \longrightarrow E_\#$ is some principal G_k-bundle and that $L\colon \xi^k \longrightarrow E$ is a bundle map. Just as in the proof of 8.1, L can be used to construct a morphism of normal systems

$$\lambda\colon \xi \longrightarrow \xi_k \times_{G_k} E$$

where $\xi_k \times_{G_k} E$ denotes the $\mathscr{I}_{n,k}$-normal system

$\left\{\xi_k^i \times_{G_k} E,\ \times f_k^{ij}\colon \xi_k^{ij} \times_{G_k} E \longrightarrow \partial_i\, \xi_k^{i+j} \times_{G_k} E\right\}$. Thus, $\mathbb{X} = (X, \varphi)$ is a pullback of the following form

$$\begin{array}{ccc} X & \xrightarrow{A(\lambda)} & M(n,k) \times_{G_k} E \\ \downarrow & & \downarrow \\ B & \xrightarrow{A(\lambda)_\#} & H_+(k) \times_{G_k} E . \end{array}$$

If $EG_k \to BG_k$ is an m-universal principal G_k-bundle, where $m = \dim B$, (BG_k is a Grassmannian), then we can always find such a bundle map $\xi^k \to EG_k$. Thus, we have proved the

8.6 Corollary: Let $C_{n,k}(B)$ denote the set of isomorphism classes of $\mathscr{S}_{n,k}(B)$-manifolds. Let EG_k be m-universal, where $m = \dim B$ (B is compact). Then, for any $n \geq k$, the pullback construction defines a bijection

$$C_{n,k}(B) \cong [B; H_+(k) \times_{G_k} EG_k]/\mathfrak{A}'$$

where $\mathfrak{A}'$ is the appropriate equivalence relation on $[B; H_+(k) \times_{G_k} EG_k]$.

8.7 Remark: For any $\sigma \in [B^k; BG_k]$, let $\Omega_\sigma \subset [B; H_+(k) \times_{G_k} EG_k]$ denote the subset consisting of those maps f such that the composition

$$B \xrightarrow{f} H_+(k) \times_{G_k} EG_k \xrightarrow{\text{projection}} BG_k$$

is homotopic to σ (this makes sense since B^k is a deformation retract of B). Let $\xi^k = \sigma^* EG_k$ and let $\mathrm{Aut}(\xi^k)$ be the space of bundle automorphisms of ξ^k. Just as in the proof of 8.1, $\pi_0 \mathrm{Aut}(\xi^k)$ acts on Ω_σ. The orbits of this action define the equivalence relation $\mathfrak{A}'$.

8.8 Remark: Notice that the $G_i \times G_{k-i}$-bundle $\xi_k^i \times_{G_k} EG_k$ sits over the i^{th} stratum $B_k^i \times_{G_k} EG_k$, (where $B_k^i \sim H_+(k)^{(i)}$). Since $\xi_k^i \cong B_i^i \times G_k$ and since B_i^i is contractible, we see that $\xi_k^i \times_{G_k} EG_k \cong B_i^i \times EG_k$, which is m-connected if EG_k is m-connected. Hence, $B_k^i \times_{G_k} EG_k$ is homotopy equivalent to $B(G_i \times G_{k-i}) = BG_i \times BG_{k-i}$. Therefore, if $f: B \longrightarrow H_+(k) \times_{G_k} EG_k$ is a classifying map for X, its restriction to the i^{th} stratum

$$f\big|_{B^i} \longrightarrow B_k^i \times_{G_k} EG_k = BG_i \times BG_{k-i}$$

is just the map which classifies ξ^i, the principal bundle of the equivariant normal bundle of $X^{(i)} \subset X$.

Just as the total spaces of two principal G-bundles over the same base space can be equivariantly diffeomorphic without the bundles being equivalent, two $\mathscr{B}_{n,k}(B)$-manifolds can be G_n-diffeomorphic but not G_n-diffeomorphic over B (that is, not $\mathscr{B}_{n,k}(B)$-diffeomorphic). There is a natural action

$$\text{Diff }(B) \times C_{n,k}(B) \longrightarrow C_{n,k}(B)$$

where $C_{n,k}(B)$ denotes the set of $\mathscr{B}_{n,k}(B)$-diffeomorphism classes of $\mathscr{B}_{n,k}(B)$-manifolds. Two $\mathscr{B}_{n,k}(B)$-manifolds are equivariantly diffeomorphic if and only if they are in the same orbit. Thus, we want to compute the quotient $C_{n,k}(B)/\text{Diff }(B)$.

If the definition of this action is reformulated in terms of classifying maps, we see that it corresponds to the natural action given by composition

$$\mathrm{Diff}(B) \times [B; H_+(k) \times_{G_k} EG_k] / \mathfrak{A}' \longrightarrow [B; H_+(k) \times_{G_k} EG_k] / \mathfrak{A}'.$$

This shows that the action depends only on the $\mathcal{B}_k$-homotopy class of $\Phi \in \mathrm{Diff}(B)$.

Suppose $\mathbb{X}$ and $\mathbb{X}'$ are $\mathcal{S}_{n,k}(B)$-manifolds classified by $\mathcal{B}_k$-morphisms f and f',

$$f, f' : B \longrightarrow H_+(k) \times_{G_k} EG_k .$$

If $\Phi \in \mathrm{Diff}(B)$ is such that $f = f' \circ \Phi$, then we get a well defined map on the mapping cylinder M_Φ,

$$F : M_\Phi \longrightarrow H_+(k) \times_{G_k} EG_k,$$

such that the pullback by F is equivalent to $\mathbb{X}$ on one end and to $\mathbb{X}'$ on the other. Conversely, if W is any $\mathcal{B}_k$-space diffeomorphic to $B \times I$, and if $F: W \longrightarrow H_+(k) \times_{G_k} EG_k$ is a morphism, then by Theorem 4.1 both ends of the pullback $F^*(M(n,k) \times_{G_k} EG_k)$ will be G_n-diffeomorphic. This leads us to the following definition.

8.9 <u>Definition</u>: Two $\mathcal{B}_k$-morphisms $f, f' : B \longrightarrow C$ are <u>concordant</u> if there is a $\mathcal{B}_k$-diffeomorphism $\psi: B \times I \longrightarrow W$ and a morphism $F: W \longrightarrow C$ such that $f = F \circ \psi_0$ and $f' = F \circ \psi_1$. Let $\{B; C\}$ denote the set of concordance classes of maps.

By the above remarks, we have the following corollary of 8.6.

8.10 <u>Corollary</u>: Suppose that B is a compact $\mathcal{B}_k$-space and that $n \geq k$. The pullback construction defines a bijection between the set of equivariant diffeomorphism classes of $\mathcal{S}_{n,k}$-manifolds with orbit space diffeomorphic to B and

$$\{B; H_+(k) \times_{G_k} EG_k\}/\mathfrak{A}' .$$

8.11 Remark: It should be possible to define a "classifying space for $\mathcal{B}_k$-spaces." This would be a stratified space P with the following properties:

(i) The i^{th} stratum $P^i \sim BPG_{k-i}$,

(ii) P^i has a tubular neighborhood in P which is a fibre bundle with fibre $H_+(k-i)$.

(iii) There is a strata preserving map $q: H_+(k) \times_{G_k} EG_k \longrightarrow P$ such that the fibre of $q\big|_{B_k^i \times_{G_k} EG_k}$ is $BG_i \times BZ_{k-i}$.

(iv) If B is any $\mathcal{B}_k$-space, there is a strata preserving map $g: B \longrightarrow P$ which is transverse on a tubular neighborhood of each stratum and such that $g\big|_{B^i}$ is just the classifying map for the normal cone bundle of B^i.

(v) If $f: B \longrightarrow H_+(k) \times_{G_k} EG_k$ is any map, then the following diagram commutes up to a strata preserving homotopy

$$\begin{tikzcd} & H_+(k) \times_{G_k} EG_k \arrow[d, "q"] \\ B \arrow[ur, "f"] \arrow[r, "g"'] & P \end{tikzcd} .$$

The existence of such a P would have several attractive features. It seems that P can be constructed by successively "coning off" the various pieces $B_k^i \times_{G_k} EG_k \subset H_+(k) \times_{G_k} EG_k$, but the details have not yet been worked out.

III. MORPHISMS OF NORMAL SYSTEMS

9. The Homotopy of $\mathrm{Hom}_{\tilde{C}}(\eta, \mu)$

Let C denote one of the categories $\mathcal{B}_k$, $\mathcal{J}_{n,k}$ or $\mathcal{J}_{n,k}(B)$ and let $\tilde{C}$ be the corresponding category of normal systems $\tilde{\mathcal{B}}_k$, $\tilde{\mathcal{J}}_{n,k}$ or $\tilde{\mathcal{J}}_{n,k}(\eta_B)$. If X and Y are C-objects, let $\mathrm{Hom}_C(X, Y)$ denote the space of C-maps $X \longrightarrow Y$ endowed with the coarse C^∞ topology (essentially this is the C^∞ version of the compact open topology). For example, if $C = \mathcal{J}_{n,k}$, then $\mathrm{Hom}_C(X, Y)$ means the space of smooth isovariant maps from X to Y. Similarly, if η and μ are $\tilde{C}$-normal systems, let $\mathrm{Hom}_{\tilde{C}}(\eta, \mu)$ denote the set of normal systems $\eta \longrightarrow \mu$. The assemble functor A defines an inclusion

$$A\colon \mathrm{Hom}_{\tilde{C}}(\eta, \mu) \longrightarrow \mathrm{Hom}_C(A(\eta), A(\mu)).$$

Give $\mathrm{Hom}_{\tilde{C}}(\eta, \mu)$ the induced topology.

By definition a morphism $\theta\colon \eta \longrightarrow \mu$ is a collection of bundle maps $\theta^i\colon \eta^i \longrightarrow \mu^i$ which cover smooth maps of $\langle i \rangle$-manifolds and which are subject to a certain compatibility condition. Hence, $\mathrm{Hom}_{\tilde{C}}(\eta, \mu)$ can be naturally regarded as a subspace of the direct product

$$\mathop{\times}_{i=0}^{k} \mathrm{Hom}(\eta^i, \mu^i) .$$

The homotopy of the space of bundle maps $\mathrm{Hom}(\eta^i, \mu^i)$ is fairly well understood. By using the above description we will be able to get some hold on the homotopy of $\mathrm{Hom}_{\tilde{C}}(\eta, \mu)$ and in particular on $\pi_0 \mathrm{Hom}_{\tilde{C}}(\eta, \mu)$. Of course, what we actually want to compute is $\pi_0 \mathrm{Hom}_C(X, Y)$; however, these two problems are the same.

9.1 Proposition: $A_*: \pi_0 \mathrm{Hom}_{\tilde{C}}(\eta, \mu) \to \pi_0 \mathrm{Hom}_C(A(\eta), A(\mu))$ is a bijection.

Proof: When $C = \mathscr{S}_{n,k}(B)$ every morphism is an isomorphism, so the proposition is just a restatement of part (3) of Theorem 7.16. For $C = \mathscr{S}_{n,k}$ or $\mathscr{B}_k$, the proof is similar to the proof of Proposition 7.12. In fact, we can construct a homotopy of $f: A(\eta) \to A(\mu)$ to $A(\theta)$ by altering f (one stratum at a time) by an isotopy on $A(\eta)$. We omit the details.

□

Notice that $\pi_0 \mathrm{Hom}_{\mathscr{B}_k}(B, H_+(k))$ is just another way of writing $[B; H_+(k)]$, which we need to calculate in order to classify $\mathscr{S}_{n,k}(B)$-manifolds. We are also interested in applying the above proposition when $C = \mathscr{S}_{n,k}(B)$, in which case $\mathrm{Hom}_C(X, X) = \mathrm{Diff}_1^{G_n}(X)$. Of course, in this case a much stronger result than 9.1 is true, namely Theorem 7.15, which says that $\mathrm{Aut}(\mathcal{E})$ is a deformation retract of $\mathrm{Diff}_1^{G_n}(A(\mathcal{E}))$, where $\mathcal{E}$ is a $\tilde{\mathscr{S}}_{n,k}(\eta_B)$-object.

For the remainder of this section we shall restrict our attention to the categories $\tilde{\mathscr{S}}_{n,k}$ and $\tilde{\mathscr{B}}_k$. Let $\eta = \{\eta^i \to B^i; f^{ij}: \eta^{ij} \to \partial_i \eta^{i+j}\}$ and $\mu = \{\mu^i \to D^i; g^{ij}: \mu^{ij} \to \partial_i \mu^{i+j}\}$ be typical normal systems (in either $\tilde{\mathscr{S}}_{n,k}$ or $\tilde{\mathscr{B}}_k$). We first mention two lemmas about the spaces $\mathrm{Hom}(\eta^i, \mu^i)$. Their proofs, which are trivial exercises, are omitted.

9.2 Lemma (The Covering Homotopy Theorem for fibre bundles): The natural projection

$$p\colon \mathrm{Hom}\,(\eta^i,\mu^i) \longrightarrow \mathrm{Hom}\,(B^i,D^i)$$

is a fibration (where $\mathrm{Hom}\,(B^i,D^i)$ means the space of smooth maps of $\langle i\rangle$-manifolds $B^i \longrightarrow D^i$). The image of p consists of those maps $h\colon B^i \longrightarrow D^i$ such that $h^*(\mu^i) \cong \eta^i$.

9.3 <u>Lemma</u>: For $j < i$, the restriction map $\mathrm{Hom}\,(\eta^i,\mu^i) \longrightarrow \mathrm{Hom}\,(\partial_j\eta^i, \partial_j\mu^i)$ is a fibration. (Recall that $\partial_j\eta^i = \eta^i|_{\partial_j B^i}$.)

Of course the same result is true if we replace $\partial_j\eta^i$ and $\partial_j\mu^i$ by $\partial\eta^i$ and $\partial\mu^i$, where $\partial\eta^i = \eta^i|_{\partial B^i}$ and $\partial B^i = \cup\, \partial_j B^i$.

Recall that a morphism $\theta \in \mathrm{Hom}_{\tilde{C}}(\eta,\mu)$ is a k+1-tuple $\theta = (\theta^0,\theta^1,\ldots,\theta^k)$, $\theta^j \in \mathrm{Hom}\,(\eta^j,\mu^j)$, such that the following diagram commutes.

(9.4)

$$\begin{tikzcd}
\eta^{ij} \arrow[rr, "\times\theta^i"] \arrow[d, "f^{ij}"'] & & \mu^{ij} \arrow[d, "g^{ij}"] \\
\partial_i\eta^{i+j} \arrow[rr, "\theta^{i+j}"] & & \partial_i\mu^{i+j}
\end{tikzcd}$$

Notice that for a given θ^i this diagram determines $\theta^{i+j}\colon \partial_i\eta^{i+j} \longrightarrow \partial_i\mu^{i+j}$. In other words, there is a map

$$\Phi^{ij}\colon \mathrm{Hom}\,(\eta^i,\mu^i) \longrightarrow \mathrm{Hom}\,(\partial_i\eta^{i+j}, \partial_i\mu^{i+j})$$

defined by $\Phi^{ij}(\theta^i) = \theta^{i+j}|_{\partial_i\eta^{i+j}}$. Let

$$h^s(\eta,\mu) \subset \mathop{\times}_{i=0}^{i=s} \text{Hom}\,(\eta^i,\mu^i)$$

be the subspace consisting of those s+1-tuples $(\theta^0, \theta^1, \ldots, \theta^s)$ which satisfy diagram (9.4) whenever $i+j \leq s$. Notice that $h^0(\eta,\mu) = \text{Hom}\,(\eta^0,\mu^0)$ and that $h^k(\eta,\mu) = \text{Hom}_{\tilde{C}}(\eta,\mu)$. There is a map

$$\Phi^s : h^s(\eta,\mu) \longrightarrow \text{Hom}\,(\partial\eta^{s+1}, \partial\mu^{s+1}) \tag{9.5}$$

defined by

$$\Phi^s(\theta^0, \ldots, \theta^s) = \bigcup_{i=0}^{s} \Phi^{i,\, s-i+1}(\theta^i)$$

where the $\Phi^{i,\, s-i+1}(\theta^i) : \partial_i \eta^{s+1} \longrightarrow \partial_i \mu^{s+1}$ are bundle maps which agree on the codimension 2 faces. Also notice that there are natural projections

$$P^s : h^{s+1}(\eta,\mu) \longrightarrow h^s(\eta,\mu) \tag{9.6}$$

defined by $P^s(\theta^0, \ldots, \theta^{s+1}) = (\theta^0, \ldots, \theta^s)$.

9.7 <u>Proposition</u>: The following diagram is a pullback square.

$$\begin{array}{ccc}
h^{i+1}(\eta,\mu) & \longrightarrow & \text{Hom}\,(\eta^{i+1},\mu^{i+1}) \\
\Big\downarrow{\scriptstyle P^i} & & \Big\downarrow \\
h^i(\eta,\mu) & \xrightarrow{\Phi^i} & \text{Hom}\,(\partial\eta^{i+1},\partial\mu^{i+1})
\end{array}$$

Moreover, P^i is a fibration. If $\alpha \in h^i(\eta,\mu)$, then the fibre of P^i at α is $\text{Hom}\,(\eta^{i+1},\mu^{i+1})_{\Phi^i(\alpha)}$, the space of bundle maps $\eta^{i+1} \longrightarrow \mu^{i+1}$ which are equal to $\Phi^i(\alpha)$ on $\partial\eta^{i+1}$.

Proof: The diagram is obviously a pullback square. Since the restriction map $\mathrm{Hom}(\eta^{i+1}, \mu^{i+1}) \longrightarrow \mathrm{Hom}(\partial\eta^{i+1}, \partial\mu^{i+1})$ is a fibration, so is P^i; moreover, they have the same fibre.

□

9.8 Remark: Rephrasing the above proposition, there is a sequence of fibrations

$$h^k(\eta,\mu) \xrightarrow{P^{k-1}} h^{k-1}(\eta,\mu) \longrightarrow \cdots\cdots \xrightarrow{P^1} h^0(\eta,\mu)$$

starting with $\mathrm{Hom}(\eta^0, \mu^0)$ and converging to $h^k(\eta,\mu) = \mathrm{Hom}_{\widetilde{C}}(\eta,\mu)$. Over a given path component the fibre of P^i is a space $\mathrm{Hom}(\eta^{i+1}, \mu^{i+1})_{\Phi^i(\alpha)}$ whose homotopy groups can be calculated by standard methods. This gives us a procedure for computing $\pi_i \mathrm{Hom}_{\widetilde{C}}(\eta,\mu)$.

For example, suppose we are trying to calculate $[B; H_+(k)] = \pi_0 \mathrm{Hom}_{\widehat{\mathfrak{G}}_k}(\eta, \eta_k)$, where η and η_k are normal systems for B and $H_+(k)$, respectively. The first step is to compute $\pi_0 h^0(\eta, \eta_k) = \pi_0 \mathrm{Hom}(\eta^0, \eta_k^0) \cong [B^0, PG_k]$ (where the last isomorphism uses the fact that η^0 is a trivial bundle and that $\eta_k^0 = PG_k$). Next we try to compute $\pi_0 h^1(\eta, \eta_k)$ by using the homotopy sequence of the fibration

$$\pi_0 \mathrm{Hom}(\eta^1, \eta_k^1)_{\Phi(a)} \longrightarrow \pi_0 h^1(\eta, \eta_k^1) \xrightarrow{P^1_*} \pi_0 \mathrm{Hom}(\eta^0, \eta_k^0).$$

Since π_0 of a space only has the structure of a set, the meaning of exactness in this sequence is somewhat obscure. The fibre $\mathrm{Hom}(\eta^1, \eta_k^1)_{\Phi(a)}$ depends on the path component of $a \in \mathrm{Hom}(\eta^0, \eta_k^0)$. However, exactness at least means that $\pi_0 \mathrm{Hom}(\eta^1, \eta_k^1)_{\Phi(a)} \longrightarrow (P^1_*)^{-1}(a)$ is a surjection.

The image of P_*^1 can be decided from the pullback diagram. Hence, we can compute $\pi_0 h^1(\eta, \eta_k)$, since $\pi_0 \operatorname{Hom}(\eta^1, \eta_k^1)_{\Phi(a)}$ and its image in $\pi_0 h^1(\eta, \eta_k)$ can be computed by using Lemmas 9.2 and 9.3 and standard methods. We continue with this process until we reach the last stage

$$\pi_0 \operatorname{Hom}(\eta^k, \eta_k^k)_{\Phi(a)} \longrightarrow \pi_0 \operatorname{Hom}(\eta, \eta_k) \xrightarrow{P_*^{k-1}} \pi_0 h^{k-1}(\eta, \eta_k).$$

Notice that $\eta^k = B^k$ and that $\eta_k^k = B_k^k$. Since B_k^k is contractible, $\pi_0 \operatorname{Hom}(B^k, B_k^k) \longrightarrow \pi_0 \operatorname{Hom}(\partial B^k, \partial B_k^k)$ is onto; and so from the pullback diagram 9.7 we see that P_*^k is also onto. Finally, notice that $\operatorname{Hom}(\eta^k, \eta_k^k)_{\Phi(a)}$ is just the space of maps $B^k \longrightarrow B_k^k$ which are equal to some fixed map on the boundary. This procedure for calculating $[B, H_+(k)]$ is actually feasible if the number of nonempty strata of B is small.

There is a dual (but inelegant) method for studying the homotopy structure of $\operatorname{Hom}_{\tilde{C}}(\eta, \mu)$. This approach uses a sequence of fibrations beginning with $\operatorname{Hom}(\eta^k, \mu^k)$. The construction of this sequence is left to the reader.

10. $\pi_i \operatorname{Diff}_1^{G_n}(M)$

Suppose that η_B is a normal system for a compact $\mathcal{B}_k$-space B and that $\mathcal{E} = (\xi, \eta, \psi)$ is a $\mathcal{S}_{n,k}(B)$-normal system $\operatorname{rel} \eta_B$. Let

$$\operatorname{Aut}(\mathcal{E}) \subset \operatorname{Hom}_{\tilde{\mathcal{S}}_{n,k}(\eta_B)}(\mathcal{E}, \mathcal{E})$$

$$\operatorname{Aut}(\xi) \subset \operatorname{Hom}_{\tilde{\mathcal{S}}_{n,k}}(\xi, \xi)$$

$$\operatorname{Aut}(\eta) \subset \operatorname{Hom}_{\tilde{\mathcal{B}}_k}(\eta, \eta)$$

denote spaces of $\widetilde{\mathcal{C}}$-automorphisms in the appropriate category $\widetilde{\mathcal{C}}$. Also, let Aut (ξ^i) (respectively, Aut (η^i)) be the space of $G_i \times G_{k-i}$ face preserving self-diffeomorphisms of ξ^i (respectively, PG_{k-i}-self-diffeomorphisms of η^i). Recall that $\theta \in \mathrm{Aut}\,(\xi)$ is a (k+1)-tuple

$$\theta = (\theta^0, \theta^1, \ldots \theta^k) \in \mathop{\times}_{i=0}^{k} \mathrm{Aut}\,(\xi^i)$$

which satisfies the compatibility condition

$$\begin{tikzcd} \xi^{ij} \arrow[r, "\times\theta^i"] \arrow[d, "f^{ij}"] & \xi^{ij} \arrow[d, "f^{ij}"] \\ \partial_i \xi^{i+j} \arrow[r, "\theta^{i+j}"] & \partial_i \xi^{i+j} \end{tikzcd} .$$

Aut (η) has a similar description. The projection $\xi^i \longrightarrow \eta^i$ induces natural projections p_i: Aut $(\xi^i) \longrightarrow$ Aut (η^i) and p: Aut $(\xi) \longrightarrow$ Aut (η). Notice that Aut $(\mathcal{E}) = p^{-1}(1)$, where $1 \in \mathrm{Aut}\,(\eta)$ denotes the identity morphism. Since $\xi^i \longrightarrow \eta^i$ is a fibre bundle, it follows from the Covering Homotopy Theorem that p_i is a fibration. Analogously, we have the

10.1 <u>Theorem:</u> Aut $(\mathcal{E}) \longrightarrow$ Aut $(\xi) \longrightarrow$ Aut (η) is a principal fibration.

<u>Proof:</u> Let Y be a finite polyhedron, let h: $Y \times I \longrightarrow$ Aut (η) be a homotopy and let f: $Y \longrightarrow$ Aut (ξ) be a lift of $h|_{Y \times \{0\}}$. Regard h as a homotopy of automorphisms h: $Y \times I \times \eta \longrightarrow Y \times \eta$ ($Y \times I \times \eta$ obviously can be regarded as a normal system). Let h^i: $Y \times I \times \eta^i \longrightarrow Y \times \eta^i$ and f^i: $Y \times \xi^i \longrightarrow Y \times \xi^i$ denote the i^{th} component of h and f, respectively.

By the Covering Homotopy Theorem for fibre bundles, there is a lift of h^0 to $H^0: Y \times I \times \xi^0 \longrightarrow Y \times \xi^0$, extending f^0. From the compatibility diagram, we see that H^0 defines homotopies of automorphisms $Y \times I \times \partial_0 \xi^i \rightarrow Y \times \partial_0 \xi^i$ which extend $f^i|_{Y \times \partial_0 \xi^i}$. Next we use the usual inductive procedure. At the i^{th} stage we will have a lift of $h^i|_{Y \times I \times \partial \eta^i}$, call it $\overline{H}^i: Y \times I \times \partial \xi^i \longrightarrow Y \times \partial \xi^i$, which extends $f^i|_{Y \times \partial \xi^i}$. So by the relative form of the Covering Homotopy Theorem, $\overline{H}^i$ extends to a map $H^i: Y \times I \times \xi^i \longrightarrow Y \times \xi^i$ covering h^i. This process constructs a map $H = (H^0, \ldots. H^k): Y \times I \longrightarrow \text{Aut}(\xi)$ which covers h. Hence p is a fibration. Since the fibre $\text{Aut}(\mathcal{E})$ is a topological group acting on $\text{Aut}(\xi)$, it is routine to verify that p is, in fact, a principal fibration.

$\square$

Suppose that M is a $\mathcal{B}_{n,k}(B)$-manifold and that $\mathcal{E}$ is a normal system for M. According to Theorem 7.15, the assemble functor defines an inclusion $\text{Aut}(\mathcal{E}) \subset \text{Diff}_1^{G^n}(M)$, which is a homotopy equivalence. Comparing the above theorem with Theorem 3.12 suggests the following

10.2 <u>Conjecture</u>: The inclusions $\text{Aut}(\xi) \subset \text{Diff}^{G^n}(M)$ and $\text{Aut}(\eta) \subset \text{Diff}(B)$ are also homotopy equivalences. Furthermore, via these inclusions, the fibration of 10.1 is equivalent to the following fibration of 3.12,

$$\text{Diff}_1^{G^n}(M) \longrightarrow \text{Diff}^{G^n}(M) \longrightarrow \text{Diff}(B).$$

(We know that this conjecture is true on the π_0-level.)

In the remainder of this section techniques are developed for computing the homotopy groups $\pi_i \operatorname{Aut}(\mathcal{E})$ (particularly $\pi_0 \operatorname{Aut}(\mathcal{E})$).

<u>Automorphic bundles and normal systems</u>:

We will show how $\operatorname{Aut}(\mathcal{E})$ can be described as a space of cross sections of a certain collection of bundles (namely, cross sections of the "automorphic normal system" associated to $\mathcal{E}$). This result is based on the following well-known proposition (see [H]).

10.3 <u>Proposition</u>: If $\alpha \longrightarrow K$ is a smooth fibre bundle with structure group G, then there is a bundle $\alpha^* \longrightarrow K$ with structure group $G/(\text{center of } G)$ and fibre G such that

$$\operatorname{Aut}_K(\alpha) \cong \Gamma(\alpha^*),$$

where $\operatorname{Aut}_K(\alpha)$ denotes the space of bundle automorphisms of α and $\Gamma(\alpha^*)$ denotes the space of sections of α^*. Both $\operatorname{Aut}_K(\alpha)$ and $\Gamma(\alpha^*)$ are naturally topological groups and in the above equation $\cong$ means isomorphism of topological groups. ($\alpha^* \longrightarrow K$ is called the <u>automorphic bundle associated to α</u>.)

<u>Proof</u>: Without loss of generality we may assume that α is a principal G-bundle. Let Z = center of G and let $\alpha/Z \longrightarrow K$ denote the (weakly) associated G/Z-bundle. Define α^* by

(1) $$\alpha^* = G \times_{G/Z} \alpha/Z,$$

where G/Z acts on G by conjugation. It is easy to see that the operation of "pointwise multiplication in each fibre of α^*" is well defined and can be used to define a group structure on $\Gamma(\alpha^*)$.

Let $\{\varphi_i: V_i \times G \longrightarrow \alpha|_{V_i}\}$ and $\{g_{ij}: V_i \cap V_j \longrightarrow G\}$ be coordinate functions and coordinate transformations for α and let $\{\varphi_i^*\}$, $\{g_{ij}^*\}$ be the corresponding maps for α^*. If $f \in \mathrm{Aut}_K(\alpha)$ and $x \in K$, then let $f_x: \alpha_x \longrightarrow \alpha_x$ denote the restriction of f to α_x, the fibre of α at x. Clearly,

(2) $$\varphi_i \circ f_x \circ \varphi_i^{-1} = (\omega_j \circ f_x \circ \varphi_j^{-1}) \cdot g_{ji}^*(x).$$

We can define a homomorphism $T: \mathrm{Aut}(\alpha) \longrightarrow \Gamma(\alpha^*)$ by the formula

(3) $$T(f)(x) = \varphi_{j,x}^*(\varphi_{j,x} \circ f_x \circ \varphi_{j,x}^{-1})$$

where $\varphi_{j,x} = \varphi_j|_{\{x\} \times G}$. Using (2) we see that this definition is independent of the choice of j. A straightforward argument shows that T is indeed an isomorphism (see [H] for more details).

□

As usual, suppose that $\mathcal{E} = (\xi, \eta, \psi)$ is a $\mathcal{J}_{n,k}(B)$ normal system rel η_B. Recall that an automorphism of $\mathcal{E}$ is a k+1-tuple $(\theta^0, \ldots. \theta^k)$ where $\theta^i \in \mathrm{Aut}(\xi^i)$. The θ^i are required to satisfy a compatibility condition and each θ^i covers the identity on η^i. Since $\xi^i \longrightarrow \eta^i$ is a principal $G_i \times Z_{k-i}$ bundle, such a θ^i will correspond to a section of ξ^{i*} which lies in the "$G_i \times Z_{k-i}$ part of the fibre." The following definition makes this precise.

10.4 <u>Definition:</u> Recall that the automorphic bundle associated to ξ^i is defined by

$$\xi^{i*} = \{G_i \times G_{k-i}\} \times_{G_i/Z_i \times G_{k-i}/Z_{k-i}} \xi^i/Z_i \times Z_{k-i}.$$

Let $\zeta^i \subset \xi^{i*}$ be the subbundle defined by

$$\zeta^i = \{G_i \times Z_{k-i}\} \times_{G_i/Z_i \times G_{k-i}/Z_{k-i}} \xi^i/Z_i \times Z_{k-i}.$$

This definition makes sense since $Z_{k-i} \subset G_{k-i}$ is invariant under conjugation. In a similar fashion we can define $G_i \times G_j \times Z_{k-i-j}$ bundles $\zeta^{ij} \subset \xi^{ij*}$, etc. The f^{ij}'s induce maps $f^{ij*}: \xi^{ij*} \longrightarrow \partial_i \xi^{i+j\,*}$ which restrict to inclusions

$$h^{ij}: \zeta^{ij} \longrightarrow \partial_i \zeta^{i+j}.$$

The collection $\zeta = \{\zeta^i \longrightarrow B^i;\ h^{ij}: \zeta^{ij} \longrightarrow \partial_i \zeta^{i+j}\}$ is called the <u>automorphic normal system</u> associated to $\mathcal{E} = (\xi, \eta, \psi)$.

By construction the space of sections $\Gamma(\zeta^i)$ can be identified with the space of automorphisms of ξ^i which cover the identity on η^i. Thus,

$$\mathrm{Aut}\,(\mathcal{E}) \subset \mathop{\times}_{i=0}^{i=k} \Gamma(\zeta^i).$$

Recall that in section 9 we defined monomorphisms $\Phi^{ij}: \mathrm{Aut}(\xi^i) \longrightarrow \mathrm{Aut}(\partial_i \xi^{i+j})$ with the property that $(\theta^0, \ldots . \theta^k) \in \mathrm{Aut}\,(\xi)$ if and only if for each pair (i,j), $\Phi^{ij}(\theta^i) = \theta^{i+j}\big|_{\partial_i \xi^{i+j}}$. These maps restrict to monomorphisms $\Phi^{ij}: \Gamma(\zeta^i) \longrightarrow \Gamma(\partial_i \zeta^{i+j})$. The explicit definition of the maps Φ^{ij} in terms of the data of the automorphic normal system $h^{ij}: \zeta^{ij} \longrightarrow \partial_i \zeta^{i+j}$ is left as an exercise for the reader.

10.5 <u>Definition</u>: Let $\Gamma(\zeta, r)$ be the subspace of all $(r+1)$-tuples of sections $(s_0, s_1 \ldots s_r) \in \overset{i=r}{\underset{i=0}{\times}} \Gamma(\zeta^i)$ such that for any pair of nonnegative integers (i, j), with $0 < i+j \leq r$, $\Phi^{ij}(s_i) = s_{i+j}\big|_{\partial_i B^{i+j}}$. Let $\Gamma(\zeta) = \Gamma(\zeta, k)$. Clearly,

$$\text{(10.6)} \qquad \mathrm{Aut}\,(\mathcal{C}) = \Gamma(\zeta).$$

In other words, the automorphisms of a $\mathscr{S}_{n,k}(B)$-normal system rel η_B can be described completely in bundle theoretic terms. Notice that $\Gamma(\zeta, r)$ can be identified with the space of equivariant diffeomorphisms of $N^0 \cup N^1 \ldots\ldots \cup N^r$ which cover the identity on the orbit space and which are linear bundle maps on each N^i. We can think of $\Gamma(\zeta)$ as the space of sections of $\zeta^k = \xi^{k*}$ which over each piece $\partial_i B^k$ lie in certain subbundles (defined by the h^{ij}'s).

There is a homomorphism

$$P^i\colon \Gamma(\zeta, i+1) \longrightarrow \Gamma(\zeta, i)$$

defined by projection on the first $i+1$ factors. There is also a monomorphism

$$\Phi^i\colon \Gamma(\zeta, i) \longrightarrow \Gamma(\partial\zeta^{i+1})$$

defined by $\Phi^i(s_0, \ldots, s_i) = \cup\, \Phi^{j, i+1-j}(s_j)$. The following theorem, which is completely analogous to Proposition 9.7, was first observed by Bredon [B2] in a slightly less general context.

10.7 <u>Theorem</u> (Bredon): The following diagram is a pullback square.

$$\begin{tikzcd}
\Gamma(\zeta, i+1) \arrow[r] \arrow[d, "P^i"] & \Gamma(\zeta^{i+1}) \arrow[d] \\
\Gamma(\zeta, i) \arrow[r, "\Phi^i"'] & \Gamma(\partial\zeta^{i+1}) .
\end{tikzcd}$$

Since the right-hand arrow is clearly a fibration, so is $P^i : \Gamma(\zeta, i+1) \longrightarrow \Gamma(\zeta, i)$. The fibre of P^i is $\Gamma(\zeta^{i+1})_\partial$, the space of sections of the bundle $\zeta^{i+1} \longrightarrow B^{i+1}$ which are the identity section over ∂B^{i+1}.

In other words, we have a sequence of (principal) fibrations $\Gamma(\zeta) \xrightarrow{P^{k-1}} \Gamma(\zeta, k-1) \longrightarrow \ldots \longrightarrow \Gamma(\zeta, 1) \xrightarrow{P^1} \Gamma(\zeta^0)$ starting with $\Gamma(\zeta^0)$ and converging to $\Gamma(\zeta)$. Since the fibre of P^i, $\Gamma(\zeta^{i+1})_\partial$, is a space of cross sections of an actual bundle, the theorem obviously provides an attack on the problem of computing the homotopy type of $\Gamma(\zeta)$.

The problem is particularly tractable when there are only two non-empty strata (the k^{th} and $k-1^{th}$). In this case the pullback square is

$$\begin{tikzcd}
\Gamma(\zeta) \arrow[r] \arrow[d] & \Gamma(\zeta^k) \arrow[d] \\
\Gamma(\zeta^{k-1}) \arrow[r] & \Gamma(\partial\zeta^k).
\end{tikzcd}$$

Notice that each of the terms other than $\Gamma(\zeta)$ is the space of sections of a bundle.

10.8 <u>Example</u>: Consider the linear action $2\rho_n : O(n) \times S^{2n-1} \longrightarrow S^{2n-1}$, where $S^{2n-1} = \sum(n, 2)$ is the unit sphere in $M(n, 2)$. The orbit space is $W_+(2)$, the set of trace one 2×2 positive semidefinite symmetric matrices

$(W_+(2) \cong D^2)$. Let $\mu_2 = \{\mu_2^i \longrightarrow W_2^i,\ f_2^{11} : \mu_2^{11} \longrightarrow \partial\mu_2^2\}$ be the standard $\mathscr{S}^1_{n,2}$-normal system for S^{2n-1} (the restriction of the normal system ξ_2 defined in section 6, i.e., $\mu_2 = \xi_2|_{W_+(2)}$). It follows from Lemma 6.1 that

(i) $\mu_2^2 \longrightarrow W_2^2 \cong D^2$ is the trivial $O(2)$-bundle.

(ii) $\mu_2^1 \longrightarrow W_2^1 \cong S^1$ is the Whitney sum of two nontrivial $O(1)$-bundles.

(iii) The "characteristic reduction" $f_2^{1,1} : \mu_2^{1,1} \longrightarrow \partial\mu_2^2$ is a reduction of the structure group of $\partial\mu_2^2$ to $O(1) \times O(1)$ which is classified by a map

$$S^1 = W_2^1 \longrightarrow O(2)/O(1) \times O(1) = S^1$$

of degree 1 (notice that $\mu_2^{1,1} = \partial_0\xi_1^1 \times_{O(1)} \mu_2^1 \cong \mu_2^1$).

Let ζ be the automorphic normal system associated to μ_2. We see that

$$\zeta^1 = \mu_2^{1*} \cong O(1) \times O(1) \times S^1, \text{ and that}$$

$$\zeta^2 = \mu_2^{2*} \cong O(2) \times D^2.$$

Hence $\Gamma(\zeta^1)$ is a space consisting of 4 points. The inclusion $h^{1,1} = f_2^{1,1} : \zeta^{1,1} = \zeta^1 \longrightarrow \partial\zeta^2$ is trivial on the subbundle defined by the sections which are identically

$$+1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } -1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in O(1) \times O(1).$$

$h^{1,1}$ rotates the other two sections via a degree 1 map $S^1 \longrightarrow O(2) - SO(2) \cong S^1$ (see [B2, p.266]). Since the image of $P^1 : \Gamma(\zeta) \longrightarrow \Gamma(\zeta^1)$ consists precisely of those sections of ζ^1 which are null-homotopic in $\Gamma(\partial\zeta^2)$, it follows that there are exactly two points in the image of P^1, namely the sections $\pm 1 \in \Gamma(\zeta^1)$. By using the sequence of the fibration

$$\Gamma(\zeta^2)_\partial \longrightarrow \Gamma(\zeta) \xrightarrow{P^1} \Gamma(\zeta^1)$$

we can deduce the following

10.9 Proposition:

(1) $\mathrm{Diff}_1^{O(n)}(S^{2n-1})$ deformation retracts to $\mathbb{Z}_2$, the subgroup consisting of the identity and the antipodal map.

(2) The following are equivalent fibrations

$$\mathrm{Diff}_1^{O(n)}(S^{2n-1}) \longrightarrow \mathrm{Diff}^{O(n)}(S^{2n-1}) \longrightarrow \mathrm{Diff}\,(D^2)$$

and

$$\mathbb{Z}_2 \longrightarrow O(2) \longrightarrow O(2)\,.$$

In particular this means that every element of $\mathrm{Diff}^{O(n)}(S^{2n-1})$ is equivariantly isotopic to an equivariant linear isomorphism.

Proof: $P^1 : \Gamma(\zeta) \longrightarrow \Gamma(\zeta^1)$ is a fibration over two points; hence, it is a trivial fibration. So by 7.15 and 10.6,

$$\mathrm{Diff}_1^{O(n)}(S^{2n-1}) \sim \Gamma(\zeta) = \Gamma(\zeta^2)_\partial \times \mathbb{Z}_2\,.$$

But $\Gamma(\zeta^2)_\partial$ is just another name for $\mathrm{Map}\,(D^2, S^1; O(2), 1)$, the space of smooth maps $D^2 \longrightarrow O(2)$ which take the boundary to the base point. $\mathrm{Map}(D^2, S^1; O(2), 1)$ can obviously be identified with the identity component of the loop space $\Omega^2(SO(2))$, which is contractible. This proves (1).

Consider the fibration of Theorem 3.12

$$\mathrm{Diff}_1^{O(n)}(S^{2n-1}) \longrightarrow \mathrm{Diff}^{O(n)}(S^{2n-1}) \longrightarrow \mathrm{Diff}\,(D^2)\,.$$

It is well known that $\mathrm{Diff}\,(D^2)$ deformation retracts to $O(2)$. On the other hand, regarding $O(2)$ as the orthogonal centralizer of $2\rho_n$, $O(2) \subset \mathrm{Diff}^{O(n)}(S^{2n-1})$. Statement (2) now follows from (1).

$\square$

10.9 <u>Remark</u>: Let $\mathcal{E}$ be a normal system for a $\mathcal{S}_{n,k}(B)$-manifold M and let ζ be the automorphic normal system associated to $\mathcal{E}$. There is another way to prove Theorem 7.15, which says that $\Gamma(\zeta) = \mathrm{Aut}\,(\mathcal{E})$ is a deformation retract of $\mathrm{Diff}_1^{G_n}(M)$. It suffices to work locally. For simplicity suppose $M = M(n,k)$. By 3.5, if $\lambda\colon M(n,k) \longrightarrow M(n,k)$ is any equivariant diffeomorphism, there is a unique smooth map $\sigma\colon H_+(k) \longrightarrow GL_k$ such that

$$\lambda(x) = x \cdot \sigma(\pi(x))\,.$$

If λ covers the identity on $H_+(k)$, then for any $z \in H_+(k)$

$$\sigma(z)^* \cdot z \cdot \sigma(z) = z\,.$$

Let $S = \{(z,g) \in H_+(k) \times GL_k \mid g^* \cdot z \cdot g = z\}$. Let $p\colon S \longrightarrow H_+(k)$ be projection on the first factor. We see that $\mathrm{Diff}_1^{G_n}(M(n,k))$ can be identified with the smooth sections of p. Let $\xi_k = \{\xi_k^i \longrightarrow B_k^i;\ f_k^{ij}\colon \xi_k^{ij} \longrightarrow \partial_i \xi_k^{i+j}\}$ be the normal system for $M(n,k)$ defined in section 6. Clearly, $p^{-1}(B_k^k) \longrightarrow B_k^k$ is a smooth fibre bundle with fibre G_k. In fact, by unraveling the definitions, we see that there is a natural isomorphism $p^{-1}(B_k^k) \cong \zeta_k^k$ where $\zeta_k^k = \xi_k^{k*}$ is the automorphic bundle. If $z \in B_k^i$, then what is the singular fibre

$p^{-1}(z)$? It suffices to consider the case where $z = e$ is the $i \times i$ identity matrix in the upper left-hand corner and zero elsewhere. Notice that $g^* \cdot e \cdot g = e$ implies that g must be of the form

$$\text{(1)} \qquad g = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

where $A \in G_i$ and C is a $(k-i) \times (k-i)$ matrix. Not every such g occurs as the image of a smooth section at e. To see this we use the fact that if $\sigma : H_+(k) \longrightarrow S$ is a smooth section, then the differential of the map $z \longrightarrow \sigma(z)^* \cdot z \cdot \sigma(z)$ is the identity. First write σ in the form

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{pmatrix}$$

where σ_1 is an $i \times i$ matrix and σ_4 is a $(k-i) \times (k-i)$ matrix. According to (1), $\sigma_2(e) = 0$. Let $v \in H_+(k-i)$, and let z denote the matrix

$$z = \begin{pmatrix} e & 0 \\ 0 & tv \end{pmatrix}.$$

Multiplying, we have that $\sigma(z)^* \cdot z \cdot \sigma(z) =$

$$\begin{pmatrix} \sigma_1^* \sigma_1 + \sigma_3^* t v \sigma_3 & \sigma_1^* \sigma_2 + \sigma_3^* t v \sigma_4 \\ \sigma_2^* \sigma_1 + \sigma_4^* t v \sigma_3 & \sigma_4^* t v \sigma \end{pmatrix}.$$

From the fact that the differential at e is the identity, we see that for all $v \in H_+(k-i)$,

$$\begin{aligned} v &= \lim_{t \to 0} t^{-1} \sigma_4(z)^* \cdot tv \cdot \sigma_4(z) \\ &= \sigma_4(e)^* \cdot v \cdot \sigma_4(e) . \end{aligned}$$

Thus, $\sigma_4(e) \in Z_{k-i}$. Similarly,

$$0 = \lim_{t \to 0} t^{-1}(\sigma_1(z)^* \sigma_2(z) + \sigma_3(z)^* tv\sigma_4(z))$$

$$= \sigma_3(e)^* \cdot v \cdot \sigma_4(e).$$

Hence, $\sigma_3(e) = 0$, and so $\sigma(e) \in G_i \times Z_{k-i}$. This shows that the sections of $p^{-1}(B_k^i) \longrightarrow B_k^i$ which extend to smooth section on $H_+(k)$ actually lie in a $G_i \times Z_{k-i}$ bundle, which can obviously be identified with ζ_k^i. Visibly, $\Gamma(\zeta_k)$ is a deformation retract of $\mathrm{Diff}_1^{G_n}(M(n,k))$. (In fact, if we recall that the decomposition of $H_+(k)$ in section 6 is defined relative to a sequence $\rho = (\rho_0, \rho_1, \ldots, \rho_k)$, the deformation retraction can be defined by letting $|\rho| \longrightarrow 0$.)

10.10 <u>Remark</u>: The dual approach, mentioned at the end of section 9, can also be used to study the homotopy structure of $\mathrm{Aut}(\mathcal{E}) = \Gamma(\zeta)$. We can define a decreasing sequence of $\mathcal{B}_k$-spaces

$$B = {}_0B \supset {}_1B \supset \ldots\ldots \supset {}_{k-2}B \supset {}_{k-1}B = B^k$$

$${}_iB = \text{closure of } \left\{B - \bigcup_{j=0}^{j=i-1} C^j\right\}.$$

Also, define $\delta({}_iB)$ by

$$\delta({}_iB) = \bigcup_{j=i}^{j=k} \partial_{i-1} C^j$$

$$= \bigcup_{j=i}^{j=k} \partial_j C^{i-1}.$$

The restriction of a normal system to a $\mathcal{B}_k$-subspace is defined in the obvious manner. As an easy consequence of the fact that ${}_iB \subset {}_{i-1}B$ is a cofibration and the usual inductive procedure we have the following

10.11 <u>Proposition</u>: Let ζ be a automorphic normal system associated to a $\widetilde{\mathcal{J}}_{n,k}(\eta_B)$-object. The following diagram is a pullback square:

$$\begin{tikzcd}
\Gamma(\zeta|_{{}_{i-1}B}) \arrow[r] \arrow[d] & \Gamma(\zeta^{i-1}) \arrow[d] \\
\Gamma(\zeta|_{{}_iB}) \arrow[r] & \Gamma(\zeta|_{\delta({}_iB)}),
\end{tikzcd}$$

where the top arrow is projection on the $(i-1)^{th}$ component, the left hand and bottom arrows are restriction maps and the right hand arrow is defined using the $\Phi^{i-1,j}$'s. The top, bottom and left hand arrows are all fibrations.

This proposition gives us a sequence of fibrations

$$\Gamma(\zeta) \longrightarrow \Gamma(\zeta|_{{}_iB}) \longrightarrow \dots\dots \longrightarrow \Gamma(\zeta^k)$$

starting with the automorphisms of principal orbit bundle and converging to $\Gamma(\zeta)$.

<u>Obstruction theory</u>: Let ζ be an automorphic normal system for a $\mathcal{J}_{n,k}(B)$-manifold. We are going to develop an obstruction theory for $\Gamma(\zeta)$. Let $|B|^q$ denote the q-skeleton of B. Suppose that $s = (s_0, s_1, \dots, s_k) \in \Gamma(\zeta|_{|B|^q})$ is a section defined on the q-skeleton. There is a sequence of obstruction cocycles

$$c_i(s) \in C^{q+1}(B^i; \pi_q(G_i \times Z_{k-i})),$$

which are the obstructions to extending $s_i \in \Gamma(\zeta|_{|B^i|^q})$ to a section defined on the (q+1)-skeleton of B^i. (In the case $G_n = O(n)$, we are using local coefficients.)

These obstructions are not unrelated. Recall that there are monomorphisms $\Phi^{ij}: \Gamma(\zeta^i) \longrightarrow \Gamma(\partial_i \zeta^{i+j})$ such that

$$\Phi^{ij}(s_i) = s_{i+j}\big|_{\partial_i B^{i+j}} .$$

Previously the definition of the Φ^{ij}'s had been left to the reader. We now give such a definition. Recall that

$$\partial_i B^{i+j} \cong \partial_0 B^j_j \times_{G_j} \zeta^i / G_i \times G_{k-i-j} .$$

The right hand expression is a bundle over B^i with fibre $\partial_0 B^j_j \times_{G_j} G_k/G_i \times G_{k-i-j}$. Let $\pi: \partial_i B^{i+j} \longrightarrow B^i$ be the projection and let $\hat{\pi}: \Gamma(\zeta^i) \longrightarrow \Gamma(\pi^* \zeta^i)$ be the induced map. There is a natural inclusion $q: \pi^* \zeta^i \subset \xi^{i+j}$. Also, we have an inclusion $h^{ij}: \zeta^{ij} \longrightarrow \partial_i \zeta^{i+j}$ which is part of the data of the normal system ζ. If $b \in \partial_i B^{i+j}$, then define $\Phi^{ij}(s_i): \partial_i B^{i+j} \longrightarrow \partial_i \zeta^{i+j}$ by the formula

(10.12) $$\Phi^{ij}(s_i)(b) = h^{ij} \circ q \circ \hat{\pi}(s_i)(b).$$

Let $\pi^*: C^*(B^i; \pi_q(G_i \times Z_{k-i})) \longrightarrow C^*(\partial_i B^{i+j}; \pi_q(G_i \times Z_{k-i}))$ be the map induced by π. Let $(h^{ij} \circ q)_*: C^*(\partial_i B^{i+j}; \pi_q(G_i \times Z_{k-i})) \longrightarrow C^*(\partial_i B^{i+j}; \pi_q(G_{i+j} \times Z_{k-i-j})$ be the coefficient homomorphism induced by $h^{ij} \circ q$. Let

$$\Phi^{ij}{}_* = (h^{ij} \circ q)_* \circ \pi^* .$$

If $J_{ij}: \partial_i B^{i+j} \subset B^{i+j}$ is the inclusion, then it follows from the definition of

Φ^{ij} and the naturality of obstructions that

(10.13) $$\Phi^{ij}_*(c_i(s)) = J^*_{ij}(c_{i+j}(s)).$$

This relation between the obstruction cocycles motivates the following definition.

10.14 Definition: Let $C^*(B; \pi_q; r)$ denote the subcomplex of

$$\bigoplus_{j=0}^{j=r} C^*(B^j; \pi_q(G_j \times G_{k-j}))$$

defined by the conditions that

$$\Phi^{ij}_*(\alpha_i) = J^*_{ij}(\alpha_{i+j})$$

whenever $(\alpha_0, \alpha_1, \ldots, \alpha_r)$ is an element of the direct sum. The cochain complex $C^*(B; \pi_q; r)$ is the obstruction theory for $\Gamma(\zeta, r)$. Its cohomology is denoted by $\mathcal{H}^*(B; \pi_q; r)$. Let $C^*(B; \pi_q) = C^*(B; \pi_q; k)$ be the obstruction theory for $\Gamma(\zeta)$. Notice that if $s \in \Gamma(\zeta|_{|B|^q})$, then by (10.13) the obstruction cocycle $c(s) = (c_0(s), c_1(s), \ldots, c_k(s))$ actually lies in $C^{q+1}(B; \pi_q)$.

Of course, there is a relative version of this obstruction theory; that is, if $D \subset B$ is a $\mathfrak{B}_k$-subspace, then for each $r \leq k$ we can define a cochain complex $C^*(B, D; \pi_q; r)$. These obstructions satisfy all the usual properties. For example, every theorem in sections 31 through 37 of [S] can be translated into this theory. In particular, if s and s' are sections in $\Gamma(\zeta|_{|B|^q}, r)$ and if $h \in \Gamma(\zeta \times I|_{|B|^{q-1} \times I}, r)$ is a homotopy

$$h: s|_{|B|^{q-1}} \sim s'|_{|B|^{q-1}},$$

then we can define a <u>difference cocycle</u> in $C^q(B; \pi_q; r)$,

$$d(s,h,s') = (d_0(s,h,s'), \ldots, d_r(s,h,s')) .$$

The following proposition lists the two most basic properties of these obstruction theories. Its proof is almost word for word the same as the proofs in [S] of the analogous results for ordinary obstruction theory.

10.15 <u>Proposition:</u>

(1) Let $s \in \Gamma(\zeta|_{|B|^{q-1}}, r)$ and suppose that s is extendable to $s' \in \Gamma(\zeta|_{|B|^q}, r)$. The set $\{c(s')\}$ of $q+1$ dimensional obstruction cocycles of all such extensions s' forms a single cohomology class, which we also denote by $c(s) \in \mathcal{H}^{q+1}(B; \pi_q; r)$. s extends to the $q+1$ skeleton (that is, to an element of $\Gamma(\zeta|_{|B|^{q+1}}, r)$) if and only if $c(s) = 0$.

(2) Let s and s' be elements of $\Gamma(\zeta|_{|B|^q}, r)$ and let

$$h: s|_{|B|^{q-2}} \sim s'|_{|B|^{q-2}}$$

be a homotopy which is extendable to $h': s|_{|B|^{q-1}} \sim s'|_{|B|^{q-1}}$. The set $\{d(s,h',s')\}$ of difference cocycles of all such extensions h' forms a single cohomology class $d(s,h,s') \in \mathcal{H}^q(B; \pi_q; r)$. h is extendable to a homotopy

$$s|_{|B|^q} \sim s'|_{|B|^q}$$

if and only if $d(s,h,s') = 0$.

This obstruction theory would be useless if we could not compute $\mathcal{H}^*(B; \pi_q)$. As the next proposition shows, there is a practical inductive method for computing $\mathcal{H}^*(B, \pi_q)$ from ordinary cohomology.

10.16 Proposition: There is an exact sequence

$$\longrightarrow H^{i-1}(\partial B^r; \pi_q(G_r \times Z_{k-r})) \longrightarrow \mathcal{H}^i(B; \pi_q; r)$$

$$\longrightarrow \mathcal{H}^i(B; \pi_q, r-1) \oplus H^i(B^r; \pi_q(G_r \times Z_{k-r})) \to H^i(\partial B^r; \pi_q(G_r \times Z_{k-r})) \longrightarrow \cdots$$

Proof: There is a short exact sequence of cochain complexes

$$0 \longrightarrow C^*(B; \pi_q; r) \longrightarrow C^*(B; \pi_q; r-1) \oplus C^*(B^r; \pi_q(G_r \times Z_{k-r})$$
$$\longrightarrow C^*(\partial B^r; \pi_q(G_r \times Z_{k-r}) \longrightarrow 0 .$$

The last map is given by $\Phi^r_* - J^*_r$, where $J_r: \partial B^r \longrightarrow B^r$ is the inclusion and $\Phi^r_* = \bigoplus_i \Phi^{i, r-i}_*$. $\square$

Now suppose that the following two conditions are satisfied

(a) $\mathcal{H}^i(B; \pi_i) = 0$ for all $i \neq q$, and

(b) $\mathcal{H}^{i-1}(B, \pi_i) = 0$ for all $i < q$.

Let $s, s' \in \Gamma(\zeta)$. By (a) there is a homotopy

$$h: s\big|_{|B|^{q-2}} \sim s'\big|_{|B|^{q-2}}$$

which can be extended to $|B|^{q-1}$. By 10.15 there is a difference class

$d(s,k,s') \in \mathcal{H}^q(B;\pi_q)$ and by (b) this class is independent of the choice of h. It makes sense to call this the primary difference between s and s' and write it as $d(s,s')$. Notice that from (a), $\mathcal{H}^0(B,\pi_0) = 0$, and so $\pi_0(G_i \times Z_{k-i}) = 0$. Hence, these conditions will only apply in the unitary and symplectic cases and the coefficients will automatically be untwisted.

10.17 Proposition: Let ζ be an automorphic normal system for a $\mathcal{S}_{n,k}(B)$-manifold and suppose that B satisfies conditions (a) and (b) above. Let $e \in \Gamma(\zeta)$ be the identity section. If $s \in \Gamma(\zeta)$ denotes an arbitrary section, then the map defined by $s \longrightarrow d(s,e)$ induces a group isomorphism

$$\pi_0\,\Gamma(\zeta) \cong \mathcal{H}^q(B;\pi_q).$$

Proof: The above is an analog of a well-known result in ordinary obstruction theory. We can translate the proof of this result in [S, p. 186] to show that the map is a bijection. Since multiplication in $\pi_q(G_i \times Z_{k-i})$ is induced by pointwise multiplication in $G_i \times Z_{k-i}$ and since the group structure of $\Gamma(\zeta^i)$ is induced by pointwise multiplication in each fibre, it follows that on the cochain level

$$d(s \circ s', e) = d(s,e) + d(s',e).$$

Hence, $\pi_0\,\Gamma(\zeta) \longrightarrow \mathcal{H}^q(B;\pi_q)$ is a group homomorphism.

□

As an example of how the above techniques can be applied, we consider the problem of computing $\pi_0 \mathrm{Diff}_1^{G_n}(S^{2dn-1})$, where S^{2dn-1} is the unit sphere in $M^d(n,2)$ and $G_n = U(n)$ or $Sp(n)$ (when $G_n = O(n)$, we have already solved this problem in Proposition 10.9).

10.18 Proposition: For any $n \geq 2$,

$$\pi_0 \mathrm{Diff}_1^{U(n)}(S^{4n-1}) \cong \mathbb{Z},$$

where S^{4n-1} denotes the unit sphere in the linear action $2\rho_n$.

Proof: The orbit space of S^{4n-1} is $W_+^2(2) \cong D^3$. Let

$$U(1) \times U(1) \longrightarrow \mu_2^1 \longrightarrow S^2 \quad \text{and}$$

$$U(2) \longrightarrow \mu_2^2 \longrightarrow D^3$$

denote the bundles of the standard $\mathscr{B}_{n,2}$-normal system for S^{4n-1} (as defined in section 6). By definition, $\zeta^1 = \mu_2^{1*}$ and $\zeta^2 = \mu_2^{2*}$ are the bundles of an automorphic normal system ζ for S^{4n-1}. In this case the exact sequence of 10.16 becomes

$$H^{i-1}(S^2; \pi_q(U(2))) \longrightarrow \mathscr{H}^i(D^3; \pi_q)$$
$$\longrightarrow H^i(D^3; \pi_q(U(2))) \oplus H^i(S^2; \pi_q(U(1) \times U(1))) \longrightarrow \dots .$$

It follows that

$$\mathscr{H}^i(D^3; \pi_i) = 0 \qquad \text{for all } i \neq 3$$

$$\mathscr{H}^{i-1}(D^3; \pi_i) = 0 \qquad \text{for all } i, \text{ and that}$$

$$\mathscr{H}^3(D^3, \pi_3) \cong H^2(S^2, \pi_3(U(2)))$$
$$\cong \pi_3(U(2)) \cong \mathbb{Z}.$$

The result follows from the previous proposition.

$\square$

Of course, the above result could also be easily deduced from the exact sequence of the fibration $\Gamma(\zeta^2)_\partial \longrightarrow \Gamma(\zeta) \longrightarrow \Gamma(\zeta^1)$. The symplectic case is somewhat more interesting.

10.19 Proposition: For any $n \geq 2$,

$$\pi_0 \operatorname{Diff}_1^{Sp(n)} (S^{8n-1}) \cong \mathbb{Z}_2 ,$$

where S^{8n-1} denotes the unit sphere in the linear action $2\rho_n$.

Proof: The automorphic normal system ζ has two bundles

$$Sp(1) \times Sp(1) \longrightarrow \zeta^1 \longrightarrow S^4$$
$$Sp(2) \longrightarrow \zeta^2 \longrightarrow D^5 \cong W_+^4(2).$$

First we compute $\mathcal{H}^i(D^5, \pi_i)$. From the exact sequence of 10.16, it follows easily that $\mathcal{H}^i(D^5; \pi_i) = 0$ for $i < 4$. At $i=4$ the sequence becomes

$$0 \longrightarrow \mathcal{H}^4(D^5, \pi_4) \longrightarrow H^4(D^5; \pi_4 (Sp(2))) \oplus H^4(S^4; \pi_4(Sp(1) \times Sp(1)))$$

$$\xrightarrow{J^* - \Phi_*} H^4(S^4; \pi_4(Sp(2))) ,$$

that is,

$$0 \longrightarrow \mathcal{H}^4(D^5; \pi_4) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 .$$

Since the coefficients are untwisted, the coefficient homomorphism is induced by $j_*: \pi_4(Sp(1) \times Sp(1)) \longrightarrow \pi_4(Sp(1))$, where $j: Sp(1) \times Sp(1) \subset Sp(2)$ is the standard inclusion. Since j_* is onto, it follows that $\mathcal{H}^4(D^5; \pi_4) \cong \ker \Phi_* \cong \pi_5(Sp(2), Sp(1) \times Sp(1)) \cong \mathbb{Z}_2$. To compute $\mathcal{H}^5(D^5; \pi_5)$ we again consider the exact sequence

$$H^4(S^4;\pi_4(Sp(1)\times Sp(1))) \longrightarrow H^4(S^4;\pi_5(Sp(2))) \longrightarrow \mathcal{H}^5(D^5;\pi_5) \longrightarrow 0\,.$$

Since $j_*\colon \pi_5(Sp(1)\times Sp(1)) \longrightarrow \pi_5(Sp(2))$ is onto, we see that $\mathcal{H}^5(D^5;\pi_5) = 0$, that is,

$$\mathcal{H}^i(D^5;\pi_i) = \begin{cases} \mathbb{Z}_2, & i=4 \\ 0 & \text{otherwise}\,. \end{cases}$$

For $i < 5$, the groups $\mathcal{H}^{i-1}(D^5, \pi_i)$ are clearly zero. Thus, the result follows from Proposition 10.17, as before.

□

These two propositions show that unlike the orthogonal case, when $G_n = U(n)$ or $Sp(n)$, $\mathrm{Diff}_1^{G_n}(S^{2dn-1})$ contains elements which are not equivariantly isotopic to a linear automorphism (by an isotopy which covers the identity).

11. Equivariant Attaching

In this section we study the problem of classifying $\mathcal{S}_{n,k}(B)$-manifolds from the slightly different perspective of equivariantly pasting together two $\mathcal{S}_{n,k}$-manifolds along a common boundary.

11.1 Definition: Let X be a $\mathcal{S}_{n,k}(B)$ manifold. An invariant submanifold $M \subset X$ is called an isovariant submanifold if M is also a $\mathcal{S}_{n,k}$-manifold. Similarly, if C is a $\mathcal{B}_k$-space and $C \subset B$ is such that the intersection $C^{(i)} \cap B^{(i)}$ is transverse, then C is a $\mathcal{B}_k$-subspace of B. Let D denote the image of $M_\#$ in B ($D \cong M_\#$). Notice that $D \subset B$ is an example of a $\mathcal{B}_k$- subspace. If $\mathcal{C}$ is a normal system $\mathrm{rel}\,\eta_B$ for X, the $\mathcal{C}|_D$, the restriction of $\mathcal{E}$ to D, is defined in obvious manner. Clearly, $\mathcal{E}|_D$ will be a normal system $\mathrm{rel}\,\eta_B|_D$ for M. (As usual we suppress explicit reference to the isomorphisms $A(\mathcal{E}) \cong X$ $A(\mathcal{E}|_D) \cong M$ and $A(\eta_B) \cong B$).

11.2 Theorem: For $n \geq k$, let $M \subset X$ be an isovariant submanifold of a $\mathcal{J}_{n,k}(B)$-manifold X and let D denote the image of $M_\#$ in B. Suppose that $\mathcal{E}$ is a $\mathcal{J}_{n,k}(B)$ normal system $\mathrm{rel}\,\eta_B$ for X. Restriction defines homomorphisms

$$r: \mathrm{Diff}_1^{G_n}(X) \longrightarrow \mathrm{Diff}_1^{G_n}(M), \quad \text{and}$$

$$\mathrm{Aut}\,(\mathcal{E}) \longrightarrow \mathrm{Aut}\,(\mathcal{E}|_D)$$

which are principal fibrations. The fibre of r is the subgroup $\mathrm{Diff}_1^{G_n}(X, M)$ consisting of those diffeomorphisms of X which are the identity on M.

Proof: Let ζ be the automorphic normal system associated to $\mathcal{E}$. Since $D^i \subset B^i$ is a cofibration, the restriction map $\Gamma(\zeta^i) \longrightarrow \Gamma(\zeta^i|_{D^i})$ is a fibration. Using the usual inductive procedure it follows that $\Gamma(\zeta) \longrightarrow \Gamma(\zeta|_D)$ is a fibration. The theorem now follows since $\mathrm{Aut}\,(\mathcal{E}) = \Gamma(\zeta)$ is a deformation retract of $\mathrm{Diff}_1^{G_n}(X)$.

□

Suppose that $X = M \cup N$ where M and N are codimension zero isovariant submanifolds with common boundary $\partial M = \partial N$. If $f: \partial M \longrightarrow \partial N$ is an equivariant diffeomorphism covering the identity on $\partial M_\# = \partial N_\#$, then we can form a new $\mathcal{J}_{n,k}(B)$-manifold $\hat{X} = M \cup_f N$.

The question arises: for a given M and N, how many different $\mathscr{S}_{n,k}(B)$-manifolds $\hat{X}$ are there? (The same question could be asked in the category $\mathscr{S}_{n,k}$.)

11.3 Definition: Let M and N be as above and let $U \subset B$ and $V \subset B$ denote respectively the images of $M_{\#}$ and $N_{\#}$ in B. If Y is a $\mathscr{S}_{n,k}(B)$ manifold, let $Y|_U$ mean the inverse image of U in Y. Let $C_{n,k}(B; M, N)$ denote the set of isomorphism classes of $\mathscr{S}_{n,k}(B)$-manifolds Y such that $Y|_U$ is $\mathscr{S}_{n,k}(U)$-isomorphic to M and $Y|_V$ is $\mathscr{S}_{n,k}(V)$ isomorphic to N.

The following theorem tells us how to compute $C_{n,k}(B; M, N)$ and therefore goes a long way towards answering our questions.

11.4 Theorem: With the above hypotheses, the diagram on the next page commutes and has the following properties:

(1) Both squares are pullback squares.

(2) All four maps in the top square are fibrations.

(3) The two diagonal sequences are exact.

Proof: Statements (1) and (3) are tautologies and (2) follows immediately from the previous theorem.

□

$$\begin{tikzcd}
 & & \mathrm{Diff}_1^{G_n}(X) \arrow[dl] \arrow[dr] & & \\
 & \mathrm{Diff}_1^{G_n}(M) \arrow[dr] & & \mathrm{Diff}_1^{G_n}(N) \arrow[dl] & \\
 & & \mathrm{Diff}_1^{G_n}(\partial M) \arrow[dl] \arrow[dr] & & \\
 & \mathrm{Diff}_1^{G_n}(N)\backslash \mathrm{Diff}_1^{G_n}(\partial M) \arrow[dl] \arrow[ddr] & & \mathrm{Diff}_1^{G_n}(\partial M)/\mathrm{Diff}_1^{G_n}(M) \arrow[ddl] \arrow[dr] & \\
0 & & & & 0 \\
 & & C_{n,k}(B;\, M, N) & &
\end{tikzcd}$$

11.5 <u>Remark</u>: Notice that the path components of $\mathrm{Diff}_1^{G_n}(M)$ map to path components of $\mathrm{Diff}_1^{G_n}(\partial M)$. Hence, the quotients in the lower square are discrete, i.e., $\mathrm{Diff}_1^{G_n}(\partial M)/\mathrm{Diff}_1^{G_n}(M) \cong \pi_0\,\mathrm{Diff}_1^{G_n}(\partial M)/\pi_0\,\mathrm{Diff}_1^{G_n}(M)$.

Let L and K denote, respectively, the images of $\pi_0\,\mathrm{Diff}_1^{G_n}(M)$ and of $\pi_0\,\mathrm{Diff}_1^{G_n}(N)$ in $\pi_0\,\mathrm{Diff}_1^{G_n}(\partial M)$. The theorem says that $C_{n,k}(B;\, M, N)$ is isomorphic to the double coset space

$$K\backslash\pi_0\,\mathrm{Diff}_1^{G_n}(\partial M)/L\,.$$

An element of such a double coset space will be called an <u>attaching invariant</u>.

We are particularly interested in applying Theorem 11.4 to the case where $N = N^p$ is a tubular neighborhood of the lowest stratum in X and where $M = \overline{X - N^p}$. Let $C^p \cong N^p_{\#}$ be a tubular neighborhood of B^p and let $U = \overline{B - C^p}$. If ζ is an automorphic normal system for X, then the diagram of 11.4 becomes

$$
\begin{tikzcd}
 & \Gamma(\zeta) \arrow[dl] \arrow[dr] & \\
\Gamma(\zeta|_U) \arrow[dr] & & \Gamma(\zeta^p) \arrow[dl] \\
 & \Gamma(\zeta|_{\partial C^p}) \arrow[dl] \arrow[dr] & \\
\Gamma(\zeta^p)\backslash\Gamma\zeta|_{\partial C^p} \arrow[dr] & & \Gamma(\zeta|_{\partial C^p})/\Gamma(\zeta|_U) \arrow[dl] \\
 & C_{n,k}(B; M, N^p) &
\end{tikzcd}
$$

Notice that the top square is the same as the pullback square in Proposition 10.11.

This diagram suggests the following inductive procedure for classifying $\mathscr{S}_{n,k}(B)$-manifolds. <u>Suppose</u> by induction that we have classified $\mathscr{S}_{n,k}$-manifolds over $U = \overline{B - C^p}$, i.e., that we have computed $C_{n,k}(U)$. Next, consider all those $M \in C_{n,k}(U)$ such that $M|_{\partial C^p}$ is equivariantly diffeomorphic over ∂C^p to ∂N^p, where N^p is some modeled G_n-disk bundle of type (k, p) over B^p. For fixed M and N^p we can then use the lower square of the diagram to calculate

$$C_{n,k}(B; M, N^p).$$

If we make this calculation for all possible M's and N^p's, we will have computed $C_{n,k}(B)$.

On the other hand, according to 8.6,

$$C_{n,k}(B) \cong [B; H_+(k) \times_{G_k} EG_k]/\mathfrak{A}'.$$

Hence, the set of attaching invariants $C_{n,k}(B; M, N^p)$ acts on a subset of the set of homotopy classes of maps $B \longrightarrow H_+(k) \times_{G_k} EG_k$. The description of this action gives new insight into the classification problem.

We shall explicitly carry out this description only in the case where there are only two orbit types, the k^{th} and $(k-1)^{th}$ strata, and where the principal orbit bundle is trivial. Thus, suppose that B has only two nonempty strata, B^k and B^{k-1}. Let X be a $\mathscr{S}_{n,k}(B)$-manifold with trivial principal orbit bundle and let N^{k-1} be a closed tubular neighborhood of X^{k-1}. Let $M = \overline{X - N^{k-1}}$. By Remark 11.5, $C_{n,k}(B; M, N^{k-1})$ is isomorphic to the double coset space

$$\pi_0\Gamma(\xi^{k-1\,*})\backslash\pi_0\Gamma(\partial\xi^{k\,*})/\pi_0\Gamma(\xi^{k\,*})$$

where $\xi = \{\xi^{k-1}, \xi^k; f^{k-1,1}; \xi^{k-1,1} \longrightarrow \partial\xi^k\}$ is a $\mathscr{S}_{n,k}$-normal system for X.

If $\theta \in \Gamma(\partial\xi^{k\,*}) \cong \mathrm{Diff}_1^{G_n}(\partial M)$, then we can form a new $\mathscr{S}_{n,k}(B)$ manifold

$$X(\theta) = M \bigcup_{A(\theta)} N^{k-1}.$$

The $\mathscr{S}_{n,k}(B)$-diffeomorphism class of $X(\theta)$ depends only on the class of θ in the double coset space. A normal system ξ_θ for $X(\theta)$ can be defined by

(i) $\xi_\theta^k = \xi^k$

(ii) $\xi_\theta^{k-1} = \xi^{k-1}$

(iii) $f_\theta^{k-1,1} = \theta \circ f^{k-1,1} : \xi_\theta^{k-1,1} \longrightarrow \partial\xi_\theta^k$.

Let $c: B \longrightarrow H_+(k)$ be a classifying map for X. We want to define $c(\theta): B \longrightarrow H_+(k)$, a classifying map for $X(\theta)$. We may assume that c is a normal system morphism, that is, $c \in \mathrm{Hom}(\eta_B, \eta_k)$ $(\pi_0 \mathrm{Hom}(\eta_B, \eta_k) \cong \pi_0 \mathrm{Hom}(B, H_+(k)))$. Since B has only two nonempty strata, $\mathrm{Hom}(\eta_B, \eta_k)$ is a subset of $\mathrm{Hom}(B^{k-1}, B_k^{k-1}) \times \mathrm{Hom}(B^k, B_k^k)$. Thus, we may write $c = (a_{k-1}, a_k)$ and $c(\theta) = (a_{k-1}(\theta), a_k(\theta))$. Since B_k^k is contractible the class of c, $[c] \in [B, H_+(k)]/\mathfrak{A}$, is determined by $[a_{k-1}] \in [B^{k-1}, B_k^{k-1}]/\mathfrak{A}$, where $\mathfrak{A} = \pi_0 \Gamma(\xi^{k*}) \cong [B^k, G_k]$ is the group of isotopy classes of automorphisms of ξ^k.

Consider the fibration

$$BG_{k-1} \times BG_1 \longrightarrow BG_k.$$

We may consider the fibre to be B_k^{k-1}, since this space deformation retracts to $G_k/G_{k-1} \times G_1$. Let

$$i: B_k^{k-1} \longrightarrow BG_{k-1} \times BG_1$$

denote the inclusion of the fibre. Since a_{k-1} and $a_{k-1}(\theta)$ must both pull back ξ_k^{k-1} to a bundle which is equivalent to ξ^{k-1}, we must have that

$$i \circ a_{k-1} \sim i \circ a_{k-1}(\theta).$$

Recall that in the proof of the Classification Theorem 8.1, a_{k-1} is defined by the diagram,

$$\begin{tikzcd}
\xi^{k-1} \arrow[r] \arrow[d] & \xi^{k-1,1} \arrow[r, "f^{k-1,1}"] & \partial\xi^{k} \arrow[r] & G_k \arrow[d] \\
B^{k-1} \arrow[rrr, dashed, "a_{k-1}"'] & & & G_k/G_{k-1} \times G_1
\end{tikzcd},$$

where $L: \xi^k \longrightarrow G_k$ is some trivialization of the principal orbit bundle. $a_{k-1}(\theta)$ can be defined by the same diagram by replacing $f^{k-1,1}$ by $f_\theta^{k-1,1} = \theta \circ f^{k-1,1}$. Conversely, if $a: B^{k-1} \longrightarrow G_k/G_{k-1} \times G_1 \subset B_k^{k-1}$ is any map such that $i \circ a \sim i \circ a_{k-1}$, then it is easy to see that there is some automorphism $\theta \in \Gamma(\partial\xi^{k*})$ such that $a = a_{k-1}(\theta)$. It follows from the definition of $a_{k-1}(\theta)$, that if $\theta' \in \Gamma(\partial\xi^{k*})$ is another automorphism such that $a_{k-1}(\theta) = a_{k-1}(\theta')$, then there is an automorphism $\alpha \in \Gamma(\xi^{k-1*})$ making the following diagram commute

$$\begin{tikzcd}
\xi^{k-1} \arrow[r] \arrow[d, dashed, "\alpha"'] & \partial\xi^{k} \arrow[dr, "\theta"] & \\
 & & \partial\xi^{k} \\
\xi^{k-1} \arrow[r] & \partial\xi^{k} \arrow[ur, "\theta'"'] &
\end{tikzcd}.$$

In other words θ and θ' are members of the same coset in

$$\Gamma(\xi^{k-1*}) \backslash \Gamma(\partial\xi^{k*}),$$

where the inclusion $\Gamma(\xi^{k-1\,*}) \longrightarrow \Gamma(\partial\xi^{k\,*})$ is induced by

$$\xi^{k-1} \cong \xi^{k-1,1} \xrightarrow{f^{k-1,1}} \partial\xi^{k}.$$

As remarked above $[B, H_+(k)] \cong [B^{k-1}, B_k^{k-1}]$. Composition with $i\colon B_k^{k-1} \longrightarrow BG_{k-1} \times BG_1$ induces a map $i_*\colon[B^{k-1}, B_k^{k-1}] \to [B^{k-1}, BG_{k-1} \times BG_1]$. Let $b = i \circ a_{k-1}\colon B^{k-1} \longrightarrow BG_{k-1} \times BG_1$ be the map classifying ξ^{k-1}. Clearly,

$$C_{n,k}(B; M, N^{k-1}) \cong i_*^{-1}(b)/\mathfrak{A},$$

where $i_*^{-1}(b) \subset [B^{k-1}, B_k^{k-1}]$ and where the action of $\mathfrak{A}$ is explained in Remark 8.5. On the other hand by the above remarks

$$i_*^{-1}(b) \cong \pi_0\Gamma(\xi^{k-1\,*}) \backslash \pi_0\Gamma(\partial\xi^{k\,*}).$$

This shows how attaching invariants are related to Theorem 8.1 in the case where there are only two orbit types. We state this relationship more precisely in the following

11.6 Proposition: With the above definitions and hypotheses

(1) $C_{n,k}(B; M, N^{k-1}) \cong i_*^{-1}(b)/\mathfrak{A}$

(2) The following diagram commutes

$$\begin{array}{ccccccc}
\pi_0\Gamma(\xi^{k-1\,*}) \longrightarrow & \pi_0\Gamma(\partial\xi^{k\,*}) & \longrightarrow & \pi_0\Gamma(\xi^{k-1\,*}) \backslash \pi_0\Gamma(\partial\xi^{k\,*}) & \longrightarrow & & \\
 & \downarrow \cong & & \downarrow \cong & & & \\
\longrightarrow & [B^{k-1}, G_k] & \longrightarrow & i_*^{-1}(b) & \longrightarrow & 0 &
\end{array}$$

Here the bottom row is part of the sequence of the fibration

$$B_k^{k-1} \xrightarrow{i} BG_{k-1} \times BG_1 \longrightarrow BG_k.$$

(3) If $C_{n,k}(B)$ is identified with $[B^{k-1}, B_k^{k-1}]/\mathfrak{A}$ and if $i_*^{-1}(b)/\mathfrak{A}$ is identified with the double coset space

$$\pi_0\Gamma(\xi^{k-1\,*}) \backslash \pi_0\Gamma(\partial\xi^{k*}) / \pi_0\Gamma(\xi^{k*})$$

via (2), then the natural inclusion $C_{n,k}(B; M, N^{k-1}) \subset C_{n,k}(B)$ corresponds to the inclusion

$$i_*^{-1}(b)/\mathfrak{A} \subset [B^{k-1}, B_k^{k-1}]/\mathfrak{A}.$$

The double coset space is sometimes easier to compute than $[B; H_+(k)]$. Thus, the above proposition proves to be useful in the computations of section 15. A similar interpretation of attaching invariants in terms of homotopy classes of maps $[B; H_+(k)]$ is also possible when there are more than two orbit types.

IV. ACTIONS ON HOMOLOGY SPHERES

12. Regularity Theorems

12.1 Definition: Let R be a ring. An R cohomology sphere is a closed m-manifold M such that

$$H^*(M; R) \cong H^*(S^m; R) .$$

Similarly, we can define an R cohomology projective space, an R cohomology Grassmannian, etc.

In this section we mention a result of the Hsiangs, which justifies our emphasis on the study of modeled actions. Essentially it says that under suitable hypotheses any G_n-action on a $\mathbb{Z}_2$ cohomology sphere is modeled on $k\rho_n$.

12.2 Theorem (the Hsiangs): Let M be a $\mathbb{Z}_2$ cohomology sphere and let $G^d(n) \times M \to M$ be a smooth action. Suppose that either

(a) the principal orbits are of type $G^d(n)/G^d(n-k)$ for some $k < n$, or

(b) the principal orbits are of type $G^d(n)$ and $\dim M \geq dn^2 - 1$.

Then the action is modeled on $k\rho_n$ for some k, with $n \geq k$. As to the orbit structure there are only two possibilities; either

(1) the smallest orbits are fixed points, in which case $\dim M = dkn + \ell$, where $\ell = \dim M^{G^d(n)}$, or

(2) the smallest orbits are spheres $G^d(n)/G^d(n-1)$, in which case

$\dim M = dkn-1$.

The proof is essentially that of Theorem 3.1 in [HH1] using the modifications mentioned in [HH2].

12.3 Remark: The Hsiangs have also observed [HH2, 3.8-3.12] that if a $\mathbb{Z}_2$ cohomology sphere M admits a smooth $G^d(n)$-action and if

$$\dim M \leq \begin{cases} \frac{1}{2}n(n-1) - [\frac{1}{2}n]; & d=1 \\ n^2 - 3[\frac{1}{2}n]; & d=2 \\ 2n^2 - 2n \quad ; & d=4 \end{cases}$$

then the action always satisfies hypothesis (a) of the above theorem. In other words,when the dimension of M is relatively small compared to n, any action of $G^d(n)$ on M is modeled on $k\rho_n$. For similar reasons, if the number of orbit types is less than $\frac{1}{2}n$, then any $G^d(n)$-action is modeled on $k\rho_n$.

13. The Cohomology of the Orbit Strata: the Unitary and Symplectic Cases

This section concerns modeled G_n-actions on integral cohomology spheres. The Classification Theorem is used to show that the local assumption that the action is modeled on $k\rho_n$, together with the global assumption about the cohomology, implies that the orbit space has the same cohomological properties as does the orbit space of the restriction of the linear action $k\rho_n$ + trivial to the unit sphere. Since the arguments in the orthogonal case are somewhat more complicated, we first deal with the case $G_n = U(n)$ or $Sp(n)$, and postpone the $O(n)$ case to the next section. Our main result is the following

13.1 Theorem: Let $G_n = U(n)$ or $Sp(n)$, let B be a $\mathfrak{B}_k$-space and let η_B be a normal system for B. Suppose that M is a $\mathscr{O}_{n,k}(B)$-manifold of dimension $dkn + \ell$, $n \geq k$, $\ell \geq -1$. Then M is an integral cohomology sphere if and only if the following three conditions are satisfied:

(i) The principal orbit bundle is trivial (and so by Theorem 8.1 M is classified by a map $f: B \longrightarrow H_+(k)$ which we may assume is a normal system morphism).

(ii) $(f|_{B^i})^*: H^*(B_k^i; \mathbb{Z}) \longrightarrow H^*(B^i; \mathbb{Z})$ is an isomorphism for each $i > 0$ (recall that $B_k^i \sim H_+(k)^{(i)} \sim G_k/G_i \times G_{k-i}$, where $\sim$ means homotopy equivalent).

(iii) B^0 is an integral cohomology sphere of dimension ℓ ($B^0 = \phi$, if $\ell = -1$).

Moreover M is a homotopy sphere if and only if, in addition, B is 1-connected.

13.2 Remark: Similar results are true for modeled actions on acyclic manifolds, on h-cobordisms, etc.

The most difficult part of the proof is showing that if M is a $\mathbb{Z}$ cohomology sphere then the principal orbit bundle is trivial. Notice that for the principal orbit bundle to be trivial it is necessary and sufficient that B be $\mathbb{Z}$-acyclic. Indeed, sufficiency follows from obstruction theory. Since M is a cohomology sphere, $\pi: M \longrightarrow B$ induces the zero map on reduced cohomology. Since $B^k \sim B$, $(\pi|_{B^k})^*: \overline{H}^*(B^k; \mathbb{Z}) \longrightarrow \overline{H}^*(M^{(k)}; \mathbb{Z})$ must also be zero. Hence, if $M^{(k)} \longrightarrow B^k$ is the trivial bundle, B must be acyclic. Now if n is much greater than k, each orbit G_n/G_{n-i} will be highly connected; so it will follow from the Vietoris-Beagle Mapping Theorem that B is acyclic. To attain the lower bound n=k, however, the cohomology of each stratum must be kept track of. Our method is ultimately based on the following simple application of P. A. Smith theory.

13.3 Lemma (the Hsiangs): Let $G_n \times M \longrightarrow M$ be an action modeled on $k\rho_n$ on a $\mathbb{Z}$ cohomology sphere M of dimension $dkn+\ell$. Then

$$H^*(M^{G_{n-i}}; R) \cong H^*(S^{dki+\ell}; R),$$

where $R = \mathbb{Z}$ if $G_n = U(n)$ or $Sp(n)$ or if $G_n = O(n)$ and n-i is even, and where $R = \mathbb{Z}_{(2)}$, the integers localized at 2, if $G_n = O(n)$ and n-i is odd.

Proof: The fixed point set of G_{n-i} is the same as the fixed point set of the maximal torus of G_{n-i}, except when $G_{n-i} = O(n-i)$ and n-i is odd. In the latter case we use the maximal $\mathbb{Z}_2$-torus of O(n-i). The result follows from Smith theory.

□

We now proceed with the proof of the "only if" part of Theorem 13.1. We first consider the case where the fixed point set is nonempty (case (1) of Theorem 12.1).

13.4 Proposition: Let $G_n = U(n)$ or $Sp(n)$ and let $G_n \times M \to M$ be an action modeled on $k\rho_n$ on a $\mathbb{Z}$-cohomology sphere M of dimension $dkn+\ell$, $\ell \geq 0$. Let N^0 be a closed tubular neighborhood of M^0 in M and let $Y = \overline{M - N^0}$. Let $\Sigma = \Sigma^d(n,k)$ be a fibre of the normal sphere bundle $\partial N^0 \to M^0$ ($\Sigma_\#$ consists of those matrices in $W_+^d(k)$ of rank $\leq \min(n,k)$). Let $J: \Sigma \to Y$ be the inclusion and let $j: \Sigma_\# \to Y_\#$ be the induced map of orbit spaces. Then for each $i > 0$,

$$(J|_{\Sigma^{(i)}})^*: H^*(Y^{(i)}; \mathbb{Z}) \to H^*(\Sigma^{(i)}; \mathbb{Z})$$

and

$$(j|_{W_k^i})^*: H^*(Y_\#^{(i)}; \mathbb{Z}) \to H^*(W_k^i; \mathbb{Z})$$

are isomorphisms, where $W_k^i = \Sigma_\#^i$. (Notice that the hypothesis $n \geq k$ is not included.)

13.5 Remark: If $n \geq k$ and $B = M_\#$, then the above proposition says, in particular, that $H^*(B^k; \mathbb{Z}) \cong H^*(W_k^k; \mathbb{Z})$, i.e., B^k is acyclic (and hence the principal orbit bundle is trivial).

Proof: Let $\xi = \{{}^i \to B^i; f^{ij}: \xi^{ij} \to \partial_i \xi^{i+j}\}$ be a normal system for Y and let $\mu_k = \{\mu_k^i \to W_k^i; f_k^{ij}: \mu_k^{ij} \to \partial_i \mu_k^{i+j}\}$ be the standard normal system for Σ described in section 6.

By Lemma 13.1, M^0 is an integral cohomology ℓ-sphere. The mapping cone of $J: \Sigma \to Y$ is $Y \cup_J D(n,k)$, where $D(n,k)$ is the unit disk in $M(n,k)$. If $U \subset M^0$ is an ℓ-disk, then we can thicken the mapping cone of J to get a space

$$X = Y \cup D(n,k) \times U$$
$$= M - (N^0|_{M^0 - U})$$

which is an acyclic $\mathcal{S}_{n,k}$-manifold with boundary. The mapping cone of $J: \Sigma^{G_{n-i}} \longrightarrow Y^{G_{n-i}}$ is homotopy equivalent to $X^{G_{n-i}}$, which by Smith theory is $\mathbb{Z}$-acyclic. Hence

(1) $$J^*: H^*(Y^{G_{n-i}}) \cong H^*(\Sigma^{G_{n-i}})$$

is an isomorphism for each i (cohomology is taken with integer coefficients here and for the rest of the proof). When $i=1$, equation (1) says that the cohomology of $Y^{\langle 1 \rangle} = Y^{G_{n-1}}$ is isomorphic to that of the sphere $\Sigma^{G_{n-1}}$. Thus, $B^1 = Y^{\langle 1 \rangle}/G_1$ has cohomology isomorphic to the projective space W_k^1. Since $j: W_k^1 \longrightarrow B^1$ is covered by a bundle map, it pulls back the euler class of $Y^{\langle 1 \rangle} \longrightarrow B^1$ to the euler class of $\Sigma^{\langle 1 \rangle} \longrightarrow W_k^1$, i.e.,

$$j^*: H^d(B^1) \cong H^d(W_k^1).$$

Since the cohomology of both spaces is a truncated polynomial ring on a generator in dimension d (the euler class), it follows that

(2) $$j^*: H^*(B^1) \cong H^*(W_k^1).$$

Now suppose by induction that $j^*: H^*(B^i) \cong H^*(W_k^i)$ for all $i < s$. Let N^i and N_k^i be the equivariant normal bundles of Y^i and Σ^i, respectively. If $P \longrightarrow B^i$ and $P' \longrightarrow W_k^i$ are fibre bundles (with the same fibre) and if $P' \longrightarrow P$ is a bundle map covering $j: W_k^i \longrightarrow B^i$, then by the Comparison Theorem for spectral sequences $H^*(P) \cong H^*(P')$. In particular, this is true for the bundles $\{N^i\}^{G_{n-s}} \longrightarrow B^i$ and $\{N_k^i\}^{G_{n-s}} \longrightarrow W_k^i$. Therefore,

(3) $$J^*: H^*(\{N^i\}^{G_{n-s}}) \cong H^*(\{N_k^i\}^{G_{n-s}}).$$

From (1), (3), and the Five Lemma it follows that

$$J^*: H^*((Y, N^1)^{G_{n-s}}) \cong H^*((\Sigma, N_k^1)^{G_{n-s}})$$

where $(A, B)^{G_{n-s}} = (A^{G_{n-s}}, B^{G_{n-s}})$. By excision

$$J^*: H^*((Y - N, \partial N^1)^{G_{n-s}}) \cong H^*((\Sigma - N_k^1, \partial N_k^1)^{G_{n-s}}),$$

where $\partial N^1 = \bigcup_{i>0} \partial_i N^1$. By the Comparison Theorem for spectral sequences again

$$J^*(H^*(\{\partial N^1\}^{G_{n-s}}) \cong H^*(\{\partial N_k^1\}^{G_{n-s}}),$$

so by the Five Lemma again,

(4) $$J^*: H^*(\{Y - N^1\}^{G_{n-s}}) \cong H^*(\{\Sigma - N_k^1\}^{G_{n-s}}).$$

Nothing prevents us from continuing in this fashion. At the next stage we show that

$$J^*: H^*(\{Y - N^1 \cup N^2\}^{G_{n-s}}) \cong H^*(\{\Sigma - N_k^1 \cup N_k^2\}^{G_{n-s}})$$

and so forth. Eventually we conclude that

(5) $$J^*: H^*(Y^{\langle s \rangle}) \cong H^*(\Sigma^{\langle s \rangle})$$

where $Y^{\langle s \rangle} = Y^{G_{n-s}} - \{N^1 \cup \dots \cup N^{s-1}\}^{G_{n-s}}$ and $\Sigma^{\langle s \rangle} = \Sigma^{G_{n-s}} - \{N_k^1 \cup \dots \cup N_k^{s-1}\}^{G_{n-s}}$. Recall that $\Sigma^{\langle s \rangle} = \mu_k^s / G_{k-s} \cong W_s^s \times G_k / G_{k-s}$ which G_s-deformation retracts to the Stiefel manifold G_k/G_{k-s}. Also, W_k^s

deformation retracts to the Grassmannian $G_k/G_s \times G_{k-s}$. An easy spectral sequence argument shows that the cohomology of $Y^{\langle s \rangle}/G_s = B^s$ must be isomorphic to that of $G_k/G_s \times G_{k-s}$ and that the characteristic classes of $Y^{\langle s \rangle} \longrightarrow B^s$ generate $H^*(B^s)$ (either the Chern classes in the unitary case or the symplectic Pontrjagin classes and the Euler class in the symplectic case). Since $J: \Sigma^{\langle s \rangle} \longrightarrow Y^{\langle s \rangle}$ is a bundle map covering $j: W_k^s \longrightarrow B^s$, it follows that j pulls back these characteristic classes. Hence

$$j^*: H^*(B^s) \cong H^*(W_k^s) .$$

This completes the induction and the proof of the proposition.

$\square$

Next we turn our attention to the case where the fixed point set is empty. Two preliminary lemmas are needed. The first of these is easy and also well known (see [J], for example). Its proof is omitted.

13.6 <u>Lemma:</u> Let $\varphi: G_n \times X \longrightarrow X$ be an action modeled on $k\rho_n$. Let $G_i \times G_{n-i} \subset G_n$ be embedded in the standard fashion. Let $\psi: G_{n-1} \times X \longrightarrow X$ denote the restriction of φ to $\{1\} \times G_{n-i}$ and let $\hat{\psi}: G_{n-1} \times X^{G_i} \longrightarrow X^{G_i}$ be the restriction of ψ to the fixed point set of $G_i \times \{1\}$. Both ψ and $\hat{\psi}$ are modeled on $k\rho_{n-i}$. Let $B = X/\varphi$, $A = X/\psi$ and $E = X^{G_i}/\hat{\psi}$. Notice that $E \subset A$. There is a natural projection $p: A \longrightarrow B$. The restriction $p|_E: E \longrightarrow B$ is a strata-preserving embedding. In fact, for all $j \leq \min(k, n-i)$,

$$p|_{E^{(j)}}: E^{(j)} \longrightarrow B^{(j)}$$

is a diffeomorphism.

13,7 Lemma: Let V be the Stiefel manifold G_n/G_{n-i}, $1 \leq i < n$. If $\psi: G_{n-1} \times V \longrightarrow V$ denotes the restriction of the natural G_n-action, then the following three statements are true:

(1) ψ is modeled on $i\rho_{n-1}$.

(2) ψ has exactly two orbit types, G_{n-1} and G_{n-1}/G_{n-i}.

(3) V/ψ is diffeomorphic to D^{di}, the disk of dimension di.

Proof: The first two statements are obvious. V can be regarded as the space of orthonormal i-frames in n-space $(\mathbb{F}^d)^n$. Let $q: (\mathbb{F}^d)^n \to \mathbb{F}^d$ be projection on the first factor. If $(x_1, x_2, \ldots, x_i)$ is an arbitrary i-frame in V, then define $f: V \longrightarrow D^{di}$ by $f(x_1, \ldots, x_i) = (q(x_1), \ldots, q(X_i))$. f is invariant with respect to ψ and the induced map $f_{\#}: V/\psi \longrightarrow D^{di}$ is easily seen to be a diffeomorphism.

□

13.8 Proposition: Let $G_n = U(n)$ or $Sp(n)$. Let M^{dkn-1} be an integral cohomology sphere. Suppose that $\varphi: G_n \times M \longrightarrow M$ is an action modeled on $k\rho_n$ and let B denote the orbit space. Then

$$H^*(B^i; \mathbb{Z}) \cong H^*(W_k^i; \mathbb{Z})$$

for all i such that $1 \leq i \leq \min(k, n)$ ($B^i = \emptyset$ for $i = 0$ or $i > \min(k, n)$). In particular, if $n \geq k$ then B^k is $\mathbb{Z}$-acyclic (and hence B is acyclic).

13.9 Remark: We are mainly interested in the last sentence of this proposition. If we were willing to settle for the requirement $n \geq k+1$

(rather than $n \geq k$), then the fact that B is acyclic would follow from 13.4 and 13.7 for trivial reasons. Indeed, if $n \geq k+1$, then M/G_{n-1} is acyclic by 13.4. By Lemma 13.7, the inverse image of each point under the natural projection $p: M/G_{n-1} \longrightarrow B$ is a disk; hence, it follows from the Vietoris-Beagle Mapping Theorem that B is also acyclic. Basically the same idea can be used to prove the proposition in general; however, we must exercise considerably more care.

<u>Proof of Proposition 13.8</u>: Cohomology is taken with integer coefficients throughout. As in Lemma 13.6, let $\psi: G_{n-1} \times M \longrightarrow M$ be the restriction of φ to $G_{n-1} \subset G_n$ and let $\hat{\psi}: G_{n-1} \times M^{G_1} \longrightarrow M^{G_1}$ be the restriction of ψ to M^{G_1} (where $G_1 \times G_{n-1} \subset G_n$ is standardly embedded). Let

$$A = M/\psi \qquad \text{and}$$
$$E = M^{G_1}/\hat{\psi}.$$

Let $J: \Sigma \subset M - M^{G_{n-1}}$ be the inclusion a normal sphere to the fixed set of ψ ($\Sigma = \Sigma(n-1,k)$). J covers an inclusion $j: \Sigma_{\#} \longrightarrow A - A^0$. By Proposition 13.4,

$$j^*: H^*(A^i) \cong H^*(W_k^i) \tag{1}$$

for all i, $1 \leq i \leq \min(k, n-1)$. In a similar fashion we may consider the inclusion $J': M^{G_1} \subset M - M^{G_{n-1}}$, covering $j': E \longrightarrow A - A^0$. Since M^{G_1} is a cohomology sphere and since J' induces an isomorphism on cohomology, the proof of 13.4 also shows that

(2) $\quad j'^{*}: H^{*}(A^{i}) \cong H^{*}(E^{i})$

for all i, $1 \leq i \leq \min(k, n-1)$. Let $p: A \longrightarrow B$ be the natural projection and let $q = p|_{E}: E \longrightarrow B$. According to Lemma 13.6, $q: E^{i} \longrightarrow B^{i}$ is a diffeomorphism for all $i \leq \min(k, n-1)$. Hence, using (1), (2) and the fact that $j^{*} \circ p^{*} = j^{*} \circ (j'^{*})^{-1} \circ q$, we have that

(3) $\quad j^{*} \circ p^{*}: H^{*}(B^{i}) \cong H^{*}(W_{k}^{i})$

for all $i \leq \min(k, n-1)$. This proves the proposition except possibly on the top stratum of B (since the top stratum has index $\min(k,n)$). In particular, (3) proves the proposition whenever $n \geq k+1$.

Since $p^{-1}(B^{i}) \subset A^{i-1}$ we can decompose A^{i} into two pieces,

$$A_{-}^{i} = p^{-1}(B^{i+1}) \cap A^{i} \quad \text{and}$$

$$A_{+}^{i} = p^{-1}(B^{i}) \cap A^{i}.$$

It follows from 13.7 that $p: A_{+}^{i} \longrightarrow B^{i}$ is an open disk bundle with fibre the interior of D^{di} and that $p: A_{-}^{i-1} \longrightarrow B^{i}$ is the associated sphere bundle. Consider the principal G_{i}-bundles (the orbit bundles),

$$\lambda^{i}(\varphi) = M^{\langle i \rangle} \longrightarrow B^{i}$$

$$\lambda^{i}(\hat{\psi}) = \{M^{G_1}\}^{\langle G_{n-i-1} \rangle} \longrightarrow E^{i}$$

$$\lambda_{k}^{i} = \Sigma^{\langle i \rangle} \longrightarrow W_{k}^{i}.$$

Using 13.7 and the definition of A_{+}^{i}

$$(4) \qquad A_+^i = M^{(i)}/G_{n-1}$$

$$= \{G_n/G_{n-i} \times_{G_i} \lambda^i(\varphi)\}/G_{n-1}$$

$$= \text{int } D^{di} \times_{G_i} \lambda^i(\varphi) .$$

Hence,

$$(5) \qquad A_-^{i-1} = S^{di-1} \times_{G_i} \lambda^i(\varphi)$$

$$= G_i/G_{i-1} \times_{G_i} \lambda^i(\varphi)$$

$$= \lambda^i(\varphi)/G_{i-1} .$$

Now suppose that $n < k+1$, so that the top stratum of B is B^n, while the top stratum of A is A^{n-1} (also the top stratum of E is E^{n-1}). Let

$$D^{d(k-n+1)} \longrightarrow N \longrightarrow A_+^{n-1}$$

be the normal disk bundle of A_+^{n-1} in A^{n-1}. Since $A_+^{n-1} \sim B^{n-1}$, it follows from (3) that

$$0 = \overline{H}^* (A^{n-1}, A_+^{n-1})$$

$$= \overline{H}^* (A^{n-1}, N)$$

$$= H^*(A^{n-1} - N, \partial N) .$$

Hence,

$$(6) \qquad H^*(A_-^{n-1}) = H^*(A^{n-1} - N)$$

$$\cong H^*(\partial N) .$$

Let $\alpha \in H^*(B^i)$ be a characteristic class of $\lambda^i(\varphi)$ (in the U(n) case let α be the total Chern class and in the Sp(n) case let α be either the total symplectic Pontrjagin class (see [BH]), the total Stiefel-Whitney class or the Euler class). Notice that for $i \leq n-1$, $\lambda^i(\hat{\psi}) = \{M^{G_1}\}^{\langle G_{n-i-1}\rangle} = M^{\langle G_{n-i-1}\rangle} = \lambda^i(\varphi)$, i.e., $\lambda^i(\hat{\psi})$ and $\lambda^i(\varphi)$ are essentially the same bundle. Since $J^* \circ (J'^*)^{-1} : H^*(\lambda^i(\hat{\psi})) \cong H^*(\lambda^i_k)$ is an isomorphism covering $j^* \circ (j'^*)^{-1} : H^*(E^i) \cong H^*(W^i_k)$, it follows that for $i \leq n-1$, $j^* \circ p^*(\alpha)$ is the corresponding class of $\lambda^i_k \longrightarrow W^i_k$. In particular this means that the characteristic classes of the disk bundle $A^{n-1}_+ = D^{di} \times_{G_i} \lambda^i(\varphi)$ are just the pullbacks of the corresponding characteristic classes of λ^{n-1}_k.

Next let $D^{dk} \longrightarrow Q \longrightarrow E^{n-1}$ be the normal disk bundle of $j' : E^{n-1} \subset A^{n-1}$. Since both M^{G_1} and M are integral cohomology spheres they are π-manifolds; hence the normal bundle of M^{G_1} in M is stably trivial. From this it follows that the union of the principal orbits of $\hat{\psi}$, $\{M^{G_1}\}^{(n-1)}$ has trivial normal bundle in the union of principal orbits of ψ, and hence that $Q \longrightarrow E^{n-1}$ is a trivial disk bundle. Also notice that

$$Q \cong A^{n-1}_+ \oplus N\big|_{E^{n-1}} \tag{7}$$

where $\oplus$ means Whitney sum of two disk bundles over E^{n-1} (the inclusinn $j' : E^{n-1} \longrightarrow A^{n-1}_+$ may be regarded as the zero section of $A^{n-1}_+ \longrightarrow B^{n-1}$). Since we know the characteristic classes of $A^{n-1}_+ \longrightarrow B^{n-1}$, it follows from the Whitney product formula that we know the characteristic classes of $N\big|_{E^{n-1}}$. Since $j' : E^{n-1} \longrightarrow A^{n-1}_+$ is a homotopy equivalence, we therefore know the characteristic classes of $N \longrightarrow A^{n-1}_+$. In particular its Euler class must be a generator of

$$H^{d(k-n+1)}(A_+^{n-1}) \cong H^{d(k-n+1)}(B^{n-1})$$
$$\cong H^{d(k-n+1)}(W_k^{n-1})$$
$$\cong H^{d(k-n+1)}(G_k/G_{n-1}\times G_{k-n+1})$$
$$\cong \mathbb{Z}.$$

Using this, it follows from the Gysin sequence that,

(8) $\quad H^*(\partial N) \cong H^*(G_k/G_{n-1}\times G_{k-n})$, and so from (6),

(9) $\quad H^*(A_-^{n-1}) \cong H^*(G_k/G_{n-1}\times G_{k-n})$.

If $n=k$, this last equation says that $H^*(A_-^{k-1}) \cong H^*(G_k/G_{k-1}) = H^*(S^{dk-1})$. Since $A_-^{k-1} \longrightarrow B^k$ is a S^{dk-1}-bundle it follows immediately that B^k is acyclic. This completes the proof in this case.

The general case $k < n$ is only a little trickier. First notice that $\lambda^n(\varphi) = \lambda^n(\psi)\big|_{A_-^{n-1}}$, so using (9),

$$H^*(\lambda^n(\varphi)) \cong H^*(\lambda^{n-1}(\psi)\big|_{A_-^{n-1}})$$
$$\cong H^*(G_k/G_{k-n}).$$

A standard argument with spectral sequences now shows that

$H^*(B^n) \cong H^*(G_k/G_n\times G_{k-n}) \cong H^*(W_k^n)$, which completes the proof. $\square$

We can now prove the main result of this section.

<u>Proof of Theorem 13.1:</u> Suppose that the $\mathscr{A}_{n,k}(B)$-manifold M is a $\mathbb{Z}$

cohomology sphere of dimension $dkn+\ell$ ($n \geq k$, $d=2$ or 4). By Propositions 13.4 and 13.8, B^k is $\mathbb{Z}$-acyclic, and so the principal orbit bundle is trivial. Hence, by the Classification Theorem 8.1, there is a $\mathcal{B}_k$-morphism $f: B \longrightarrow H_+(k)$ such that

$$M \cong f^*(M(n,k)).$$

If $\ell \geq 0$, then $M^0 \cong B^0$ is an integral cohomology ℓ-sphere. As in 13.4 let $J: \Sigma \subset M$ be the inclusion of a normal sphere to M^0 and let $j: W_+(k) \subset B$ be the induced map of orbit spaces. Recall that B_k^i, the i^{th} stratum of $H_+(k)$, deformation retracts to W_k^i. The composition

$$(f \circ j|_{W_k^i})^*: H^*(W_k^i) \longrightarrow H^*(W_k^i)$$

is clearly the identity; while according to 13.4,

$$(j|_{W_k^i})^*: H^*(B^i) \longrightarrow H^*(W_k^i)$$

is an isomorphism for all $i > 0$. Thus,

$$(f|_{B^i})^*: H^*(B_k^i) \longrightarrow H^*(B^i)$$

is an isomorphism for $i > 0$.

The argument for $\ell = -1$ is not so easy since the isomorphism of 13.8, $H^*(B^i) \cong H^*(W_k^i)$, is not induced by a map such as j^*. However, since $B^0 = \emptyset$ we may assume that the image of $f: B \longrightarrow H_+(k)$ actually lies in $W_+(k)$. A routine argument shows that $\tilde{f}: M = f^*(\Sigma(n,k)) \longrightarrow \Sigma(n,k)$ is degree one and hence that

$$f^*: H^*(\Sigma(n,k)) \cong H^*(M).$$

(Indeed, since $f|_{B^1}$ pulls back the Euler class of μ_k^1/G_{k-1} to the Euler class of ξ^i/G_{k-1}, it follows from the proof of 13.8 that $f|_{B^1}$ induces a cohomology isomorphism. Hence, $\tilde{f}|_{N^1}: N^1 \longrightarrow N_k^1$ is degree one, and therefore, so is $\tilde{f}: M \longrightarrow \Sigma(n,k)$.) Using the above isomorphism and Smith theory it follows that

$$\tilde{f}^*: H^*(\Sigma^{G_{n-i}}) \cong H^*(M^{G_{n-i}}).$$

Hence, the situation is in complete analogy with that of Proposition 13.4. The inductive argument used there implies that

$$(f|_{B^i})^*: H^*(W_k^i) \cong H^*(B^i),$$

which is the desired result.

In the other direction we must show that if M is a $\mathscr{B}_{n,k}(B)$-manifold satisfying conditions (1), (2), and (3) of the theorem, then M is an integral cohomology sphere. Let ξ_k be the standard normal system for $M(n,k)$ and let $\xi = f^*\xi_k = \{\xi^i \longrightarrow B^i; f^{ij}: \xi^{ij} \longrightarrow \partial_i \xi^{i+j}\}$ be the corresponding normal system for M. It follows from condition (2) and the Comparison Theorem for spectral sequences that if $P^i \longrightarrow B^i$ is any bundle associated to ξ^i and if $P_k^i \longrightarrow B_k^i$ is the corresponding bundle associated to ξ_k^i, then

$$\tilde{f}^*: H^*(P_k^i) \cong H^*(P^i).$$

In particular this isomorphism holds when $P^i = M^i$, N^i or $\partial_j N^i$. From this and the fact that $N^0 \cong D(n,k) \times B^0$ a simple argument with exact sequences shows that M is indeed a $\mathbb{Z}$ cohomology sphere.

If M is simply connected, then so is B (this follows, for example, from II.6.3 in [B1, p.91]). Conversely, if B is simply connected, then so is B^k. Hence $M^{(k)}$ is simply connected. Since the lower strata are submanifolds of M of codimension greater than 2, M must also be simply connected. This completes the proof.

$\square$

14. The Cohomology of the Orbit Strata: the Orthogonal Case

In trying to prove a result analogous to Theorem 13.1 for O(n)-actions, the methods of the previous section break down. This is because half of the time P. A. Smith theory yields information only about the $\mathbb{Z}_{(2)}$ cohomology of $M^{O(n-i)}$ (see Lemma 13.3). Thus, we lose control over the odd torsion of the integral cohomology of the orbit strata. Further complications develop since the various bundles over the B^i's (the orbit strata) will not always be orientable. It therefore becomes necessary to keep track of cohomology with twisted as well as untwisted coefficients. Thus for arbitrary n and k, the problem of stating conditions which are at the same time both necessary and sufficient for a $\mathcal{S}^1_{n,k}(B)$-manifold to be a $\mathbb{Z}$ cohomology sphere, seems to be very complicated. Partial results are available; for example, the following theorem provides sufficient conditions. We omit its proof since it is essentially identical with the "only if" part of the proof of 13.1.

14.1 Theorem: Suppose that $O(n) \times M \longrightarrow M$ is a $\mathcal{S}^1_{n,k}(B)$ manifold of dimension $kn+\ell$, which is classified by $f: B \longrightarrow H_+(k)$, where $n \geq k$, $\ell \geq -1$. If the following conditions are satisfied, then M is an integral cohomology sphere.

(1) For $i > 0$,

$$(f|_{B^i})^*: H^*(B_k^i; \mathbb{Z}) \cong H^*(B^i, \mathbb{Z}).$$

(2) By (1), $H^1(B^i, \mathbb{Z}_2) = \mathbb{Z}_2$ for $0 < i < k$ so there is a unique nontrivial local system of integer coefficients for B^i. If this system of twisted coefficients is denoted by $\mathbb{Z}^t$, then for $0 < i < k$

$$(f|_{B^i})^*: H^*(B_k^i; \mathbb{Z}^t) \cong H^*(B^i; \mathbb{Z}^t).$$

(3) B^0 is an integral cohomology ℓ-sphere. ($B^0 = \emptyset$, if $\ell = -1$). Moreover, if, in addition, B is simply connected, then M is a homotopy sphere.

As for necessary conditions, we have the following

14.2 <u>Theorem</u>: Suppose that $O(n) \times M \longrightarrow M$ is a $\mathcal{S}^1_{n,k}(B)$-manifold of dimension $kn+\ell$ and either

(a) $\ell \geq 0$ and $n \geq k$, or

(b) $\ell = -1$ and $n > k$.

If M is an integral cohomology sphere, then the following conditions are satisfied.

(1) B^k is $\mathbb{Z}$-acyclic. Consequently, the principal orbit bundle is trivial and M is classified by a map $f: B \longrightarrow H_+(k)$.

(2) $(f|_{B^i})^*: H^*(B_k^i; \mathbb{Z}_2) \cong H^*(B^i; \mathbb{Z}_2)$ for all $i > 0$.

(3) Let $\mathbb{Z}_{(2)} = \mathbb{Z}[\frac{1}{3}, \frac{1}{5}, \ldots,]$ denote the ring of integers localized

at 2 with an involution defined by multiplication by -1. By (2), $H^1(B^1, \mathbb{Z}_2) = \mathbb{Z}_2$, so there is a unique nontrivial local $\mathbb{Z}_{(2)}$ coefficient system for B^i, which is denoted by $\mathbb{Z}^t_{(2)}$. With these definitions, then for $0 < i \leq k$

$$(f|_{B^i})^*: H^*(B^i_k; \mathbb{Z}_{(2)}) \cong H^*(B^i; \mathbb{Z}_{(2)}), \text{ and}$$

$$(f|_{B^i})^*: H^*(B^i_k; \mathbb{Z}^t_{(2)} \cong H^*(B^i; \mathbb{Z}^t_{(2)})$$

for $0 < i < k$.

(4) B^0 is a $\mathbb{Z}_2$-cohomology sphere of dimension ℓ. If n is even then B^0 is an integral cohomology sphere.

If M is, in fact, a homotopy sphere, then B is simply connected (and hence contractible).

Proof: First we consider case (1) where the fixed point set M^0 is non-empty. Condition (4) follows immediately from Lemma 13.3. As in Proposition 13.4, let $\Sigma = \Sigma(n,k)$ and let $J: \Sigma \longrightarrow M - M^0$ be the inclusion of a normal sphere to M^0. Let $j: W_+(k) \longrightarrow B$ be the induced map of orbit spaces. With only slight modifications, the proof of 13.4 can be used to show that

(i) $(J_{\Sigma^{\langle i \rangle}})^*: H^*(M^{\langle i \rangle}) \longrightarrow H^*(\Sigma^{\langle i \rangle})$ is an isomorphism

with $\mathbb{Z}_2$ or $\mathbb{Z}_{(2)}$ coefficients and is surjective with coefficients in $\mathbb{Z}$, $(i > 0)$, and

(ii) $(j|_{W^i_k})^*: H^*(B^i) \longrightarrow H^*(W^i_k)$ is an isomorphism with coefficients in $\mathbb{Z}_2$, $\mathbb{Z}_{(2)}$ or $\mathbb{Z}^t_{(2)}$ and is surjective with coefficients in $\mathbb{Z}$ or $\mathbb{Z}^t$.

As in the $U(n)$ and $Sp(n)$ cases the principal orbit bundle $\xi^k \longrightarrow B^k$ is trivial if and only if B^k is acyclic over the integers. Indeed, since M is an integral cohomology sphere, $(\pi|_{M^{(k)}})^*: \overline{H}^*(B^k; \mathbb{Z}) \longrightarrow \overline{H}^*(M^{(k)}; \mathbb{Z})$ is the zero map. Hence, the composition $\xi^k = M^{\langle k \rangle} \subset M^{(k)} \xrightarrow{\pi} B^k$ also induces the zero map on cohomology (since it factors through zero). Thus, in the Serre spectral sequence of $\xi^k \longrightarrow B^k$, $E_\infty^{p,0} = 0$ for $p \neq 0$. This says that everything in $\overline{H}^*(B^k; \mathbb{Z})$ must be in the image of some differential. Consequently B^k must be $\mathbb{Z}$-acyclic if the principal orbit bundle is trivial. In particular if the principal orbit bundle is nontrivial, the transgression must be a surjection. But according to (1)

$$(J|_{\Sigma\langle k \rangle})^*: H^*(\xi^k; \mathbb{Z}) \longrightarrow H^*(\xi_k^k; \mathbb{Z}) = H^*(\mathbb{O}(k); \mathbb{Z})$$

is a surjection, i.e., the inclusion of the fibre is a surjection on integral cohomology. This implies that the transgression is zero and hence that B^k is $\mathbb{Z}$-acyclic. This proves (1). Conditions (2) and (3) now follow immediately from (ii) as in the proof of Theorem 13.1.

Now consider case (b) where $M^0 = \phi$ $(\ell = -1)$. We proceed as in the proof of Proposition 13.8 by considering the restriction of the $O(n)$-action to $O(n-1) \times M \longrightarrow M$, which is an action modeled on $k\rho_{n-1}$ with nonempty fixed point set. If $n > k$, then the above proof shows that $M/O(n-1)$ is $\mathbb{Z}$-acyclic. By Lemma 13.7 the inverse image of each point under the map $M/O(n-1) \longrightarrow M/O(n) \cong B$ is a disk. It follows by the Vietoris-Beagle Mapping Theorem (as in 13.9), that B is also $\mathbb{Z}$-acyclic, which proves (1). Conditions (2) and (3) follow from an inductive argument as in the proof of

13.4 (see also the proof of 13.1).

The result of Bredon [B1; p. 91], mentioned previously, shows that the orbit space is simply connected whenever M is simply connected and the action has at least one orbit which is connected. This proves the last sentence of the theorem.

□

14.3 Remark: Undoubtedly, the above theorem is also true when $n=k$ and $\ell = -1$. The problem is in proving that the principal orbit bundle is trivial. The proof of 13.8 goes through to show that

$$H^*(B^i) \cong H^*(W_k^i)$$

with coefficients in $\mathbb{Z}_2$, $\mathbb{Z}_{(2)}$ or $\mathbb{Z}_{(2)}^t$. It is not obvious, however, why $H^*(B^k; \mathbb{Z})$ should have no odd torsion. Thus, we were forced to exclude this case in the hypothesis of 14.2.

There is still the gray area between the conditions in 14.1 and the conditions in 14.2 to be decided. Roughly, what happens is that $H^*(M^{O(n-i)}, \mathbb{Z})$ can have nonzero odd torsion if $(n-i)$ is odd. In general this torsion will show up in $H^*(M^{\langle i \rangle}; \mathbb{Z})$ and consequently in either $H^*(B^i; \mathbb{Z})$ or $H^*(B^i; \mathbb{Z}^t)$ depending on whether $O(i)$ acts trivially or nontrivially on it (the torsion in $H^*(M^{\langle i \rangle}; \mathbb{Z})$). Much work remains to be done on the difficult problems of where torsion can occur and of how torsion subgroups in the cohomology of the various strata effect one another.

When $k \leq 3$ the above mentioned difficulties are manageable and it is easy to give conditions which are both necessary and sufficient for M

to be an integral cohomology sphere. For $k=2$ these conditions are well known. For completeness we summarize them below without proof. First we consider the case where M^0 is empty.

14.4 Theorem (Bredon): Let $O(n) \times M^{2n-1} \longrightarrow M^{2n-1}$ be an action modeled on $2\rho_n$, $n \geq 2$, and let $B = M_{\#}$. M is an integral cohomology sphere if and only if the following two conditions are satisfied:

(1) B is diffeomorphic to D^2, the 2-disk. (This implies that the principal orbit bundle is trivial and that M is simply connected if $n > 2$).

(2) If $f: B \longrightarrow W_+(2) = D^2$ is a classifying map, then f has degree ± 1 if n is even; while f has odd degree if n is odd.

14.5 Remarks: Recall that $\mathcal{S}^1_{n,2}(D^2)$-manifolds are classified by $[D^2, D^2]/\mathfrak{A}$. Thus, the action is classified by the absolute value of the degree of $f: D^2 \longrightarrow D^2$. Let $M(r) = f^*\Sigma(n,2)$, where $r = (\deg f)$. Of course, the $M(r)$ are isomorphic to the Brieskorn-Hirzebruch examples. Hence, if n is even, $M(r)$ is a highly connected manifold with middle dimensional homology isomorphic to $\mathbb{Z}_r$. If n is odd, $M(r)$ is diffeomorphic to either the standard sphere, the Kervaire sphere, $S^n \times S^{n-1}$, the Stiefel manifold $O(n+1)/O(n-1)$, or $S^n \times S^{n-1}$ # Kervaire sphere, depending on the congruence class of r modulo 8.

When M^0 is nonempty, actions modeled on $2\rho_n$ are called "knot manifolds."

14.6 Theorem (the Hsiangs): Let $O(n) \times M^{2n+\ell} \longrightarrow M^{2n+\ell}$ be an action modeled on $2\rho_n$, $n \geq 2$, $\ell \geq 0$, and let $B = M_{\#}$. M is an integral cohomology sphere if and only if the following conditions are satisfied.

(1) B^2 is a $\mathbb{Z}$-acyclic $(3+\ell)$-manifold.

(2) If M is classified by $f: B \longrightarrow H_+(2)$, then

$$(f|_{B^1})^*: H^*(B_2^1) \cong H^*(S^1) \longrightarrow H^*(B^1)$$

is an isomorphism with integer coefficients for n even and with twisted integer coefficients for n odd.

(3) B^0 is a R-cohomology ℓ-sphere, where $R = \mathbb{Z}$ if n is even, and $R = \mathbb{Z}_{(2)}$ if n is odd.

Moreover, M is a homotopy sphere if and only if B is 1-connected.

14.7 Remarks: In fact, in the above case M is classified by its orbit space since for any B satisfying the above conditions, $[B, H_+(2)]$ has exactly one element. If B is 1-connected, then the diffeomorphism type of M is determined by the index or arf invariant of the knot $(\partial B, B^0)$ (see [B2]).

Finally, we consider $O(n)$-actions modeled on $3\rho_n$. We only deal with the case where the fixed point set is empty.

14.8 Theorem: Let $O(n) \times M^{3n-1} \longrightarrow M^{3n-1}$ be an action modeled on $3\rho_n$, $n \geq 3$, and let $B = M_{\#}$. M is an integral cohomology sphere if and only if the following conditions are satisfied.

(1) B^3 is a $\mathbb{Z}$-acyclic 5-manifold with boundary. Since this means that the principal orbit bundle is trivial, there is a classifying map $f: B \longrightarrow W_+(3)$.

(2) B^1 is diffeomorphic to $\mathbb{RP}^2 = O(3)/O(1) \times O(2)$ and $f|_{B^1}: B^1 \longrightarrow W_3^1 = \mathbb{RP}^2$ induces an isomorphism on π_1 and has twisted degree ± 1.

(3) $(f|_{B^2})^*: H^*(W_3^2) \longrightarrow H^*(B^2)$ is an isomorphism with integer coefficients. If n is even, it must be an isomorphism with twisted integer coefficients as well.

M is a homotopy sphere if and only if, in addition, B is 1-connected.

As we shall see in the next section, several of these conditions are superfluous.

Proof: First suppose that M is an integral cohomology sphere. Notice that by Lemma 13.1 $M^{O(n-1)}$ is a $\mathbb{Z}_{(2)}$-cohomology 2-sphere; hence, it is a 2-sphere. It follows that B^1 is diffeomorphic to $\mathbb{RP}^2$. If $n > 3$, then B^3 is $\mathbb{Z}$-acyclic by condition (1) of Theorem 14.2. If $n=3$, a different approach must be used to deduce this fact.

Let $\xi = \{\xi^i \longrightarrow B^i,\ f^{ij}: \xi^{ij} \longrightarrow \partial_i \xi^{i+j}\}$ be a normal system for M. By Remark 14.3, B^3 is $\mathbb{Z}_{(2)}$-acyclic, i.e., its integral cohomology has only odd torsion. Since $H^1(B^3, \mathbb{Z}_2) = 0$, the principal orbit bundle $O(3) \longrightarrow \xi^3 \longrightarrow B^3$ is orientable. The only obstruction to the triviality of ξ^3 lies in $H^4(B^3; \pi_3(O(3))) = H^4(B^3; \mathbb{Z})$, which is at most an odd torsion group. Since B^2 is a 4-manifold with boundary, $H^4(B^2;\mathbb{Z}) = H^4(\partial_2 B^3;\mathbb{Z}) = 0$. It follows from the naturality of obstructions that $\partial_2\xi^3 \longrightarrow \partial_2 B^3$ is the trivial bundle. Since $n=3$, $\partial_2\xi^3$ can be identified with the boundary of a tubular neighborhood of M^2 in M. Since M^2 is of codimension one in M, this tubular neighborhood is a line bundle. Since $\partial_2\xi^3 \cong \partial_2 B^3 \times O(3)$, which is disconnected, this line bundle must be trivial; hence,

$$\text{(1)} \qquad \begin{aligned} M^2 &\cong \partial_2 B^3 \times SO(3) \\ &\cong B^2 \times O(3)/O(1)\ . \end{aligned}$$

Let $h^*(X)$ denote the odd torsion part of $H^*(X,\mathbb{Z})$. Since $h^*(B^1) = h^*(\mathbb{R}P^2) = 0$, it follows that $h^4(B,B^1) = h^4(B)$. So by Poincaré duality for the triple (B, B^2, B^1) and the Universal Coefficient Theorem

$$\begin{aligned} h^2(B^3, \partial_2 B^3) &\cong h^2(B,B^2) \\ &\cong h^4(B) \\ &\cong h^4(B^3)\ . \end{aligned}$$

Thus,

$$\text{(2)} \qquad h^2(\xi^3, \partial_2\xi^3) \cong h^4(B^3) \oplus h^4(B^3)\ .$$

Consider the exact sequence of the pair $(M - M^1, M^2)$. By excision,

$$H^*(M - M^1, M^2; \mathbb{Z}) \cong H^*(\xi^3, \partial_2\xi^3; \mathbb{Z}).$$

Since $B^1 \cong \mathbb{R}P^2$, $M^1 \cong O(3)/O(2) \times_{O(1)} S^2$, and so the integral cohomology of $M - M^1$ has no odd torsion. By (1), $h^1(M^2) \cong h^1(B^2) = 0$. Therefore, in the sequence of the pair we have

$$0 \longrightarrow h^2(\xi^3, \partial_2 \xi^3) \longrightarrow h^2(M - M^1),$$

and so $h^2(\xi^3, \partial_2 \xi^3) = 0$. By (2), this implies that $h^4(B^3) = 0$. Hence ξ^3 is trivial and consequently B^3 is $\mathbb{Z}$-acyclic, which proves condition (1).

Let $f: B \longrightarrow W_+(3)$ be a classifying map. Let $\mu_3 = \{\mu_3^i \longrightarrow W_3^i ; \mu_3^{ij} \longrightarrow \partial_i \mu_3^{i+j}\}$ be the standard normal system for $\Sigma(n,3)$ (the action is the restriction of $3\rho_n$ to the unit sphere). We may assume that f is a morphism between the normal systems covered by ξ and by μ_3. According to Theorem 14.2,

$$(f|_{B^1})^*: H^*(W_3^1) \longrightarrow H^*(B^1)$$

is an isomorphism with $\mathbb{Z}_2$ coefficients (the proof of 14.2 goes through in the case $n=3$ since we have already shown B^3 to be $\mathbb{Z}$-acyclic).

Consider the normal cone bundle $C^1 \longrightarrow B^1$ of B^1 in B. From the definitions,

$$\begin{aligned} \partial_3 C^1 &= W_2^2 \times_{O(2)} \eta^1 \\ &= \partial_1 B^3, \end{aligned}$$

where $\eta^1 = \xi^1/O(1) \times Z(2)$, $Z(2) =$ center of $O(2)$, and where W_2^2 is just the 2-disk. Since B^3 is acyclic, ∂B^3 is an integral cohomology 4-sphere. By a theorem of Massey [M1] the normal bundle of any embedding $\mathbb{RP}^2 \subset S^4$ is unique and is determined by the fact that its twisted Euler class must be ± 2 generator of $H^2(\mathbb{RP}^2, \mathbb{Z}^t) = \mathbb{Z}$. The proof of Massey's theorem also works for integral cohomology 4-spheres. Since f is a map of normal systems, $f|_{\partial_3 C^1}: \partial_3 C^1 \longrightarrow \partial_3 C_3^1$ is a bundle map and so $f|_{B^1}$ pulls back $\mu_3^1/O(1) \times Z(2)$ to η^1. It follows that $f|_{B^1}: B^1 \longrightarrow W_3^1$ has twisted degree ± 1, which proves condition (2), (for further details of this argument see section 15).

Next we consider the second stratum B^2. Since ∂B is a $\mathbb{Z}$ cohomology sphere and since $B^{(2)} \cong \partial B - B^1$, we see from Alexander duality that B^2 has cohomology isomorphic to that of $W_3^2 \sim \mathbb{R}P^2$ both with $\mathbb{Z}_2$ and $\mathbb{Z}$ coefficients (untwisted). By condition (2) of 14.2,

$$(f|_{B^2})^*: H^*(W_3^2; \mathbb{Z}_2) \cong H^*(B^2; \mathbb{Z}_2).$$

Since $\overline{H}^*(B^2, \mathbb{Z})$ is 2-torsion, it follows that $(f|_{B^2})^*$ is also an isomorphism with untwisted integer coefficients.

It remains to be shown that if n is even, then $(f|_{B^2})^*$ is an isomorphism with $\mathbb{Z}^t$ coefficients, as well. If n is even, then by 13.3, $M^{O(n-2)}$ is a $\mathbb{Z}$-cohomology S^5. Hence $M^{O(n-2)}/SO(2)$ is a $\mathbb{Z}$-cohomology $\mathbb{C}P^2$. $O(1)$ operates on $M^{O(n-2)}/SO(2)$ with fixed point set B^1 and with orbit space ∂B. In other words, $\widetilde{B}^2 = M^{O(n-2)}/SO(2) - B^1$ is the double cover of B^2. By Alexander duality

$$H^i(\widetilde{B}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z}; & \text{if } i=0,2 \\ 0; & \text{otherwise} \end{cases} .$$

Since $H^2(B^2; \mathbb{Z}) = 0$, it follows that $O(1)$ operates nontrivially on $H^2(\widetilde{B}^2: \mathbb{Z})$. Thus, from the Gysin sequence,

$$H^{i-1}(B^2; \mathbb{Z}^t) \longrightarrow H^i(B^2; \mathbb{Z}) \longrightarrow H^i(\widetilde{B}^2; \mathbb{Z}) \longrightarrow ,$$

we can deduce that

$$H^i(B^2;\mathbb{Z}^t) \cong H^i(W_3^2, \mathbb{Z}^t) \cong \begin{cases} \mathbb{Z} \; ; & \text{if } i=1 \\ \mathbb{Z}_2 \; ; & \text{if } i=1 \\ 0 \; ; & \text{otherwise} \end{cases} .$$

Next we assert that the twisted euler class of the $O(2)$-bundle $\xi^2/O(1) = M^{\langle 2\rangle} \longrightarrow B^2$ is a generator of $H^2(B^2;\mathbb{Z}^t)$. To see this first notice that the ordinary euler class of the $SO(2)$-bundle $M^{O(n-2)} \longrightarrow M^{O(n-2)}/SO(2)$ is a generator of $H^2(M^{O(n-2)}/SO(2);\mathbb{Z}) \cong \mathbb{Z}$. The inclusion map induces $H^2(M^{O(n-2)}/SO(2);\mathbb{Z}) \longrightarrow H^2(\tilde{B}^2;\mathbb{Z})$, which is multiplication by 2 , so the euler class of the $SO(2)$-bundle $M^{\langle 2\rangle} \longrightarrow \tilde{B}^2$ is $\pm 2 \cdot$ generator of $H^2(\tilde{B}^2;\mathbb{Z})$. It follows from the ordinary Gysin sequence that $M^{\langle 2\rangle}$ has integral cohomology isomorphic to that of $O(3)/O(1)$ and then from a spectral sequence argument that the twisted Euler class of $M^{\langle 2\rangle} \longrightarrow B^2$ is a generator of $H^2(B^2;\mathbb{Z}^t)$. In particular, the twisted Euler class of $\mu_3^2/O(1) \longrightarrow W_3^2$ is also a generator of $H^2(W_3^2;\mathbb{Z}^t)$. Since f pulls back orbit bundles, $f|_{B^2}$ pulls back the twisted Euler class of $\mu_3^2/O(1)$ to the twisted Euler class of $\xi^2/O(1)$, i.e.,

$$(f|_{B^2})^*: H^*(W_3^2;\mathbb{Z}^t) \cong H^*(B^2;\mathbb{Z}^t),$$

which proves condition (3) and, therefore, the "only if" part of the theorem.

In the other direction, suppose that $O(n) \times M \longrightarrow M$ is an action modeled on $3\rho_n$, satisfying conditions (1), (2), and (3) of the theorem. If n is even, then the action satisfies the conditions of Theorem 14.1, so M is an integral cohomology sphere.

Now suppose that n is odd. Let $\tilde{f}: M \longrightarrow \Sigma(n,3)$ be the isovariant map covering the classifying map. Using conditions (1), (2), and the Comparison Theorem for spectral sequences, we find that $\tilde{f}$ induces the following isomorphisms of cohomology groups (with integer coefficients),

(3) $\tilde{f}^*: H^*(\Sigma^3) \cong H^*(M^3)$

(4) $\tilde{f}^*: H^*(\Sigma^3, \partial\Sigma^3) \cong H^*(M^3, \partial M^3)$

(5) $\tilde{f}^*: H^*(\Sigma^1) \cong H^*(M^1)$.

The fibre bundle $M^2 \longrightarrow B^2$ can be defined by

$$M^2 = O(n)/O(n-2) \times_{O(2)} \xi^2/O(1).$$

Since n is odd, $O(2)$ acts trivially on the $\mathbb{Z}$ cohomology of $O(n)/O(n-2)$. Since $f|_{B^2}: B^2 \longrightarrow W_3^2$ induces a cohomology isomorphism with integer coefficients, the Comparison Theorem shows that

(6) $\tilde{f}^*: H^*(\Sigma^2; \mathbb{Z}) \cong H^*(M^2; \mathbb{Z})$

is also an isomorphism. It follows from excision and (4) that

$$\tilde{f}^*: H^*(\Sigma - \Sigma^1, \Sigma^2) \cong H^*(M - M^1, M^2)$$

and so from (6)

$$\tilde{f}^*: H^*(\Sigma - \Sigma^1) \cong H^*(M - M^1)$$

(integer coefficients). Using excision and (5), $\tilde{f}^*: H^*(\Sigma; \mathbb{Z}) \cong H^*(M; \mathbb{Z})$. This completes the proof.

□

15. Actions Modeled on $3\rho_n$

Let M be an integral cohomology sphere of dimension $3dn-1$. In this section we consider actions $G^d(n) \times M \longrightarrow M$ which are modeled on $3\rho_n$, $n \geq 3$. In the two previous sections we showed that the orbit space $B = M_{\#}$ is an acyclic topological manifold of dimension $3d+2$ and that the lowest stratum, B^1, is a cohomology projective space of codimension $(d+1)$ in ∂B. Conversely, given a $\mathcal{B}^d_3$-space B with the above two properties, we should ask the following questions:

(A) Is B the orbit space of a $\mathcal{S}^d_{n,3}$-manifold?

(B) In the cases $d=2$ or 4, which $\mathbb{Z}$ cohomology projective planes can occur as the first stratum of such a B?

(C) How many $\mathcal{S}^d_{n,3}(B)$-manifolds are there?

(D) Which of these $\mathcal{S}^d_{n,3}(B)$-manifolds are homology spheres?

Using the fact that B is acyclic, it follows from the Classification Theorem 8.1 and from Remarks 8.4 and 8.5 that

$$C^d_{n,3}(B) \cong [B; W^d_+(3)],$$

where $C^d_{n,3}(B)$ means the set of isomorphism classes of $\mathcal{S}^d_{n-3}(B)$-manifolds. Therefore, we can reformulate questions (C) and (D) as:

(C′) Compute $[B; W^d_+(3)]$, and

(D′) For which $f \in [B; W^d_+(3)]$ is the pullback $f^*\Sigma(n,3)$ a $\mathbb{Z}$ cohomology sphere?

As for question (A), notice that B is diffeomorphic to the orbit space of some $\mathcal{S}^d_{n,3}$-manifold if and only if $[B; W^d_+(3)]$ is nonempty.

Our best result is in the $O(n)$ case, where we succeed in answering all these questions.

15.1 Theorem: Let B be a space modeled on $H^1_+(3)$ such that

(a) B^3 is a $\mathbb{Z}$ acyclic 5-manifold, and

(b) the lowest stratum, B^1, is diffeomorphic to $\mathbb{R}P^2$.

Then for any $n \geq 3$, the following two statements are true.

(1) $C^1_{n,3}(B) \cong \mathbb{Z}_2 \times T$, where T = torsion of $H^2(B^2; \mathbb{Z}^t)$.

(2) If n is odd, then any $\mathcal{B}^1_{n,3}(B)$-manifold M is an integral cohomology $(3n-1)$-sphere; while if n is even, such an M is a $\mathbb{Z}$ cohomology sphere if and only if $H^2(B^2; \mathbb{Z}^t)$ is torsion free.

Before proving this, we consider the corresponding problems in the $U(n)$ and $Sp(n)$ cases. First we answer question (B) in the unitary case.

15.2 Proposition: Let X be any integral cohomology $\mathbb{C}P^2$. Then there is a $\mathcal{B}^2_3$-space B such that

(a) B^3 is a contractible 8-manifold with boundary.

(b) B^1 is diffeomorphic to X.

Proof: By obstruction theory there is a map $g: X \longrightarrow \mathbb{C}P^2$ inducing an isomorphism on integral cohomology. The normal bundle of an orientable 4-manifold is determined by its Stiefel-Whitney classes and its first Pontrjagin class. It follows from Wu's formula that g pulls back the Stiefel-Whitney classes of $\mathbb{C}P^2$ to those of X and from the Index Theorem that it pulls back the Pontrjagin class. Hence, g can be covered by a map of normal bundles, i.e., g is a normal map.

According to Sullivan (see [W; p. 191]), there is a bijection $[\mathbb{C}P^2; G/O] \cong \mathbb{Z}$ given by the surgery obstruction (the index). Thus, g is normally cobordant to the identity. Let $f': V' \longrightarrow \mathbb{C}P^2 \times I$ be such a normal cobordism. We can do surgery on f' (in the interior of V') to a cohomology isomorphism $f: V \longrightarrow \mathbb{C}P^2 \times I$. Thus,

(1) $\quad \partial V = X \cup \mathbb{C}P^2$, and

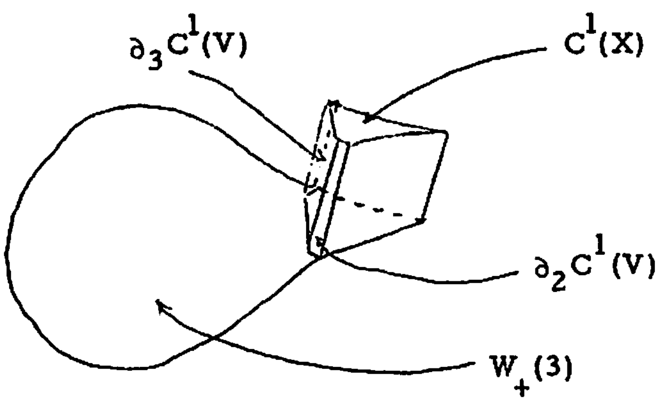
(2) $\quad f^*: H^*(\mathbb{C}P^2 \times I; \mathbb{Z}) \cong H^*(V; \mathbb{Z})$.

Let $W_3^1 \subset W_+^2(3)$ be the first stratum. Recall that $W_3^1 \cong \mathbb{C}P^2$. Let C_3^1 be the normal cone bundle of W_3^1. Let $C^1(V) = f^*(C_3^1 \times I)$. Thus, $C^1(V)$ is a $H_+^2(2)$-bundle over V and $C^1(V)|_{\mathbb{C}P^2} = C_3^1$. Consider the $\mathcal{B}_3^2$-cobordism

$$W_+(3) \times I \bigcup_{C_3^1 \times 1} C^1(V).$$

The "top" of this cobordism is a $\mathcal{B}_3^2$-space,

$D = (W_+(3) - C_3^1) \cup \partial_3 C^1(V) \cup \partial_2 C^1(V) \cup C^1(X)$, where $C^1(X) = C^1(V)|_X$.



By construction the first stratum of D is X. Moreover, since V is a cohomology h-cobordism, D is $\mathbb{Z}$-acyclic. We can do surgery on the interior of D^3 to obtain a new $\mathcal{B}_3^2$-space B such that B is contractible. (If X is 1-connected, then B^3 will actually be an 8-disk.)

□

Notice that the space D (or B) which we constructed will actually be the orbit space of a $\mathcal{S}^2_{n,3}$-manifold. For we can use the map $f: V \longrightarrow \mathbb{C}P^2 \times I$ together with the identity map on $W_+(3)$ to construct a morphism $D \longrightarrow W_+(3)$. However, the next theorem, which answers questions (A) and (D), shows that this result is true more generally.

15.3 <u>Theorem</u> (U(n) case): Let B be a space modeled on $H^2_+(3)$ such that

(a) B^3 is a $\mathbb{Z}$-acyclic 8-manifold,

(b) the lowest stratum, B^1, is an integral cohomology $\mathbb{C}P^2$.

Then for any $n \geq 3$, the following two statements are true.

(1) There is an action $U(n) \times M^{6n-1} \longrightarrow M^{6n-1}$, modeled on $3\rho_n$ such that $M_\#$ is diffeomorphic to B, i.e., $C^2_{n,3}(B)$ is nonempty.

(2) Any such $\mathcal{S}^2_{n,3}(B)$-manifold is an integral cohomology sphere.

In the symplectic case, not every cohomology quaternionic projective plane can occur as the first stratum of the orbit space of a modeled Sp(n)-action on a cohomology sphere. Indeed, we have the following

15.4 <u>Proposition</u>: Let X be an integral cohomology QP^2 (quaternionic projective plane). For X to be diffeomorphic to the first stratum of an $Sp(n)$-action, modeled on $3\rho_n$, on a homology sphere M^{12n-1}, it is necessary and sufficient that

$$p_1(X) = 2 \cdot \text{generator of } H^4(X;\mathbb{Z}),$$

where $p_1(X)$ is the first Pontrjagin class of X.

15.5 Remark: Recall that the total Pontrjagin class of QP^2 is given by

$$p = (1+u)^6 (1+4u)^{-1}$$

where u is the generator of $H^4(QP^2;\mathbb{Z})$ (see [BH]). Thus, $p_1(QP^2) = 2u$ and $p_2(QP^2) = 7u^2$. An arbitrary integral cohomology QP^2, X, may have different Pontrjagin classes (although they are still subject to the relation of the Index Theorem, $(7p_2 - p_1^2)[X] = 45$).

Necessity of the condition in 15.4 follows from the following slightly stronger result.

15.6 Lemma: Let B be a $\mathfrak{B}_3^4$-space such that

(a) B^3 is a $\mathbb{Z}$-acyclic 14-manifold,

(b) the lowest stratum, B^1, is an integral cohomology QP^2.

If there is a $\mathfrak{B}_3^4$-morphism $f: B \longrightarrow W_+^4(3)$, then

(c) $p_1(B^1) = 2 \cdot \text{generator of } H^4(B^1;\mathbb{Z})$.

Proof: Let η^1 and η_3^1 be the $PSp(2)$-bundles associated to the normal cone bundles of B^1 and W_3^1, respectively. ($PSp(2) \cong SO(5)$).

Since f is transverse on normal cone bundles,

(1) $\quad (f|_{B^1})^* (\eta_3^1) \cong \eta^1$.

Let u and v be generators of $H^4(\mathbb{C}P^2; \mathbb{Z})$ and $H^4(B^1, \mathbb{Z})$, respectively. Suppose that $(f|_{B^1})^*(u) = mv$ for some integer m. Then by (1) and the above remark,

$$p_1(B^1) = 2mv \ ,$$
$$p_2(B^1) = 7m^2v^2 .$$

It follows from the Index Theorem that $49\,m^2 - 4\,m^2 = 45$. Hence, $m = \pm 1$, which proves (c).

□

Conversely,

15.7 Lemma: If X is a $\mathbb{Z}$ cohomology QP^2 such that

(c) $\quad p_1(X) = 2 \cdot$ generator of QP^2,

then there is a $\mathcal{B}_3^4$-space B such that

(a) $\quad B^3$ is a contractible 14-manifold,

(b) $\quad B^1$ is diffeomorphic to X.

Moreover, there is a morphism $f: B \to W_+(3)$ such that $f^*\Sigma^4(n, 3)$ is a homotopy $(12n-1)$-sphere.

Proof: By adding 1-handles we may kill $\pi_1(X)$. The new manifold will have a free abelian second homology group and we kill this by adding 2-handles. Thus, we have constructed a cobordism V of X to a new 1-connected integral cohomology quaternionic projective plane X′. By construction V is 1-connected 9-manifold with $\partial V = X \cup X'$ and with

$H^*(V;\mathbb{Z}) \cong H^*(QP^2;\mathbb{Z})$. Moreover, we may assume that $p_1(V) =$ $2 \cdot$ generator of $H^4(V;\mathbb{Z})$ (and in particular, that $p_1(X') = 2 \cdot$ generator of $H^4(X';\mathbb{Z})$). This implies, by a result of Eells and Kuiper [EK], that

(1) $$X' \cong QP^2 \# \Sigma^8$$

where Σ^8 is a homotopy 8-sphere. According to Table 7.4 in [L;p.48] Σ^8 embeds in S^{13}. Cut out a tubular neighborhood of Σ^8 in D^{14} and paste $\Sigma^8 \times H_+^4(2)$ back in. This gives us a $\mathcal{B}_3^4$-space, A, with Σ^8 as lowest stratum and with $A^3 \cong D^{14}$. Recall that the first stratum $W_3^1 \subset W_+^4(3)$ is diffeomorphic to QP^2. We can form the boundary connected sum $W_+^4(3) \# A$ to get a new $\mathcal{B}_3^4$-space with first stratum diffeomorphic to X' (taking the connected sum at points in W_3^1 and in Σ^8). Let $C^1(X')$ be the normal cone bundle of X' in $W_+^4(3) \# A$. Since X' and V are both 1-connected and since the inclusion $X' \subset V$ is a homology isomorphism, it follows from the theorem of J. H. C. Whitehead that there is a retraction

(2) $$r: V \longrightarrow X'.$$

Let $C^1(V) = r^* C^1(X')$. As in the proof of Proposition 15.2, we form the $\mathcal{B}_3^4$-cobordism

$$(W_+^4(3) \# A) \times I \underset{C^1(X') \times 1}{\cup} C^1(V)$$

by pasting $C^1(V)$ onto $(W_+^4(3) \# A) \times I$. The "top" of this cobordism is a $\mathcal{B}_3^4$-space B such that

(a) B^3 is $\mathbb{Z}$-acyclic, and

(b) $B^1 \cong X$.

Using the retraction r given in (2), we get a retraction of the $\mathcal{B}_3^4$-cobordism to $W_+^4(3) \# A$. There is a natural map $c: W_+^4(3) \# A \longrightarrow W_+^4(3)$ which collapses A to a point. The composition of this retraction with c defines a $\mathcal{B}_3^4$-morphism $f: B \longrightarrow W_+^4(3)$. It is easily checked that

$$(f|_{B^i})^*: H^*(W_3^i; \mathbb{Z}) \longrightarrow H^*(B^i; \mathbb{Z})$$

is an isomorphism for $i=1,2$ and hence that $f^*\Sigma^4(n,3)$ is a homology $(12n-1)$-sphere (by Theorem 13.1). By doing 1 and 2-surgeries in the interior of B^3 the argument can easily be modified so that B^3 is contractible (and consequently $f^*\Sigma^4(n,3)$ will be a homotopy sphere). This completes the proof of the lemma and of Proposition 15.4.

$\square$

15.8 <u>Remark</u>: Notice that the retraction $r: V \longrightarrow X'$ defined in (2) of the above proof restricts to a map $r|_X: X \longrightarrow X'$, inducing a cohomology isomorphism. Thus, if X is any $\mathbb{Z}$ cohomology QP^2 such that $p_1(X)$ is twice a generator of $H^4(X, \mathbb{Z})$, it follows that there is a map

$$g: X \longrightarrow QP^2$$

inducing an isomorphism on integral cohomology.

In the symplectic case we run into homotopy theoretic difficulties in answering question (A). For the sake of completeness we state the following result, which at least locates the problem.

15.9 <u>Theorem</u> (Sp(n) case): Let B be a space, modeled on $H^4_+(3)$, such that

(a) B^3 is a $\mathbb{Z}$-acyclic 14-manifold,

(b) B^1 is a $\mathbb{Z}$ cohomology QP^2, and

(c) $p_1(B^1) = 2 \cdot$ generator of $H^4(B^1; \mathbb{Z})$.

By 15.8 there is a map $g: B^1 \longrightarrow W^1_3$ inducing a cohomology isomorphism and which can be covered by a map of circle bundles, $h: \partial B^2 \longrightarrow \partial W^2_3$. Suppose we know, in addition, that

(d) h extends to a map $B^2 \longrightarrow W^2_3$.

Then the following two statements are true.

(1) For any $n \geq 3$, $C^4_{n,3}(B)$ is nonempty.

(2) Any $\mathcal{B}^4_{n,3}(B)$-manifold is an integral cohomology (12n-1)-sphere.

The first step towards proving these three theorems is the study of the possible normal cone bundles of a cohomology projective plane B^1 embedded in an acyclic $\mathcal{B}_3$-space B. Let $\eta = \{\eta^i \to B^i; g^{ij}: \eta^{ij} \to \partial_i \eta^{i+j}\}$ be a normal system for B and let $\eta_3 = \{\eta^i_3 \to W^i_3; g^{ij}_3: \eta^{ij}_3 \to \partial_i \eta^{i+j}_3\}$ be the standard $\mathcal{B}_3$-normal system for $W_+(3)$. The map g^{12} covers a smooth embedding

$$B^1 \longrightarrow \partial B^3 = \partial_1 B^3 \cup \partial_2 B^3.$$

The normal bundle of this embedding is the disk bundle $D^{d+1} \to \partial_1 B^3 \to B^1$, which has associated principal bundle η^1 and structure group $PO(2) = O(2)$, $PU(2) = SO(3)$, or $PSp(2) = SO(5)$ as $d = 1, 2$, or 4. Massey has observed in [M2] that any embedding $\mathbb{R}P^2 \subset S^4$, $\mathbb{C}P^2 \subset S^7$ or $QP^2 \subset S^{13}$ has a unique normal bundle in each of the three cases. This observation is also applicable to an embedding $B^1 \subset \partial B^3$ of a cohomology projective plane in a cohomology sphere.

15.10 Proposition (Massey): Let

$$X^{2d} = \begin{cases} \mathbb{R}P^2 & ; \text{if } d=1 \\ \text{a } \mathbb{Z}\text{-cohomology } \mathbb{C}P^2 & ; \text{if } d=2 \\ \text{a } \mathbb{Z}\text{-cohomology } QP^2 & ; \text{if } d=4 . \end{cases}$$

(If $d=4$ we also require that $p_1(X) = 2 \cdot$ generator of $H^4(X;\mathbb{Z})$.) Let $X^{2d} \subset HS^{3d+1}$ be any embedding of X in an integral cohomology $(3d+1)$-sphere. Then, in each of the three cases, the normal bundle of X is unique, i.e., it is independent of the embedding and of the cohomology sphere.

15.11 Remark: In each case the bundle is determined by certain characteristic classes which are described below.

(1) If $d=1$, the bundle is determined by the fact that its first Stiefel-Whitney class is nonzero and by the fact that its twisted Euler class is $2 \cdot$ generator of $H^2(\mathbb{R}P^2;\mathbb{Z}^t)$.

(2) If $d=2$, the bundle is determined by the facts that $w_1 = 0$, $w_2 \neq 0$ and $p_1 = -3u^2$ where u is a generator of $H^2(X^4;\mathbb{Z})$.

(3) If $d=4$, the bundle is determined by the Pontrjagin classes of X^8.

Proof: First note that since $\mathbb{Z}$-cohomology spheres are π-manifolds, the normal bundle of $X^{2d} \subset HS^{3d+1}$ is stably equivalent to the normal bundle of X embedded in a standard sphere. In the cases $d=1$ or $d=2$, Massey's argument completes the proof. As Massey also points out, Sasao and Tamura have proved that an $SO(5)$-bundle over QP^2 is determined by its Pontrjagin classes. Since there is a map $g: X^8 \longrightarrow QP^2$ inducing an isomorphism on cohomology, the same result is true for X^8 by obstruction theory. See [M2; 801-802] for further details and references.

□

Let $[X;Y]$ mean the set of free homotopy classes of maps from X to Y.

15.12 Lemma:

(1) $[\mathbb{R}P^2;\mathbb{R}P^2] \cong \mathbb{Z}_2 \cup \{\text{positive odd integers}\}$. The $\mathbb{Z}_2$ corresponds to the two homotopy classes which induce the zero map on the fundamental group. The maps which induce an isomorphism on π_1 are classified by the absolute value of their twisted degree, which must be an odd integer.

(2) If X^4 is a $\mathbb{Z}$-cohomology $\mathbb{C}P^2$, then $[X^4;\mathbb{C}P^2] \cong \mathbb{Z}$. Two maps are homotopic if and only if they induce the same map on H^2.

(3) If X^8 is a $\mathbb{Z}$-cohomology QP^2 with the correct Pontrjagin classes, then $[X^8;QP^2] \cong \mathbb{Z} \cong H^4(X^8;\mathbb{Z})$.

Proof: The first statement follows from [O; p. 464]. The second statement follows immediately from obstruction theory (see, for example, [Sp; p. 452]).

To prove (3), first notice that there is a homotopy equivalence

$$X \sim K \cup_p D^8$$

where K is a 4-dimensional CW complex with the homology of S^4 and where the 8-cell is attached via a map $p: S^7 \longrightarrow K$ of "Hopf invariant one" (i.e., X has the correct cohomology). The sequence of the cofibration $K \subset X$ is the top row of the following commutative diagram,

$$\begin{tikzcd}
{[\Sigma K; QP^2]} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \pi_8(QP^2) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & {[X; QP^2]} \arrow[r] & {[K; QP^2]} \arrow[r] & \pi_7(QP^2) \\
\pi_4(S^3) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \pi_7(S^3) \arrow[d, phantom, "\cong" rotate=90] & & & \\
\mathbb{Z}_2 & \mathbb{Z}_2 & & &
\end{tikzcd}$$

The isomorphisms with the homotopy groups of S^3 are via the sequence of the fibration $S^3 \longrightarrow S^{11} \longrightarrow QP^2$ and by obstruction theory. Notice that $[K, QP^2] \cong \mathbb{Z}$, where the isomorphism is defined by the induced map on H^4. Since, by Remark 15.8, there is a map $g: X \longrightarrow QP^2$ inducing a cohomology isomorphism, it follows that $[X; QP^2] \longrightarrow [K; QP^2]$ is a surjection. The map $\pi_4(S^3) \longrightarrow \pi_7(S^3)$ is just composition with the Hopf map $S^7 \longrightarrow S^4$; therefore it is an isomorphism (see [H2; p. 330]). Hence, $[\Sigma K; QP^2] \longrightarrow \pi_8(QP^2)$ is also an isomorphism. It follows that $[X; QP^2] \longrightarrow [K; QP^2]$ is a bijection, i.e., $[X; QP^2] \cong \mathbb{Z}$. This proves the lemma.

$\square$

Recall that $W_3^1 \cong \mathbb{R}P^2$, $\mathbb{C}P^2$ or QP^2 as d=1,2, or 4. Let $PG_2 \longrightarrow \eta_3^1 \longrightarrow W_3^1$ classify the normal cone bundle of W_3^1 in $W_+^d(3)$. There is the following simple corollary of the last two results and Lemma 15.6.

15.13 <u>Corollary</u>: Let B be a $\mathcal{B}_3^d$-space satisfying the hypotheses of either 15.1, 15.3, or 15.9, as d=1, 2 or 4. In other words, we are assuming that

(a) B^3 is $\mathbb{Z}$-acyclic,

(b) B^1 is an integral cohomology projective plane,

(c) if d=4, $p_1(B^1)$ is twice a generator of $H^4(B^1, \mathbb{Z})$.

If η^1 classifies the normal cone bundle of B^1 in B, then there is a map $g: B^1 \longrightarrow W_3^1$ such that

$$g^*(\eta_3^1) \cong \eta_3^1 .$$

Moreover, any such g induces an isomorphism on integral cohomology and g is unique up to homotopy.

<u>Proof</u>: Obvious.

□

To prove that $C_{n,3}^d(B)$ is nonempty we must construct a map $B \longrightarrow W_+(3)$. Our procedure is to do this one stratum at a time starting with the smallest. The above corollary shows that we can always find such a map on a normal cone bundle of the first stratum. The map of normal cone bundles restricts to a map of W_2^1-bundles

$$h: \partial_1 B^2 \longrightarrow \partial_1 W_3^2 .$$

Thus question (A) becomes: does h extend to a map $h': B^2 \longrightarrow W_3^2$ (possibly after altering h by a bundle automorphism)? If the answer is affirmative, then there is clearly no problem in extending h' first to a collared neighborhood of ∂B and then finally to a map $f: B \longrightarrow W_+(3)$ (using the fact that W_3^3 is contractible).

In the unitary case we solve this extension problem below. The proof involves a formula of Peterson and Stein in [PS] for computing the indeterminancy of the tertiary obstruction to extending maps into $\mathbb{C}P^2$. The idea of applying this formula to this problem was suggested to me by Massey.

<u>Proof of Theorem 15.3</u> <u>(the U(n) case)</u>: Let $Y = \partial_1 B^2$ and let $h: Y \to \partial_1 W_3^2$ be the bundle map defined above (care of 15.13). Recall that according to Lemma 6.1, W_3^2 deformation retracts to $U(3)/U(2) \times U(1) = \mathbb{C}P^2$. Let $g: Y \longrightarrow \mathbb{C}P^2$ be the composition

$$Y \xrightarrow{h} \partial_1 W_3^2 \subset W_3^2 \sim \mathbb{C}P^2 .$$

We must show that g extends to a map $B^2 \longrightarrow \mathbb{C}P^2$. The first obstruction lies in $H^3(B^2, Y; \pi_2(\mathbb{C}P^2)) \cong \mathbb{Z}$. Let ι be a generator of $H^2(\mathbb{C}P^2; \mathbb{Z})$ and let u be a generator of $H^2(B^2; \mathbb{Z})$. Let $j: Y \subset B^2$ be the inclusion. Since ∂B is a cohomology sphere and since $h: Y \longrightarrow \partial_1 W_3^2$ is a map of circle bundles (i.e., W_2^1-bundles), an argument with Mayer-Vietoris sequences shows that

(1) $$g^*(\iota) = \pm j^*(u).$$

Since $g^*(\iota)$ is in the image of j^*, the first obstruction vanishes. For simplicity suppose that the sign in (1) is positive.

By part (2) of Lemma 15.12, there is a map $r: B^2 \longrightarrow \mathbb{C}P^2$, unique up to a homotopy, such that $r^*(\iota) = u$. Let $g' = r|_Y : Y \longrightarrow \mathbb{C}P^2$. If we can show that g and g' are homotopic, then g extends to B^2 (since g' extends). The obstruction to finding such a homotopy is a difference class $d(g,g')$ which lies in some quotient of $H^6(Y; \pi_6(\mathbb{C}P^2)) \cong \mathbb{Z}_2$ (we must divide out by the indeterminancy). Essentially we are going to show that the indeterminancy is nonzero, from which it follows that g and g' are homotopic. To show it is nonzero we must first define it.

The first step is to define a two-stage Postnikov system for $\mathbb{C}P^2$. Let $\mathcal{P} \longrightarrow K(\mathbb{Z},2)$ be the principal fibre space classified by $\theta = \iota^3 \in H^6(\mathbb{Z},2;\mathbb{Z})$, where ι is the natural generator of $H^2(\mathbb{Z},2;\mathbb{Z})$,

$$\begin{array}{ccc} K(\mathbb{Z},5) \longrightarrow & \mathcal{P} & \\ & \Big\downarrow p & \\ & K(\mathbb{Z},2) & \xrightarrow{\theta} K(\mathbb{Z},6)\,. \end{array}$$

We also have a principal fibre space

$$\begin{array}{ccc} K(\mathbb{Z},6) \longrightarrow & \mathcal{P}' & \\ & \Big\downarrow p' & \\ & \mathcal{P} & \xrightarrow{\phi} K(\mathbb{Z}_2,7) \end{array}$$

classified by the nonzero element $\phi \in H^7(\mathcal{P};\mathbb{Z}_2) \cong \mathbb{Z}_2$. Clearly, $\mathcal{P}'$ is 6-equivalent to $\mathbb{C}P^2$. Since $g^*(\iota) = j^*(u) = j^*r^*(\iota) = g'^*(\iota)$, we can find a homotopy $Y \times I \longrightarrow K(\mathbb{Z},2)$ of $p \circ p' \circ g$ to $p \circ p' \circ g'$. Since

$H^6(Y \times (I, \partial I); \mathbb{Z}) = 0$, we can lift this to a homotopy $F: Y \times I \longrightarrow \mathcal{O}$ such that the following diagram commutes

$$\begin{tikzcd} Y \times \partial I \arrow[r] \arrow[d, phantom, "\cap"] & \mathcal{O}' \arrow[d, "p'"] \\ Y \times I \arrow[r] & \mathcal{O} \end{tikzcd} .$$

We encounter an obstruction $c(F) \in H^7(Y \times (I, \partial I); \mathbb{Z}_2)$ to lifting F rel $Y \times \partial I$ to a homotopy $Y \times I \longrightarrow \mathcal{O}'$. If we had chosen a different lift $\overline{F}: Y \times I \longrightarrow \mathcal{O}$ to begin with then we would get a possibly different obstruction $c(\overline{F})$. The obstruction to finding a homotopy between g and g' lies in $H^7(Y \times (I, \partial I); \mathbb{Z}_2)/Q$ where Q is the subgroup consisting of all elements of the form $\{c(F) - c(\overline{F})\}$. We will show that $Q \neq 0$, and hence that it is the whole group.

Since F and $\overline{F}$ agree on $Y \times \partial I$, they piece together to define a map

$$G = F_0 \cup F_1 : Y \times S^1 \longrightarrow \mathcal{O},$$

where

$$Y \times S^1 = Y \times I \underset{Y \times \partial I}{\cup} Y \times I.$$

Let $\Delta: H^7(Y \times S^1; \mathbb{Z}_2) \longrightarrow H^7(Y \times (I, \partial I); \mathbb{Z}_2)$ be defined by the diagram

$$\begin{tikzcd} H^7(Y \times S^1) \arrow[r, "\cong"] \arrow[dr, "\Delta"'] & H^7(Y \times S^1, Y \times I) \arrow[d, phantom, "\wr\| \text{ excision}"] \\ & H^7(Y \times (I, \partial I)) \end{tikzcd} .$$

According to [Sp, p.449],

(2) $$\Delta G^*(\phi) = c(F) - c(\overline{F}).$$

Let $\mu: K(\mathbb{Z},5) \times \theta \longrightarrow \theta$ be fibre multiplication on the principal fibre space θ. There is a map $\gamma: Y \times (I, \partial I) \longrightarrow (K(\mathbb{Z},5), *)$ such that

(3) $$\mu(\gamma, F) = \overline{F}.$$

Let $G' = F \cup F: Y \times S^1 \longrightarrow \theta$ be the double of F. We can extend γ to a map $\gamma: Y \times S^1 \longrightarrow K(\mathbb{Z},5)$ (denoted by the same letter) by mapping the second copy of $Y \times I$ to the base point $*$. By construction

(4) $$\mu(\gamma, G') = G.$$

According to the formula of Peterson and Stein [PS; 290-291],

(5) $$\begin{aligned} G^*(\phi) &= \mu(\gamma, G')^*(\phi) \\ &= Sq^2(\gamma) + \gamma \cup G'^*(\iota) \end{aligned}$$

where $\iota \in H^2(\mathbb{Z},2;\mathbb{Z})$ is 2-characteristic for $K(\mathbb{Z},2)$ and where the cup product

$$\cup: H^5(Y \times S^1; \mathbb{Z}) \otimes H^2(Y \times S^1; \mathbb{Z}) \longrightarrow [H^7(Y \times S^1; \mathbb{Z}_2)$$

is relative to the nontrivial Whitehead product pairing of coefficients

$$\begin{array}{ccc} \pi_5(\theta') \otimes \pi_1(\theta') & \longrightarrow & \pi_6(\theta') \\ \wr\| & & \wr\| \\ \mathbb{Z} \otimes \mathbb{Z} & \longrightarrow & \mathbb{Z}_2. \end{array}$$

It is not difficult to see that $Sq^2: H^5(Y \times S^1; \mathbb{Z}) \longrightarrow H^7(Y \times S^1; \mathbb{Z}_2)$ is the zero map. On the other hand, if $\pi: Y \times S^1 \longrightarrow Y$ is the projection, then

$G'^*(\iota) = \pi^* g^*(\iota) = \pi^* j^*(u)$, so $G'^*(\iota)$ reduces mod 2 to a nonzero element. It follows from Poincaré duality that we can find a $\gamma \in H^5(Y \times S^1; \mathbb{Z})$ such that $\gamma \cup G'^*(\iota) = G^*(\phi)$ is nontrivial. Hence, $\Delta G^*(\phi)$ is also nontrivial, so it follows from (2) that $Q = H^7(Y \times S^1; \mathbb{Z}_2)$. Thus, $g \sim g'$. It follows from the remarks immediately preceding the beginning of this proof, that if B is any $\mathcal{B}_3^2$-space satisfying the hypothesis of 15.3, then there is a smooth strata-preserving map

$$f: B \longrightarrow W_+(3).$$

This proves statement (1) of the theorem.

By 15.12 and 15.13,

$$(f|_{B^1})^*: H^*(W_3^1; \mathbb{Z}) \cong H^*(B^1, \mathbb{Z}).$$

By the above argument

$$g^* = f(|_{B^2})^*: H^*(W_3^2; \mathbb{Z}) \longrightarrow H^*(B^2; \mathbb{Z})$$

is also an isomorphism. Thus, by Theorem 13.1, $f^*\Sigma(n, 3)$ is always an integral cohomology sphere. This completes the proof.

$\square$

As for Theorem 15.9 (the Sp(n) case) nothing remains to be proved, since in this case, the hypothesis of the theorem already includes the assumption that the (more difficult) extension problem can be solved. With these distractions out of the way we can now begin our attack on Theorem 15.1 in earnest.

<u>The Orthogonal Case:</u> For the remainder of this section B will always

denote a space modeled on $H^1_+(3)$, satisfying the hypothesis of Theorem 15.1, i.e., B^3 will be a $\mathbb{Z}$-acyclic 5-manifold and B^1 will be diffeomorphic to $\mathbb{R}P^2$. We start by collecting some information in the next four lemmas.

15.14 Lemma: Let $O(2) \longrightarrow \gamma \longrightarrow X$ be a nonorientable principal $O(2)$-bundle. If the twisted euler class $\chi(\gamma) \in H^2(X; \mathbb{Z}^t)$ satisfies $2\chi(\gamma) \neq 0$, then γ admits precisely two automorphisms, one of which is the identity and the other being multiplication by -1 on each fibre.

Proof: The automorphic bundle

$$\gamma^* = O(2) \times_{O(2)/Z} \gamma/Z$$

has two components, which are denoted by

$$\gamma^*_+ = SO(2) \times_{O(2)/Z} \gamma/Z, \text{ and}$$

$$\gamma^*_- = \{O(2) - SO(2)\} \times_{O(2)/Z} \gamma/Z$$

(where Z = center of $O(2) = \{\pm 1\}$). First consider γ^*_+. If $a \in O(2)$ and $b \in SO(2)$, then

$$a^{-1}ba = \begin{cases} b \ ; & \text{if } a \in SO(2) \\ b^{-1} \ ; & \text{if } a \notin SO(2). \end{cases}$$

Thus, the structure group of γ^*_+ is actually $O(2)/SO(2) = \mathbb{Z}_2$, i.e.,

$$\gamma^*_+ = SO(2) \times_{\mathbb{Z}_2} \gamma/SO(2).$$

Since $\mathbb{Z}_2$ fixes $\{\pm 1\} = Z \subset SO(2)$, γ^*_+ has at least two sections. On the other hand, since $\gamma/SO(2) \longrightarrow X$ is a nontrivial $\mathbb{Z}_2$-bundle, γ^*_+ has no

Wrong

other sections.

It remains to be shown that γ_-^* admits no sections. There is a canonical identification of $O(2) - SO(2)$ with $\mathbb{R}P^1$ which is equivariant with respect to conjugation on $O(2) - SO(2)$ and the standard $O(2)$-action on $\mathbb{R}P^1$ (an element of $\mathbb{R}P^1$ determines a reflection). If we replace $\mathbb{R}P^1$ by its double cover S^1, we see that

$$\gamma_-^* \cong S^1 \times_{O(2)} p_*(\gamma),$$

where $O(2)$ acts standardly on S^1, where $p: O(2) \longrightarrow O(2)$ is the homomorphism with kernel $\{\pm 1\}$ and where $p_*(\gamma)$ is the pullback of the universal $O(2)$-bundle via the classifying map for γ composed with the induced map $p_*: BO(2) \longrightarrow BO(2)$. Since $(p_*)^*: H^2(BO(2); \mathbb{Z}^t) \rightarrowtail H^2(BO(2); \mathbb{Z}^t)$ is multiplication by 2, it follows that $\chi(p_*(\gamma)) = 2\chi(\gamma) \neq 0$. Noting that $\chi(p_*(\gamma))$ is the primary obstruction to a cross section of γ_-^* completes the proof.

$\square$

15.15 Lemma: Suppose that B satisfies the hypothesis of 15.1. Then B^2 has cohomology isomorphic to that of $\mathbb{R}P^2$ with either $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients. Since this says that $H^1(B^2; \mathbb{Z}_2) \cong \mathbb{Z}_2$, there is a unique nontrivial system of twisted integer coefficients over B^2, denoted by $\mathbb{Z}^t$. The integral cohomology (twisted or untwisted) of B^2, $\partial_1 B^2$ or $(B^2, \partial_1 B^2)$ is given according to the following table:

	$H^i(B_2, \partial_1 B^2)$	$H^i(B^2)$	$H^i(\partial_1 B^2)$	
i=0	0	$\mathbb{Z}$	$\mathbb{Z}$	
1	0	0	0	
2	0	$\mathbb{Z}_2$	$\mathbb{Z}_2 + \mathbb{Z}_2$	$\mathbb{Z}$ coefficients
3	$\mathbb{Z}_2$	0	$\mathbb{Z}$	
4	$\mathbb{Z}$	0	$\mathbb{Z}$	
i=0	0	0	0	
1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
2	$\mathbb{Z} + T$	$\mathbb{Z} + T$	$\mathbb{Z}_2$	$\mathbb{Z}^t$ coefficients
3	T	T	$\mathbb{Z}_2$	
4	$\mathbb{Z}_2$	0	0	

where T is some odd torsion group.

<u>Proof</u>: Since $B^2 \sim \partial B - B^1$, it follows from Alexander duality that the $\mathbb{Z}$ or $\mathbb{Z}_2$ cohomology ring of B^2 is isomorphic to that of $\mathbb{RP}^2$. The $\mathbb{Z}$ or $\mathbb{Z}^t$ cohomology of the nonorientable circle bundle can be computed from a Gysin sequence with twisted coefficients $(\partial_1 B^2 \cong O(3)/O(1) \times O(1) \times O(1))$. Since B^2 has Euler characteristic one, its double cover $\widetilde{B}^2$ has Euler characteristic two; hence, $H^2(\widetilde{B}^2; \mathbb{Q})$ has rank one. It follows that $\widetilde{B}^2$ has the $\mathbb{Z}_{(2)}$-cohomology of a 2-sphere. Our computation of $H^*(B^2; \mathbb{Z}^t)$ uses the Gysin sequence of the double cover $\widetilde{B}^2 \longrightarrow B^2$ (with twisted integer coefficients). The computation of $H^*(B, \partial_1 B^2)$ follows from Lefschetz duality.

$\square$

15.16 Lemma:

$$[B^2;\mathbb{R}P^2] \cong H^2_{odd}(B^2;\mathbb{Z}^t)/\mathbb{Z}_2 \cup H^1(B^2;\mathbb{Z}_2)$$

$$\cong \{\text{positive odd integers}\} \times T \cup \mathbb{Z}_2$$

where $H^2_{odd}(B^2;\mathbb{Z}^t)$ means the inverse image of the nontrivial element under the coefficient homomorphism $H^2(B^2;\mathbb{Z}^t) \to H^2(B^2;\mathbb{Z}_2)$ and where $\mathbb{Z}_2$ acts on $H^2_{odd}(B^2;\mathbb{Z}^t)$ by multiplication by -1.

Proof: Our principal tool is the sequence of the fibration

$$\mathbb{R}P^2 \xrightarrow{i} BO(2) \times BO(1) \xrightarrow{\pi} BO(3) .$$

This yields the following exact sequence of homotopy classes of maps,

$$(1) \quad \to [B^2;O(2)\times O(1)] \to [B^2;O(3)] \to [B^2,\mathbb{R}P^2] \xrightarrow{i_*} [B^2;BO(2)\times BO(1)]$$

$$\xrightarrow{\pi_*} [B^2;BO(3)].$$

The set $[B^2;BO(2)\times BO(1)]$ is completely determined by characteristic classes; namely, the Stiefel-Whitney class of an $O(1)$-bundle together with the first Stiefel-Whitney class and the (possibly twisted) Euler class of an $O(2)$-bundle. It follows from the product formula that image $i_* = \ker \pi_* \subset [B^2;BO(2)\times BO(1)]$ consists precisely of

(a) the trivial bundle, and

(b) those bundles such that both w_1's are nonzero and such that the twisted Euler class X is not divisible by two, i.e., such that $X \in H^2_{odd}(B^2;\mathbb{Z}^t)$.

Let $u \in H^2(\mathbb{R}P^2; \mathbb{Z}^t)$ be the pullback via i of the universal twisted Euler class in $H^2(BO(2), \mathbb{Z}^t)$. If $f: B^2 \longrightarrow \mathbb{R}P^2$ is a map which is surjective on π_1, then $\chi(i \circ f) = f^*(u)$, where $\chi(i \circ f)$ means the twisted Euler class of the bundle classified by $i \circ f$. Consider the self-diffeomorphism φ_g of $\mathbb{R}P^2 = O(3)/O(2) \times O(1)$ which is left multiplication by

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in O(3).$$

Clearly, φ_g is base point preserving and has twisted degree -1. Since g is in the same path component of $O(3)$ as the antipodal map, it follows that φ_g is freely homotopic to the identity, and so $i \circ f \sim i \circ \varphi_g \circ f$. On the other hand, since φ_g has twisted degree -1, $\chi(i \circ \varphi_g \circ f) = -f^*(u) = -\chi(i \circ f)$. Thus, the twisted Euler class of a bundle is well defined only up to a choice of sign, i.e., the $O(2)$-bundle is determined by an element $H^2(B^2; \mathbb{Z}^t)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by multiplication by -1.

By obstruction theory,

$$(2) \qquad [B^2; SO(3)] \cong H^1(B^2; \mathbb{Z}_2) = \mathbb{Z}_2,$$

and hence $[B^2; O(3)] \cong \mathbb{Z}_2 + \mathbb{Z}_2$. The only problem in using sequence (1) to complete the proof is that it is not clear what exactness means in this context. To clarify this point we recall the method of equivariant attaching and in particular Proposition 11.6. Let $b \in [B^2; BO(2) \times BO(1)]$ be in the image of i_*, and let $\xi^2 \longrightarrow B^2$ be the bundle classified by b. Let $\xi^3 \longrightarrow B^3$ be the trivial $O(3)$-bundle. According to 11.6, we have the following commutative diagram

$$\begin{array}{ccccccc}
\pi_0\Gamma(\xi^{2*}) \longrightarrow & \pi_0\Gamma(\partial_2\xi^{3*}) & \longrightarrow & \pi_0\Gamma(\partial_2\xi^{3*})/\pi_0\Gamma(\xi^{2*}) & \longrightarrow \\
& \Big\downarrow \cong & & \Big\downarrow \cong & \\
\longrightarrow & [B^2; O(3)] & \longrightarrow & i_*^{-1}(b) & \longrightarrow 0
\end{array}$$

where $i_*^{-1}(b) \subset [B^2; \mathbb{R}P^2]$.

Next, we assert that

$$\text{cokernel}\,[\pi_0\Gamma(\xi^{2*}) \longrightarrow \pi_0\Gamma(\partial_2\xi^{3*})]$$

$$= \begin{cases} \mathbb{Z}_2; & \text{if } b \text{ is trivial} \\ 0\ ; & \text{if } b \text{ is nontrivial} \end{cases}$$

This is obvious when b is trivial (i.e., when $\xi^2 \longrightarrow B^2$ is trivial) for then the image contains only those elements represented by the constant sections $\pm 1 \in \pi_0\Gamma(\partial_2\xi^{3*}) \cong [B^2; O(3)]$. If $\xi^2 \longrightarrow B^2$ is nontrivial then by Lemma 15.14, ξ^{2*} has precisely 4 sections, namely the sections which are identically one of the four elements $(\pm 1, \pm 1) \in O(2) \times O(1)$ on each fibre. It is not difficult to see that the sections $(+1, -1)$ and $(-1, +1)$ go to the nontrivial elements of $[B^2; O(3) - SO(3)]$ and $[B^2; SO(3)]$, respectively. Thus when b is nontrivial $\pi_0\Gamma(\xi^{2*}) \cong \pi_0\Gamma(\partial_2\xi^{3*})$, as claimed. In other words, i_*^{-1} (trivial bundle) $= \mathbb{Z}_2$; while i_*^{-1} (nontrivial bundle) is either empty or a singleton. Combining this with our computation of the image of i_* in $[B^2; BO(2) \times BO(1)]$ completes the proof.

□

The next project is to compute $[\partial B^2; \mathbb{R}P^2]$. As before, we have the fibration sequence

$$[\partial B^2; \mathbb{R}P^2] \xrightarrow{i_*} [\partial B^2; BO(2) \times BO(2)] \longrightarrow [\partial B^2; BO(3)].$$

The following lemma is a simple argument about characteristic classes.

15.17 Lemma: The image of i_* in $[\partial B^2; BO(2) \times BO(1)]$ can be identified with $\mathbb{Z}_2 + \mathbb{Z}_2$, since it consists precisely of the trivial bundle and those $O(2) \times O(1)$-bundles satisfying the following two conditions:

(a) $0 \neq w_1 \in H^1(\partial B^2; \mathbb{Z}_2) = \mathbb{Z}_2 + \mathbb{Z}_2$, where w_1 is the first Stiefel-Whitney class of the $O(1)$ part of the bundle.

(b) $0 \neq \chi \in H^2(\partial B^2; \mathbb{Z}^t) = \mathbb{Z}_2$, where χ is the twisted Euler class of the $O(2)$ part of the bundle and where the coefficient system is determined by w_1.

15.18 Proposition: If $i_*: [\partial B^2; \mathbb{R}P^2] \rightarrow \mathbb{Z}_2 + \mathbb{Z}_2 \subset [\partial B^2; BO(2) \times BO(1)]$ is the map considered above, then

(1) $i_*^{-1}(0,0)$ can be identified with $\mathbb{Z} + \mathbb{Z}_2 + \mathbb{Z}_2$.

(2) $i_*^{-1}(1,0)$ can be identified with $\mathbb{Z} + \mathbb{Z}_2$ (similarly for $i_*^{-1}(0,1)$ and $i_*^{-1}(1,1)$).

Proof: Let $\alpha \in [\partial B^2; \mathbb{R}P^2]$ and let $\partial\xi^2$ be the $O(2) \times O(1)$-bundle classified by $i_*(\alpha)$. Let ξ^3 be the trivial $O(3)$-bundle. In analogy with the proof of the previous lemma, the diagram of 11.6 is considered.

$$\begin{array}{ccccccc}
\pi_0\Gamma(\partial_1\xi^{2*}) \longrightarrow & \pi_0\Gamma(\partial_1\partial_2\xi^{3*}) & \longrightarrow & \pi_0\Gamma(\partial_1\partial_2\xi^{3*})/\pi_0\Gamma(\partial_1\xi^{2*}) & \longrightarrow 0 \\
& \wr\| & & \cap & \\
\longrightarrow & [\partial B^2; O(3)] & \longrightarrow & [\partial B^2; \mathbb{R}P^2] & \xrightarrow{i_*} \mathbb{Z}_2 + \mathbb{Z}_2
\end{array}$$

The proposition follows easily from the next two lemmas.

15.19 <u>Lemma</u>:

(1) Any $g \in [\partial B^2; SO(3)]$ is determined by two invariants:

(a) $g_*: \pi_1(\partial B^2) \longrightarrow \pi_1(SO(3))$ (g_* may be regarded as an element of $H^1(\partial B^2; \mathbb{Z}_2)$.)

(b) $\deg(g)$.

(2) The degree of g is always even.

(3) $[\partial B^2; SO(3)] \cong \mathbb{Z} + \mathbb{Z}_2 + \mathbb{Z}_2$.

15.20 <u>Lemma</u>: cokernel $[\pi_0\Gamma(\partial_1\xi^{2*}) \longrightarrow \pi_0\Gamma(\partial_1\partial_2\xi^{3*})]$

$$= \begin{cases} \mathbb{Z} + \mathbb{Z}_2 + \mathbb{Z}_2 & ; \text{ if } \partial_1\xi^2 \text{ is trivial} \\ \mathbb{Z} + \mathbb{Z}_2 & ; \text{ if } \partial_1\xi^2 \text{ is nontrivial.} \end{cases}$$

<u>Proof of Lemma 15.19</u>: Notice that ∂B^2 and $SO(3)$ are both orientable 3-manifolds. The proof of the first statement can be found in [O]. Moreover, it is shown in [O] that if g_* represents some fixed homomorphism on π_1, then $\deg(g)$ is either even or odd (depending on the homomorphism). If g_* is the zero map on π_1, then g can be lifted to a map into S^3, the

double cover of SO(3); hence, in this case deg (g) is even.

Now consider the case where $g_*: \pi_1(\partial B^2) \longrightarrow \pi_1(SO(3)) = \mathbb{Z}_2$ is a nonzero homomorphism. Clearly there is a map $\partial B^2 \longrightarrow \mathbb{R}P^2$ such that the composition $\partial B^2 \longrightarrow \mathbb{R}P^2 \subset \mathbb{R}P^3 \cong SO(3)$ induces g_* on π_1. Since this composition is obviously degree zero, it follows that deg (g) is even in this case as well.

Recalling that $\mathrm{Hom}\,(\pi_1(\partial B^2), \pi_1(SO(3))) = H^1(\partial B^2, \mathbb{Z}_2) \cong \mathbb{Z}_2 + \mathbb{Z}_2$, one checks easily that the map $[\partial B^2; SO(3)] \longrightarrow \mathbb{Z} + H^1(\partial B^2; \mathbb{Z}_2)$ defined by

$$g \longrightarrow (\tfrac{1}{2}\deg(g), \; g_*)$$

is an isomorphism (of abelian groups).

□

<u>Proof of Lemma 15.20</u>: The proof is similar to that of 15.16. The result is obvious if $\partial_1 \xi^2$ is trivial. If $\partial_1 \xi^2$ is nontrivial, then by Lemma 15.14 $\partial_1 \xi^{2*} \longrightarrow \partial_1 B^2$ has exactly four sections. Let S^1 be a fibre of $\partial_1 B^2 \longrightarrow B^1$ and notice that

$$\partial_1 \xi^2|_{S^1} \cong O(2) \times_{O(1)} \xi_2^1$$

where $\xi_2^1 \longrightarrow W_2^1 = S^1$ is a bundle for the linear action $2\rho_n$. That the sections represented by $(1,-1)$ and $(-1,1)$ are inessential in $\Gamma(\partial_1\partial_2\xi^{3*})$ follows from the corresponding fact for $\Gamma(\xi_2^{1\,*}) \subset \Gamma(\partial_1 \xi_2^{2\,*})$ (see [B1, 226]).

□

This completes the proof of Proposition 15.19.

15.21 <u>Lemma</u>: Let $\alpha: B^2 \longrightarrow \mathbb{R}P^2$ be a map inducing a surjection of fundamental groups. Let $j\colon \partial B^2 \subset B^2$ be the inclusion and let $i\colon \mathbb{R}P^2 \longrightarrow BO(2)\times BO(1)$ be inclusion of a fibre. Then $i\circ\alpha\circ j$ represents a unique nontrivial element $c \in \mathbb{Z}_2 + \mathbb{Z}_2 = \text{image } i_* \subset [\partial B^2; BO(2)\times BO(1)]$.

<u>Proof</u>: In view of Lemmas 15.16 and 15.17 and Proposition 15.18, to show that $i\circ\alpha\circ j$ is inessential it suffices to show that $j^*\colon H^2(B^2;\mathbb{Z}^t) \longrightarrow H^2(\partial B^2;\mathbb{Z}^t)$ is onto, where the coefficient system over ∂B^2 is induced by j. This follows immediately from the table in Lemma 15.15. (It can also be proved by an easy argument using the sequence of the pair $(B^2, \partial B^2)$ with both $\mathbb{Z}$ and $\mathbb{Z}^t$ coefficients and the Gysin sequences relating these two sequences.) According to 15.16, the image of $\alpha\circ j$ in $[\partial B^2;\ BO(2)\times BO(1)]$ is determined by the induced homomorphism of fundamental groups and this homomorphism is clearly independent of α. Thus, c is unique.

□

15.22 <u>Lemma:</u> Let $h\colon \partial B^2 \longrightarrow \partial W_3^2$ be a map of circle bundles covering a homotopy equivalence $B^1 \longrightarrow W_3^1$ (such an h exists by 15.13). Let h' denote the composition

$$\partial B^2 \xrightarrow{h} \partial W_3^2 \subset W_3^2 \sim \mathbb{R}P^2.$$

Then, $[i\circ h'] = c \in [\partial B^2; BO(2)\times BO(1)]$ where c is the element defined in the previous lemma (15.21).

<u>Proof</u>: First notice that $\partial B^2 \cong O(3)/O(1)\times O(1)\times O(1) \cong S^3/K$ where K is the unit quaternion group of order 8. The homotopy equivalence

$h: \partial B^2 \longrightarrow \partial W_3^2$ induces an isomorphism of fundamental groups. Let $j: \partial B^2 \subset B^2$ be the inclusion and $j_*: \pi_1(\partial B^2) \longrightarrow \mathbb{Z}_2 \subset \pi_1(B^2)$ the induced homomorphism. It follows from van Kampen's Theorem that

$$h_*(\ker j_*) = \ker(\pi_1(\partial W_3^2) \longrightarrow \pi_1(W_3^2)).$$

Hence, $h'_* = j_*: \pi_1(\partial B^2) \longrightarrow \mathbb{Z}_2$. So by the proof of the previous lemma, $[i \circ h'] = c$.

□

The next proposition is the homotopy theoretic core of the proof of Theorem 15.1.

15.23 Proposition:

(1) Let $\alpha \in H^2_{odd}(B^2; \mathbb{Z}^t)$ and let u be a generator of $H^2(\mathbb{R}P^2; \mathbb{Z}^t)$. According to Lemma 15.16, there is a map $g: B^2 \longrightarrow \mathbb{R}P^2$ such that $g^*(u) = \alpha$. Let $p: H^2(B^2; \mathbb{Z}^t) \longrightarrow H^2(B^2; \mathbb{Z}^t)/\text{torsion} \cong \mathbb{Z}$ be the natural projection and let $j: \partial B^2 \subset B^2$ be the inclusion. Then

$$[g \circ j] = (p(\alpha)^2, 1) \in \mathbb{Z} + \mathbb{Z}_2 \cong i_*^{-1}(c)$$

where $c \in [\partial B^2, BO(2) \times BO(1)]$ is defined in Lemma 15.21.

(2) As in the previous lemma, let $h: \partial B^2 \longrightarrow \partial W_3^2$ be a map of circle bundles covering a homotopy equivalence and let h' denote the composition $\partial B^2 \longrightarrow \partial W_3^2 \subset W_3^2 \sim \mathbb{R}P^2$. Then

$$[h'] = (1, 1) \in \mathbb{Z} + \mathbb{Z}_2 \cong i_*^{-1}(c).$$

Proof: The situation is somewhat more tractable if we consider the corresponding maps on the double covers $\tilde{B}^2$ and $\partial\tilde{B}^2$. Since g and h' are both surjective on π_1, they are covered by maps

$$\tilde{g}: \tilde{B}^2 \longrightarrow S^2 \qquad \text{and}$$

$$\tilde{h}': \partial\tilde{B}^2 \longrightarrow S^2.$$

Let $\tilde{j}: \partial\tilde{B}^2 \longrightarrow \tilde{B}^2$ be the inclusion. Since ∂B^2 is diffeomorphic to S^3/K (K is the unit quaternion group of order 8), it follows that $\partial\tilde{B}^2$ is the lens space $S^3/\mathbb{Z}_4 = L(4)$.

By a well-known theorem of Pontrjagin, we can compute $[X; S^2]$ whenever X has the homotopy type of a 3-complex. Recall that this theorem says that there is a surjection $\lambda: [X, S^2] \longrightarrow H^2(X, \mathbb{Z})$ defined by $\lambda(\theta) = \theta^*(\tilde{u})$ where $\tilde{u}$ is a generator of $H^2(S^2; \mathbb{Z})$. Moreover, if $y \in H^2(X; \mathbb{Z})$, then

$$\lambda^{-1}(y) \cong H^3(X, \mathbb{Z})/y \cup H^1(X; \mathbb{Z}).$$

When $X = L(4)$ or $\tilde{B}^2$, $H^1(X; \mathbb{Z}) = 0$, so that

$$(1) \qquad [L(4); S^2] \cong H^3(L(4); \mathbb{Z}) \times H^2(L(4); \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}_4$$

$$(2) \qquad [\tilde{B}^2; S^2] \cong H^3(\tilde{B}^2; \mathbb{Z}) \times H^2(\tilde{B}^2; \mathbb{Z}).$$

The invariant in $H^3(L(4); \mathbb{Z})$ has a description analogous to that of the Hopf invariant. If $\theta \in [L(4); S^2]$ then both $H^2(\theta; \mathbb{Z})$ and $H^4(\theta; \mathbb{Z})$ are infinite cyclic groups ($H^*(\theta)$ means the cohomology of the mapping cone of θ.) Let $\delta: H^3(L(4); \mathbb{Z}) \cong H^4(\theta; \mathbb{Z})$ be the connecting homomorphism. If v = generator of $H^2(\theta; \mathbb{Z})$ and w = generator of $H^3(L(4); \mathbb{Z})$, then

$v^2 = m\,\delta(w)$ for some integer m. The invariant in $H^3(L(4), \mathbb{Z})$ corresponding to θ can be taken to be mw.

For $x \in H^2(\widetilde{B}^2; \mathbb{Z})$, notice that $\tilde{j}^*(x) = p(x) \pmod 4$, where $\tilde{j}^*: H^2(\widetilde{B}^2; \mathbb{Z}) \longrightarrow H^2(\partial\widetilde{B}^2; \mathbb{Z}) = \mathbb{Z}_4$ is induced by the inclusion and where $p: H^2(\widetilde{B}^2; \mathbb{Z}) \longrightarrow H^2(\widetilde{B}^2; \mathbb{Z})/\text{torsion} \cong \mathbb{Z}$ is the natural projection.

Let $\tilde{u}$ be a generator of $H^2(S^2; \mathbb{Z})$. Since $g^*(u) = \alpha$, we have that $\tilde{g}^*(\tilde{u}) = \tilde{\alpha}$, where $\tilde{\alpha}$ is the inverse image of α under the transfer homomorphism $H^2(\widetilde{B}^2, \mathbb{Z}) \longrightarrow H^2(B^2; \mathbb{Z}^t)$. Let M be the mapping cone of $\tilde{g} \circ \tilde{j}: L(4) \longrightarrow S^2$. $\tilde{g}$ induces a map $\widetilde{B}^2/\partial\widetilde{B}^2 \longrightarrow M$ which is an isomorphism on H^4 and which pulls back a generator of $H^2(M; \mathbb{Z})$ to the inverse image of $\tilde{\alpha}$ in $H^2(\widetilde{B}^2; \partial\widetilde{B}^2; \mathbb{Z})$. It follows from Lefschetz duality for the oriented manifold pair $(\widetilde{B}^2, \partial\widetilde{B}^2)$ that the invariant for $\tilde{g} \circ \tilde{j}$ in $H^3(L(4); \mathbb{Z})$ is just $p(\tilde{\alpha})^2 \cdot w$, where w is a generator of $H^3(L(4); \mathbb{Z})$ (and $p(\tilde{\alpha})^2 = p(\alpha)^2$ is an integer). If we are careful with the signs, we see that

$$[\tilde{g} \circ \tilde{j}] = (p(\alpha)^2,\ p(\alpha) \pmod 4) \in \mathbb{Z} \times \mathbb{Z}_4 \cong [L(4); S^2].$$

Statement (1) now follows from Proposition 15.18 and Lemma 15.21.

To prove statement (2), recall that

$$W_3^2 \cong W_2^2 \times_{O(2)} O(3)/O(1),$$

i.e., W_3^2 is a 2-disk bundle over $O(3)/O(2) \times O(1) = \mathbb{R}P^2$. We therefore have the pullback square

$$\begin{tikzcd} L(4) = \partial\widetilde{W}_3^2 \arrow[r] \arrow[d, "H"] & \partial W_3^2 \arrow[d] \\ S^2 \arrow[r] & \mathbb{R}P^2 \end{tikzcd},$$

where $H: L(4) \longrightarrow S^2$ is the projection of an S^1-bundle. Since the linear S^1 and $\mathbb{Z}_4$ actions on S^3 commute, we have a map $L(4) \longrightarrow S^2$ and this map can be identified with H; thus, H is essentially the "Hopf map." Clearly,

$$[H] = (1,1) \in \mathbb{Z} + \mathbb{Z}_4 \cong [\partial \widetilde{W}_3^2; S^2].$$

Let $\tilde{h}: \partial \tilde{B}^2 \longrightarrow \partial \widetilde{W}_3^2$ be a lift of h to the double covers. By definition $\tilde{h}' = H \circ \tilde{h}$. Since h is a homotopy equivalence so is $\tilde{h}$. It follows that

$$[\tilde{h}'] = [H \circ \tilde{h}] = (1,1) \in \mathbb{Z} \times \mathbb{Z}_4 = [\partial \tilde{B}^2, S^2].$$

Statement (2) follows easily from this. $\square$

15.24 <u>Remark:</u> The above proposition solves the extension problem in the O(n) case, that is, it shows that $[B; W_+(3)]$ is not empty. Indeed, by Corollary 15.13, there is a homotopy equivalence $B^1 \longrightarrow W_3^1$ which can be covered by a map $h: \partial B^2 \longrightarrow \partial W_3^2$ of circle bundles. If $g: B^2 \longrightarrow W_3^2 \sim \mathbb{R}P^2$ is any map such that $p(g^*(u)) = 1$, then, by the above proposition, $g \circ j \sim h': \partial B^2 \longrightarrow W_3^2$. Hence, h' extends to B^2 and this new map further extends to $f: B \longrightarrow W_+(3)$.

<u>Proof of Theorem 15.1:</u> Suppose that B is modeled on $H_+^1(3)$ and that

(a) B^3 is a $\mathbb{Z}$-acyclic 5-manifold, and

(b) $B^1 \cong \mathbb{R}P^2$.

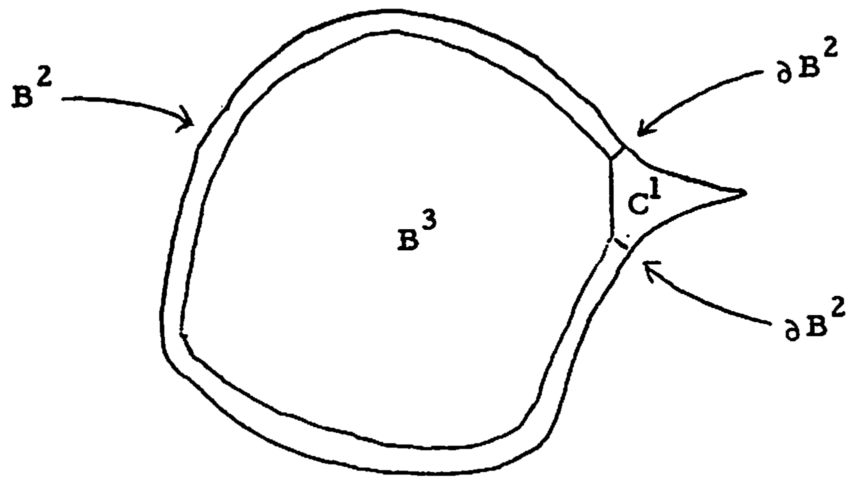
We must compute $[B; W_+(3)]$. Let $\eta = \{\eta^i \longrightarrow B^i; \eta^{ij} \longrightarrow \partial_i \eta^{i+j}\}$ be a normal system for B and let $\eta_3 = \{\eta_3^i \longrightarrow W_3^i; \eta_3^{ij} \longrightarrow \partial_i \eta_3^{i+j}\}$ be the standard normal system for $W_+(3)$. Suppose that f and g are morphisms $B \longrightarrow W_+(3)$. The question is: when is f $\mathfrak{B}_3$-homotopic to g? First of all, by Proposition 9.1, we may assume that f and g are both morphisms of normal systems, i.e., that $f = A(\theta)$ and that $g = A(\lambda)$, where $\theta: \eta \longrightarrow \eta_3$ and $\lambda: \eta \longrightarrow \eta_3$ are $\widetilde{\mathfrak{B}}_3$-morphisms, and where A is the assemble functor.

By Corollary 15.13, $f|_{B^1} \sim g|_{B^1}$ (and both maps are homotopy equivalences). Let $H^1: B^1 \times I \longrightarrow W_3^1$ be such a homotopy. Let

$$C^1 = F_2 \times_{O(2)} \eta^1 \qquad \text{and}$$
$$C_3^1 = F_2 \times_{O(2)} \eta_3^1$$

be the normal cone bundles of B^1 in B and W_3^1 in $W_+(3)$, respectively (F_2 is defined by 6.5). Is $f|_{C^1} \sim g|_{C^1}$? By the Covering Homotopy Theorem, H^1 can be covered by a homotopy $\widetilde{H}^1: \eta^1 \times I \longrightarrow \eta_3^1$ such that $\widetilde{H}^1|_{\eta^1 \times 0} = \theta^1$ (Recall that $f^1|_{C^1} = \times\theta^1: C^1 \longrightarrow C_3^1$.) Any such homotopy $\widetilde{H}^1$ is unique. Indeed, any two such homotopies differ by an automorphism of $\eta^1 \times I$ which is the identity on $\eta^1 \times 0$. But $\eta^1 \times I \longrightarrow B^1 \times I$ is a non-orientable $O(2)$-bundle with nonzero twisted Euler class in $H^2(B^1; \mathbb{Z}^t)$; so by Lemma 15.14 $\widetilde{H}^1$ is unique (since $\mathrm{Aut}\,(\eta^1 \times I) = \Gamma(\eta^{1*} \times I) = \mathbb{Z}_2$). On the other hand, it may not be the case that $\widetilde{H}^1|_{\eta^1 \times I} = \lambda^1$ (they might differ by an automorphism of η^1). Thus, there is an obstruction in $\Gamma(\eta^{1*}) = \mathbb{Z}_2$ to finding a homotopy between $f|_{C^1}$ and $g|_{C^1}$. This obstruction is obviously realized. For suppose that $f|_{C^1} \sim g|_{C^1}$. If $\sigma \in \Gamma(\eta^{1*})$ is the nontrivial automorphism, then σ induces a circle bundle automorphism $\bar{\sigma}: \partial B^2 \longrightarrow \partial B^2$ (recalling that $\partial B^2 \cong \partial_3\partial_2 C^1$). The new map

$g|_{\partial B^2} \circ \bar{\sigma}: \partial B^2 \longrightarrow \partial W_3^2$ extends to a map $B^2 \longrightarrow W_3^2$ by Proposition 15.23. Hence, it extends to a morphism $\bar{g}: B \longrightarrow W_+(3)$ with the property that $\bar{g}|_{C^1}$ is not homotopic to $f|_{C^1}$.



Now suppose that $\times\widetilde{H}^1: C^1 \times I \longrightarrow C_3^1$ is a homotopy between $f|_{C^1}$ and $g|_{C^1}$. This restricts to a homotopy $\partial B^2 \times I \longrightarrow \partial W_3^2$ between $f|_{\partial B^2}$ and $g|_{\partial B^2}$. Consider the problem of extending this to a homotopy $H^2: B^2 \times I \longrightarrow W_3^2$ between $f|_{B^2}$ and $g|_{B^2}$. Let u be a generator of $H^2(W_3^2; \mathbb{Z}^t) \cong \mathbb{Z}$. Let $\alpha_f = (f|_{B^2})^*(u)$ and $\alpha_g = (g|_{B^2})^*(u)$. Let $p: H^2(B^2; \mathbb{Z}^t) \longrightarrow \mathbb{Z}$ be the map which divides out torsion. By Proposition 15.23 and Lemma 15.10,

(1) $$p(\alpha_f) = \pm\, p(\alpha_g) = \pm\, 1,$$

and subject to this condition any such class α_f can occur as the pullback of u. Moreover, again by 15.16, $f|_{B^2}$ and $g|_{B^2}$ are homotopic (not necessarily rel ∂B^2) if and only if

(2) $$\alpha_f = \pm \alpha_g .$$

In other words, if T = torsion $H^2(B^2; \mathbb{Z}^t)$, then $\alpha_f - \alpha_g$ may represent an arbitrary element of $\{\pm 1\} + T/\mathbb{Z}_2 \cong T$.

In fact, if $f|_{\partial B^2} \sim g|_{\partial B^2}$ and if equation (2) is satisfied, then $f|_{B^2}$ and $g|_{B^2}$ are homotopic rel ∂B^2 (i.e., relative the homotopy $\times \tilde{H}^1|_{\partial B^2 \times I} : \partial B^2 \times I \longrightarrow \partial W_3^2$). To see this, notice that the first obstruction to finding such a homotopy is a difference class lying in $H^2(B^2, \partial B^2; \pi_2(\mathbb{R}P^2)) = H^2(B^2, \partial B^2; \mathbb{Z}^t)$. By Lemma 15.15 this group is isomorphic to $\mathbb{Z} + T$, and by condition (2) the obstruction vanishes. There are two further obstructions lying in

(3) $$H^3(B^2, \partial B^2; \pi_3(\mathbb{R}P^2)) \cong H^3(B^2; \partial B^2; \mathbb{Z}) = \mathbb{Z}_2,$$

(4) $$H^4(B^2, \partial B^2; \pi_4(\mathbb{R}P^2)) \cong H^4(B^2; \partial B^2; \mathbb{Z}_2) = \mathbb{Z}_2,$$

where the computation of these cohomology groups is done in Lemma 15.15. (Notice that in (3) the coefficients are untwisted.) By considering a 3-stage Postnikov decomposition of $\mathbb{R}P^2$ it follows, as in the proof of Theorem 15.3, that the indeterminancy of the obstruction lying in (3) is the image of

$$Sq^1 : H^2(B^2, \partial B; \mathbb{Z}_2) \longrightarrow H^3(B^2, \partial B^2; \mathbb{Z}_2)$$

(where $H^3(B^2, \partial B^2; \mathbb{Z}_2)$ is identified with $H^3(B^2, \partial B^2; \mathbb{Z})$). Similarly, if the second obstruction is also zero, the indeterminancy of the final obstruction (lying in (4)) is the image of

$$Sq^2 : H^2(B^2, \partial B^2; \mathbb{Z}_2) \longrightarrow H^4(B^2, \partial B^2; \mathbb{Z}_2).$$

We claim that both Sq^1 and Sq^2 are nonzero and consequently that each indeterminancy is the entire group. To prove this it suffices to consider $(W_3^2, \partial W_3^2)$, since this pair has the same $\mathbb{Z}_2$-cohomology as does $(B^2, \partial B^2)$. But $W_3^2/\partial W_3^2$ is the Thom space of the 2-disk bundle $W_3^2 \longrightarrow \mathbb{R}P^2$ which has nonzero first and second Stiefel-Whitney classes. According to Wu's formula,

$$w_i \cup U = Sq^i(U),$$

where $U \in H^2(W_3^2, \partial W_3^2; \mathbb{Z}_2)$ is the Thom class and w_i is the pullback of the i^{th} Stiefel-Whitney class to W_3^2. It follows from Lefschetz duality for $(W_3^2, \partial W_3^2)$, with $\mathbb{Z}_2$ coefficients, that $w_i \cup U \neq 0$ for $i=1$ or 2. Hence, $Sq^i : H^2(W_3^2, \partial W_3^2; \mathbb{Z}_2) \longrightarrow H^2(W_3^2, \partial W_3^2; \mathbb{Z}_2)$ is nonzero for $i = 1, 2$. Since the same is true for $(B^2, \partial B^2)$, it follows that $f|_{B^2}$ is homotopic to $g|_{B^2}$ rel ∂B^2 if and only if $\alpha_f = \pm \alpha_g$.

Once we have succeeded in constructing a homotopy between f and g on $B^2 \cup C^1$, it automatically extends to $C^2 \cup C^1$, since $C^2 \cong B^2 \times I$. This defines a homotopy $\partial B^3 \times I \longrightarrow \partial W_3^2$ between $f|_{\partial B^3}$ and $g|_{\partial B^3}$. This homotopy extends to a homotopy $B^3 \times I \longrightarrow W_3^3$, since W_3^3 is contractible (it is a 5-disk). Therefore,

$$[B; W_+(3)] \cong \mathbb{Z}_2 \times T,$$

where $\mathbb{Z}_2 = \Gamma(\eta^{1*})$ and where $T = \text{torsion } H^2(B^2; \mathbb{Z}^t)$ measures the difference between α_f and α_g. This proves statement (1) of the theorem.

If $f\colon B \longrightarrow W_+(3)$ is a $\mathcal{B}_3$-morphism, then we know that $f|_{B^1}\colon B^1 \longrightarrow W_3^1$ is a homotopy equivalence and that $f|_{B^2}\colon B^2 \longrightarrow W_3^2$ induces an isomorphism on integral cohomology. Since $p(\alpha_f) = \pm 1$, $f|_{B^2}$ is an isomorphism on cohomology with twisted integer coefficients if and only if $H^2(B^2; \mathbb{Z}^t)$ is torsion free. Statement (2) of the theorem now follows immediately from Theorem 14.8. $\square$

15.25 <u>Remark:</u> The above proof essentially uses the approach of section 9. The theorem can also be proved by using the methods of equivariant attaching. Let B be a $\mathcal{B}_3^1$-space satisfying hypotheses (a) and (b) of 15.1. Let C^1 be the normal cone bundle of B^1 in B. Let $C_{n,3}(C^1)$ denote the set of isomorphism classes of $\mathcal{S}_{n,3}(C^1)$-manifolds. Clearly, any $\mathcal{S}_{n,3}$-manifold over C^1 is isomorphic to a G_n-disk bundle (the normal bundle of the first stratum). Hence, by 5.6, it is determined by a principal $O(1) \times O(2)$-bundle $\xi^1 \longrightarrow B^1$. Let η^1 be the principal $PO(2)$-bundle associated to C^1. By 15.10 (Massey's Theorem), the twisted Euler class of η^1 is $2 \cdot$ generator of $H^2(B^1; \mathbb{Z}^t)$. Since $\eta^1 \cong \xi^1/O(1) \times Z(1)$, it follows that the twisted Euler class of the $O(2)$-bundle $\xi^1/O(1) \longrightarrow B^1$ is a generator of $H^2(B^1; \mathbb{Z}^t)$. If the principal orbit bundle of the $\mathcal{S}_{n,k}(C^1)$-manifold is required to be trivial, then by 5.13 the bundle

$$O(3) \times_{O(1) \times O(2)} \xi^1$$

is also trivial. It follows that $\xi^1/O(2) \longrightarrow B^1$ is the unique nontrivial $O(1)$-bundle over B^1. Since these two facts determine ξ^1, there is a unique object of $\mathcal{C}_{n,k}(C^1)$ with trivial principal orbit bundle; namely, the isomorphism class of

$$N^1 = \{O(n) \times_{O(n-1)} D(n-1,2)\} \times_{O(1) \times O(2)} \xi^1 .$$

Next we consider $\mathcal{B}_{n,k}$-manifolds over the complement $\overline{B - C^1}$. Clearly,

$$\begin{aligned} \mathcal{C}_{n,3}(\overline{B - C^1}) &\cong [\overline{B - C^1}; W_+(3) - C_3^1] \\ &\cong [B^2, W_3^2] \\ &\cong H^2_{odd}(B^2; \mathbb{Z}^t)/\mathbb{Z}_2 \cup H^1(B^2; \mathbb{Z}_2), \end{aligned}$$

where the last isomorphism is via Lemma 15.16. We only want to consider those $\mathcal{B}_{n,3}(\overline{B - C^1})$-manifolds with boundary isomorphic to ∂N^1. By 15.23 these are in one-to-one correspondence with elements $\alpha \in H^2_{odd}(B^2; \mathbb{Z}^t)/\mathbb{Z}_2$ such that $p(\alpha) = \pm 1$. Thus, for every $\alpha \in T$ = torsion $H^2(B^2, \mathbb{Z}^t)$ we get a $\mathcal{B}_{n,3}(\overline{B - C^1})$-manifold $M(\alpha)$ such that $\partial M(\alpha) \cong \partial N^1$.

It remains to consider the different ways of pasting $M(\alpha)$ and N^1 together, i.e., to compute $\mathcal{C}_{n,3}(B; M(\alpha), N^1)$ for each $\alpha \in T$. Let $\mu = \{\mu^2, \mu^3; f^{2,3}: \mu^{2,1} \longrightarrow \partial_2\mu^3\}$ be a $\mathcal{B}_{n,3}$-normal system for ∂N^1, i.e.,

$$\mu^2 = \xi^{1,1} = \partial_0 \xi_2^1 \times_{O(2)} \xi^1$$

$$\mu^3 = \xi^{1,2} = \partial_0 \xi_2^2 \times_{O(2)} \xi^1 .$$

Let ζ be the associated automorphic normal system. Let $\xi = \{\xi^2, \xi^3; f^{2,3}: \xi^{2,1} \longrightarrow \partial_2 \xi^3\}$ be a normal system for $M(\alpha)$ and let $\bar{\zeta}$ be the associated automorphic normal system. By our assumption about

$M(\alpha)$, there is an isomorphism of normal systems $\theta\colon \mu \longrightarrow \xi|_{\partial C^1}$ covering the identity on ∂C^1 (corresponding to some equivariant diffeomorphism over ∂C^1, $\partial N^1 \longrightarrow \partial M(\alpha)$). According to the remarks following Theorem 11.4, $\mathcal{C}_{n,3}(B; M(\alpha), N^1)$ is isomorphic to the double coset space

$$\pi_0\,\Gamma(\zeta^1)\backslash\pi_0\,\Gamma(\zeta)/\pi_0\,\Gamma(\bar{\zeta})$$

where $\zeta^1 \subset \xi^{1*}$ is the subbundle with fibre $O(1) \times Z(2)$ and where the map $\tau\colon \Gamma(\bar{\zeta}) \longrightarrow \Gamma(\zeta)$ is the composition

$$\Gamma(\bar{\zeta}) \longrightarrow \Gamma(\bar{\zeta}|_{\partial C^1}) \cong \Gamma(\zeta).$$

First we compute the cokernel of τ. Consider the commutative diagram,

(1)

$$\begin{tikzcd}
0 \arrow[r] & \pi_0\,\Gamma(\mu^{3*})_1 \arrow[r] \arrow[d] & \pi_0\,\Gamma(\zeta) \arrow[r] \arrow[d, "\tau"] & \pi_0\,\Gamma(\mu^{2*}) \arrow[d] \\
0 \arrow[r] & \pi_0\,\Gamma(\xi^{3*})_1 \arrow[r] & \pi_0\,\Gamma(\bar{\zeta}) \arrow[r] & \pi_0\,\Gamma(\xi^{2*})
\end{tikzcd}$$

The rows are the fibration sequences defined in Theorem 10.7 and the vertical maps are defined by restriction to ∂C^1 composed with θ. The subscript 1 means the subspace consisting of those sections which are the identity section on the boundary, i.e.,

(2) $$\pi_0\,\Gamma(\mu^{3*})_1 \cong [(\partial_1 B^3, \partial_2\partial_1 B^3); (O(3), 1)]$$

(3) $$\pi_0\,\Gamma(\xi^{3*})_1 \cong [(B^3, \partial_2 B^3); (O(3), 1)].$$

It follows from Lemma 15.8 that the spaces of sections $\Gamma(\mu^{2*})$ and $\Gamma(\xi^{2*})$ each consist of exactly four points. Thus zeroes appear on the left in diagram (1) since both of these spaces are discrete. It follows from an argument in the proof of 15.16, that the image of $\Gamma(\bar{\zeta})$ in $\Gamma(\xi^{2*})$ is exactly the two sections which are identically +1 or -1 in each fibre. Similarly, the image of $\Gamma(\zeta)$ in $\Gamma(\mu^{2*})$ is just ± 1. Thus, in diagram (1) the right-hand vertical map $\pi_0 \Gamma(\xi^{2*}) \longrightarrow \pi_0 \Gamma(\mu^{2*})$ takes the image of $\pi_0 \Gamma(\bar{\zeta})$ bijectively to the image of $\pi_0 \Gamma(\zeta)$.

A routine but fairly complicated argument using obstruction theory and cofibration sequences shows that

(4) $\qquad \pi_0 \Gamma(\mu^{3*})_1 \cong \mathbb{Z}_2 + \mathbb{Z}_2$

(5) $\qquad \pi_0 \Gamma(\xi^{3*})_1 \cong \mathbb{Z}_2$

(6) $\qquad \text{cokernel } [\pi_0 \Gamma(\xi^{3*})_1 \longrightarrow \pi_0 \Gamma(\mu^{3*})_1] \cong \mathbb{Z}_2 .$

Combining (6) with the above remarks, it follows that in diagram (1)

$$\begin{aligned} \text{cokernel } \tau &= \pi_0 \Gamma(\zeta)/\pi_0 \Gamma(\bar{\zeta}) \\ &\cong \mathbb{Z}_2 . \end{aligned}$$

Finally we claim that the image of $\pi_0 \Gamma(\zeta^1)$ in $\pi_0 \Gamma(\zeta)$ is contained in the image of $\Gamma(\bar{\zeta})$. To see this first notice that the trivial $O(1) \times Z(2)$-bundle ζ^1 has precisely four sections $(Z(2) \cong O(1))$, which are identically one of the following four matrices in each fibre:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In $\Gamma(\zeta)$ the first two are clearly homotopic as are the last two. Since the composition $\Gamma(\zeta^1) \longrightarrow \Gamma(\zeta) \longrightarrow \Gamma(\mu^{2*})$ takes the section -1 to -1, it follows that the image of $\Gamma(\zeta^1)$ is contained in the image of τ. Therefore, $C_{n,3}(B; M(\alpha), N^1) \cong \pi_0\Gamma(\zeta^1)\backslash\pi_0\Gamma(\zeta)/\pi_0\Gamma(\bar{\zeta}) \cong \mathbb{Z}_2$, from which it follows that

$$C_{n,3}(B) \cong T \times \mathbb{Z}_2.$$

This completes an alternate proof of Theorem 15.1.

Although Theorem 15.1 classifies $O(n)$-manifolds over B, there is still the possibility that two such $\mathscr{S}^1_{n,3}(B)$-manifolds might be $\mathscr{S}^1_{n,3}$-isomorphic without being $\mathscr{S}^1_{n,3}(B)$-isomorphic. Recall that to solve this problem one considers the action of $\mathrm{Diff}\,(B)$ on $[B; W_+(3)]$.

Let $f \in [B; W_+(3)] \cong T \times \mathbb{Z}_2$. We are going to consider the effect of $\mathrm{Diff}\,(B)$ on the $\mathbb{Z}_2$ factor. First recall how to alter f by the $\mathbb{Z}_2$ factor. Let $\sigma: C^1 \longrightarrow C^1$ be the nontrivial bundle automorphism which is multiplication by -1 on each fibre. Let $\bar{\sigma} = \sigma|_{\partial B^2}: \partial B^2 \longrightarrow \partial B^2$. The composition $f|_{\partial B^2} \circ \bar{\sigma}$ extends to a map $g': B^2 \longrightarrow W^2_3$ and hence $f|_{C^1} \circ \sigma: C^1 \to C^1$ extends to a new map $g: B \longrightarrow W_+(3)$ which is not homotopic to f.

On the other hand suppose that $\bar{\sigma}$ extends to a self diffeomorphism $\varphi^1: B^2 \longrightarrow B^2$. Further suppose that $\varphi^1 \cup \sigma: B^2 \cup C^1 \longrightarrow B^2 \cup C^1$ extends to a diffeomorphism $\varphi: B \longrightarrow B$ (this further supposition will always be the case, for example, if B^3 is a 5-disk). If such a φ exists, then for some g, not homotopic to f, the following diagram commutes.

$$\begin{tikzcd} B \arrow[d, "\varphi"'] \arrow[dr, "g"] & \\ B \arrow[r, "f"'] & W_+(3) \end{tikzcd}$$

Hence, $f^*\Sigma(n,3)$ and $g^*\Sigma(n,3)$ will be $\mathscr{B}_{n,3}$-equivalent. It is not unreasonable to suspect that the above situation does, in fact, occur.

It follows from section 6 that the circle bundle $\partial B^2 \longrightarrow B^1$ is equivalent to $O(3)/O(1)\times O(1)\times O(1) \longrightarrow O(3)/O(1)\times O(2)$. The bundle automorphism $\overline{\sigma}\colon \partial B^2 \longrightarrow \partial B^2$ can be described more explicitly as follows. Let

$$d = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in O(3).$$

Since d is in the normalizer of $O(1)\times O(1)\times O(1)$, right translation by d, defined by $R_d(g[O(1)\times O(1)\times O(1)]) = g\,d[O(1)\times O(1)\times O(1)]$ is a well defined diffeomorphism of $O(3)/O(1)\times O(1)\times O(1)$. Clearly, $R_d = \overline{\sigma}$. The following conjecture should not be difficult to decide.

15.26 <u>Conjecture:</u> $R_d = \overline{\sigma}\colon \partial B^2 \longrightarrow \partial B^2$ is isotopic to the identity.

In view of the above remarks this would imply the following

15.27 <u>Conjecture:</u> If B satisfies the hypothesis of 15.1 then the number of $\mathscr{B}^1_{n,3}$-isomorphism classes of $O(n)$-manifolds with orbit space diffeomorphic to B is less than or equal to $|T|$, where $T = \text{torsion } H^2(B^2; \mathbb{Z}^t)$.

16. Equivariant Bordism

Which homotopy spheres admit G_n-actions modeled on $k\rho_n$? In this section we indicate some methods for answering this type of question. Our results are of an extremely preliminary nature. The first step in trying to answer this intriguing question should be to decide whether or not such a homotopy sphere must bound a π-manifold.

Two $\mathscr{B}_{n,k}$-manifolds M and N are said to be $\underline{\mathscr{B}_{n,k}\text{-cobordant}}$ if there is a $\mathscr{B}_{n,k}$-manifold W such that $\partial W = M \cup N$. If $f: M_\# \longrightarrow H_+(k) \times_{G_k} EG_k$ and $g: N_\# \longrightarrow H_+(k) \times_{G_k} EG_k$ are classifying maps for M and N, then finding a $\mathscr{B}_{n,k}$-cobordism between them is the same thing as finding a morphism $h: W_\# \longrightarrow H_+(k) \times_{G_k} EG_k$ such that $h|_{M_\#} = f$ and $h|_{N_\#} = g$.

Suppose that a $\mathscr{B}_{n,k}$-manifold M is also a homotopy sphere. If M is $\mathscr{B}_{n,k}$-cobordant to the restriction of the linear action $k\rho_n$ + trivial to the unit sphere, and if the cobordism is parallelizable, then M clearly bounds a π-manifold. (Moreover, the π-manifold admits a G_n-action modeled on $k\rho_n$.) Since for $n > k$ the principal orbit bundle of M is trivial, we might also wish to assume that the $\mathscr{B}_{n,k}$-cobordism has trivial principal orbit bundle. A pertinent observation along this line is due to Bredon [B2].

16.1 <u>Proposition</u> (Bredon): For $n \geq k$, if X is a $\mathscr{B}_{n,k}$-manifold with trivial principal orbit bundle, then X is a π-manifold if and only if $X_\#$ is a π-manifold (in view of section 2, it makes sense to say that the stable tangent bundle of $X_\#$ is trivial).

Although Bredon states this proposition only for actions modeled on $2\rho_n$, the simple proof, which he gives, is valid in general. Further relationships between the tangent bundle of the principal orbit bundle and the tangent bundle of an arbitrary $\mathscr{S}_{n,k}$-manifold can be derived.

Next, let us consider the O(n)-actions of the previous section. Let M^{3n-1} be a $\mathscr{S}^1_{n,3}(B)$-manifold, where B satisfies the hypothesis of Theorem 15.1. Let $f: B \longrightarrow W_+(3)$ classify M. We may assume that $f|_{B^1}: B^1 \longrightarrow W^1_3$ is a diffeomorphism and hence that $f|_{\partial B^2}: \partial B^2 \longrightarrow \partial W^1_3$ is also a diffeomorphism. Since $f|_{B^2}: (B^2, \partial B^2) \longrightarrow (W^2_3, \partial W^2_3)$ is a $\mathbb{Z}$ cohomology equivalence, it is in particular a degree one map. Since both B^2 and W^2_3 are π-manifolds, $f|_{B^2}$ can be covered by a map of stable normal bundles, i. e., $f|_{B^2}$ is a normal map. The following proposition suggests an attack on the problem of whether or not M equivariantly bounds a π-manifold.

16.2 <u>Proposition:</u> Let B be a $\mathcal{B}^1_3$-space satisfying the hypothesis of 15.1 and let M be a $\mathscr{S}^1_{n,3}(B)$-manifold classified by $f: B \longrightarrow W_+(3)$, as above. Suppose that $f|_{B^2}: (B^2, \partial B^2) \longrightarrow (W^2_3, \partial W^2_3)$ is normally cobordant rel ∂B^2 to a diffeomorphism. Then there is a $\mathscr{S}^1_{n,3}$-manifold U such that

(1) $\partial U = M$,

(2) U is a π-manifold,

(3) U has trivial principal orbit bundle, and

(4) U has exactly one fixed point.

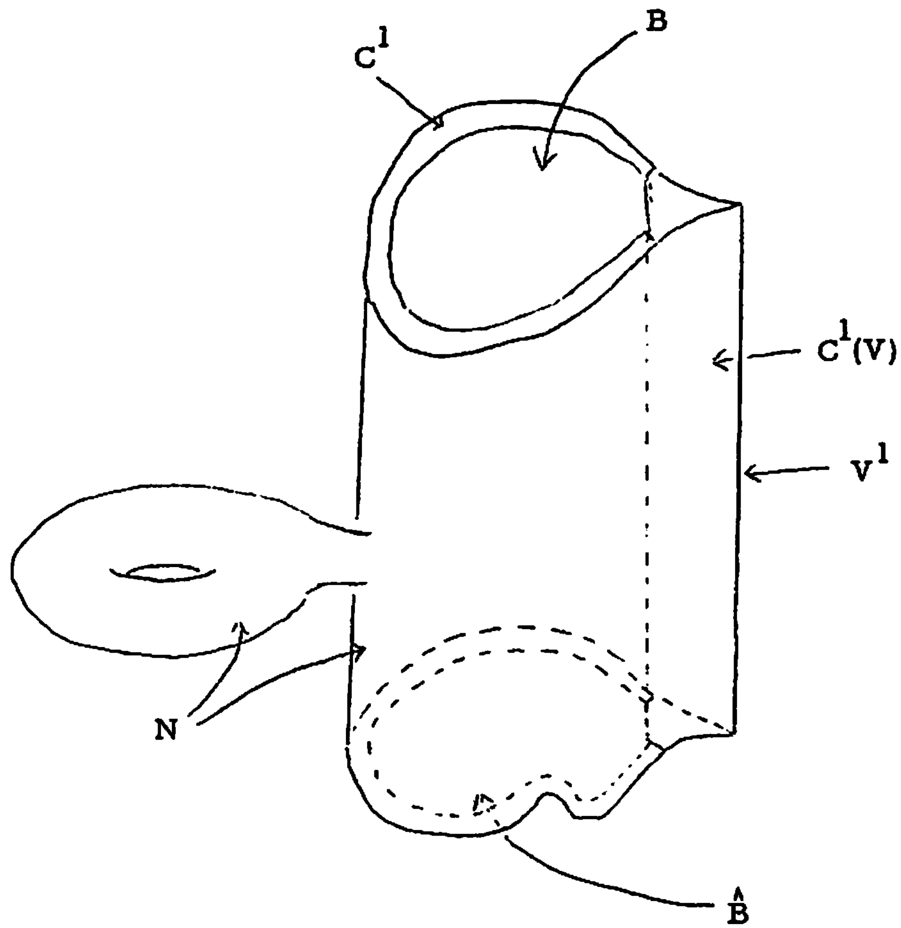
<u>Proof</u>: Let $H: (N, T) \longrightarrow (W_3^2 \times I, \partial W_3^2 \times I)$ be the normal cobordism of $f|_{B^2}$, where $T = \partial B^2 \times I$. We shall construct a cobordism $F: V \longrightarrow W_+(3)$ such that $F|_B = f$ and $F|_{\hat{B}}$ is a diffeomorphism, where $\hat{B}$ denotes the other end of the cobordism.

(i) The first stratum: $V^1 = B^1 \times I$; the normal cone bundle of V^1 is $C^1(V) = C^1(B) \times I$. A map $F': C^1(V) \longrightarrow C_3^1 \times I$ is defined by $F' = f|_{C^1(B)} \times I$.

(ii) The second stratum: $V^2 = N$, $C^2(V) = N \times I$ and $F': C^2(V) \longrightarrow C_3^2 \times I$ is defined by $F' = H \times 1$.

(iii) The third stratum: the 6-manifold V^3 is supposed to be a bordism of B^3. Thus, one part of its boundary must be $\partial(C^2(V) \cup C^1(V))$, while the other part is the disjoint union $B^3 \cup W_3^3$, where W_3^3 is attached to $C^1(V)$ by the diffeomorphism $f|_{\partial_3 C^1(B)}: \partial_3 C^1(B) \longrightarrow \partial_3 C_3^1$, and to $C^2(V)$ using the diffeomorphism which is the other end of the normal cobordism H. Hence, we have already constructed a closed 5-manifold $\partial V^3 = \partial(C^1(V) \cup C^2(V)) \cup B^3 \cup W_3^3$ and a map $G: \partial V^3 \longrightarrow S^5 = \partial(W_3^3 \times I)$ defined by using F' on $\partial(C^1(V) \cup C^2(V))$, $f|_{B^3}$ on B^3, and the identity on W_3^3. Clearly, G is a normal map. We can do surgery on G to a homotopy equivalence. Therefore, ∂V^3 bounds a π-manifold V^3 and G extends to a normal map $F': (V^3, \partial V^3) \longrightarrow (D^6, S^5) = (W_3^3, \partial W_3^3) \times I$.

The three maps denoted by F' piece together to define :
$F': V \longrightarrow W_+(3) \times I$. If $p: W_+(3) \times I \longrightarrow W_+(3)$ is projection on the first factor, then $F = p \circ F': V \longrightarrow W_+(3)$ is the desired cobordism.



$F^*\Sigma(n,3)$ is a $\mathscr{S}^1_{n,3}$-cobordism between M and $(F|_{\hat{B}})^*\,\Sigma(n,3)$, where $\hat{B}$ is the other end of V. Since V is a π-manifold, it follows from 16.1 that $F^*\Sigma(n,3)$ is a π-manifold. There is a natural map $\varphi{:}: (F|_{\hat{B}})^*\Sigma(n,3) \longrightarrow \Sigma(n,3)$ defined by the pullback square. Since $F|_{\hat{B}}$ is a diffeomorphism, φ is an $O(n)$-equivariant diffeomorphism. Therefore, set

$$U = F^{*}\Sigma(n, 3) \underset{\varphi}{\cup} D(n, 3).$$

U clearly has the four desired properties. □

16.3 Remark: Since $f|_{B^2}: (B^2, \partial B^2) \longrightarrow (W_3^2, \partial W_3^2)$ is a homology equivalence, it seems plausible to suppose that it is always normally cobordant to a diffeomorphism; and hence that for such a B any $\mathscr{B}^1_{n,3}(B)$-manifold equivariantly bounds a π-manifold. In fact in this case (of O(n)-actions modeled on $3\rho_n$), by doing surgery on the normal cobordism to a homology h-cobordism, it should be possible to prove that the π-manifold is actually contractible (at least when B is 1-connected and $H^2(B; \mathbb{Z}^t)$ is torsion free). Thus, we conjecture that the only homotopy (3n-1)-sphere which admits an O(n)-action modeled on $3\rho_n$ is the standard sphere.

Suppose a homotopy sphere M admits a G_n-action modeled on $k\rho_n$, $n \geq k$. We can ask: does M bound a $\mathscr{B}_{n,k}$ π-manifold? (That is, does the action on M extend to a modeled G_n-action on a π-manifold cobounding M?) The answer is yes when k=2 (see [B2]). However, what the answer is in general is unclear.

An $\underline{s}$ - $\underline{\mathscr{B}_{n,k}}$-cobordism between M and N is a $\mathscr{B}_{n,k}$-cobordism W which is also an s-cobordism. Thus, if $\dim M > 4$, an s-$\mathscr{B}_{n,k}$-cobordism is essentially a modeled G_n-action on $M \times I$. (Notice the distinction between our definition which only requires the inclusion $M \subset W$ to be a simple homotopy equivalence and the requirement that the inclusion is an isovariant simple homotopy equivalence.)

On the orbit space level an s-$\mathcal{B}_{n,k}$-cobordism corresponds to the notion of "stratified cohomology h-cobordism." In other words, if W is an s-$\mathcal{B}_{n,k}$-cobordism, then, as in Theorems 13.1 and 14.2, the inclusions $M^i_\# \subset W^i_\#$ induce cohomology isomorphisms (with $\mathbb{Z}$ coefficients in the $U(n)$ and $Sp(n)$ cases and $\mathbb{Z}_{(2)}$ coefficients in the $O(n)$ case). Also, if the principal orbit bundle of M is trivial, then so is the principal orbit bundle of W, since the inclusion $M^k_\# \subset W^k_\#$ induces a $\mathbb{Z}$ cohomology isomorphism (assume $n > k$ in the $O(n)$ case). Conversely, just as in 13.1 and 14.1, we have the following

16.4 Proposition: Let B be a simply connected $\mathfrak{B}_k$-space and let V be a $\mathfrak{B}_k$-cobordism of B (the definition of this is left to the reader). Let $j\colon B \longrightarrow V$ be the inclusion and suppose that

(1) $(j|_{B^i})^*\colon H^*(V^i) \longrightarrow H^*(B^i)$ is an isomorphism with integral coefficients (and with twisted integral coefficients, as well, in the $O(n)$ case).

(2) V^k is simply connected.

Then, for $n \geq k$, any $\mathcal{B}_{n,k}(V)$-manifold W is diffeomorphic to $W|_V \times I$ (i.e., W is an s-$\mathcal{B}_{n,k}$-cobordism).

The proof is essentially the same as that of Theorem 13.1 and that of Theorem 14.1.

Let $\Theta^d(k\rho_n, \ell)$ be the set of s-$\mathcal{B}^d_{n,k}$-cobordism classes of modeled $G^d(n)$-actions on homotopy $(dnk+\ell)$-spheres. Let Θ_m be the group of homotopy m-spheres. There is a function

$$T : \Theta^d(k\rho_n, \ell) \longrightarrow \Theta_{dnk+\ell}$$

which forgets the action. It follows from the previous proposition that $\Theta^d(k\rho_n, \ell)$ is essentially determined by the "stratified cohomology h-cobordism classes" of the various classifying maps. In other words, if the homotopy spheres M and N are $\mathcal{S}_{n,k}$-manifolds classified by $f: B \longrightarrow H_+(k)$ and $g: D \longrightarrow H_+(k)$, respectively, then for M and N to be equivalent in $\Theta^d(k\rho_n, \ell)$ it is sufficient that there is a $\mathcal{B}_k$-cobordism V between B and D, satisfying conditions (1) and (2) of 16.5, and that there is a morphism $F: V \longrightarrow H_+(k)$ such that $F|_B = f$ and $F|_D = g$ (in the $U(n)$ and $Sp(n)$ cases this condition is also necessary).

Studying this relationship (e.g. studying T) between the differential topological invariants of the orbit space and the differential structure of the ambient manifold is one of the most exciting prospects in this area of transformation groups. For example, in the knot manifold case, $\Theta^d(2\rho_n, \ell)$, the index or arf invariant of the orbit knot $B^0 \subset \partial B$ (with a framing determined by the classifying map) is the same as the index or arf invariant of a $\mathcal{S}^d_{n,2}$-manifold cobounding the homotopy sphere. Working out this relationship for $k > 2$ will surely prove to be interesting.

V. FURTHER QUESTIONS AND REMARKS

We shall first mention two ways of applying the ideas of Chapter II. In the case of actions modeled on $2\rho_n$ both of these applications are due to Bredon (in [B2]), and his proofs extend easily to the general case. The first application is to show how actions modeled on $k\rho_n$ $(n \geq k)$ can be embedded in a linear representation of a relatively small codimension. As for the second application, if M is a $\mathscr{S}^1_{2n,k}(B)$-manifold, then by restricting the $O(2n)$-action to $U(n) \subset O(2n)$, we can give M the structure of a $\mathscr{S}^2_{n,k}$-manifold. We will show how $M/U(n)$ can be obtained from B and from a classifying map by using a pullback construction (and similarly for the restriction of a $U(2n)$-action to $Sp(n)$).

<u>An embedding theorem:</u>

<u>Theorem</u> (Bredon): Let M be a $\mathscr{S}^d_{n,k}(B)$-manifold $(n \geq k)$, and suppose that the principal orbit bundle of M is trivial. Suppose further that there is a smooth embedding $B \longrightarrow \mathbb{R}^m$. Then M equivariantly embeds in the linear representation $M^d(n,k) \times \mathbb{R}^m$ (where $G^d(n)$ acts trivially on $\mathbb{R}^m$).

<u>Proof:</u> By the Classification Theorem (Theorem 8.1) there is a morphism $f\colon B \longrightarrow H_+(k)$ such that $M \cong f^*M^d(n,k)$, where $f^*M^d(n,k) \subset B \times M^d(n,k)$. Since B embeds in $\mathbb{R}^m$, the result follows. $\square$

<u>Remark:</u> If B is, say, contractible, then it is probably true that B embeds in Euclidean space in codimension one. This would imply that if

the $\mathscr{B}^d_{n,k}(B)$-manifold M is a homotopy sphere, then M equivariantly embeds in a linear representation in codimension $p+1$, where $p = \dim H^d(k)$.

Actually there is some reason to believe that this embedding theorem can be substantially improved. We conjecture that if B is contractible, then any $\mathscr{B}^d_{n,k}(B)$-manifold M embeds in codimension $dk+1$ in a linear action on a sphere.

Restricting actions: One way to deepen our understanding of smooth G-actions is to study the restriction of the action to various subgroups. In our situation it is natural to consider the restriction of a modeled O(2n)-action to the subgroup $U(n) \subset O(2n)$. It is clear that if $O(2n) \times M \longrightarrow M$ is modeled on $k\rho_{2n}$, then the restricted action $U(n) \times M \longrightarrow M$ is modeled on $k\rho_n$. (Similar statements are true for the restriction of a U(2n)-action to $Sp(n)$ or for the restriction of an O(4n)-action to $Sp(n)$).

If B is the orbit space of an O(2n)-manifold M, then what does $M/U(n)$ look like? If $x = (x_{ij})$ is a positive semidefinite complex hermitian matrix, notice that the matrix

$$Re(x) = (Re(x_{ij}))$$

is positive semidefinite symmetric (where $Re(u+iv) = u$). Thus, there is a map $\pi_2 : H^2_+(k) \longrightarrow H^1_+(k)$ defined by $\pi_2(x) = Re(x)$. The following observation answers our question.

Proposition (Bredon): Let M be an O(2n)-manifold modeled on $k\rho_{2n}$, $2n \geq k$. Let $B = M/O(2n)$ and let $B' = M/U(n)$. Suppose that the principal orbit bundle of M is trivial and that $f: B \longrightarrow H_+^1(k)$ classifies M as a $\mathscr{S}^1_{2n,k}(B)$-manifold. Let $\overline{H_+^2(k)^{(r)}}$ denote the set of positive semidefinite hermitian matrices of rank less than or equal to $r = \min(k, n)$. Then the following diagram is a pullback square,

$$\begin{tikzcd} B' \arrow[r, "f'"] \arrow[d] & \overline{H_+^2(k)^{(r)}} \arrow[d, "\pi_2"] \\ B \arrow[r, "f"'] & H_+^1(k) \end{tikzcd},$$

where as a U(n)-manifold, $M \cong f'^* M^2(n,k)$.

This same idea can be used to construct the orbit space of the restriction of a modeled G_n-action to any subgroup $H \subset G_n$. Thus, the following diagram is a pullback square:

$$\begin{tikzcd} M/H \arrow[r] \arrow[d] & M(n,k)/H \arrow[d] \\ M/G_n = B \arrow[r] & H_+(k) \end{tikzcd}$$

where $M/H \longrightarrow B$ and $M(n,k)/H \longrightarrow H_+(k)$ are the natural maps.

Two uses of this result are envisioned. First, it should enable us to recognize when, say, a U(n)-action extends to an O(2n)-action (see [B2]). Secondly, it should be useful in determining the differential structure of the ambient manifold. For example, if we consider the case where $H \subset G_n$ is the maximal torus, it may be easier to relate the differential structure of M to M/H than to M/G_n.

Actions of an arbitrary compact connected Lie group: It should be evident that the definitions, as well as the statements of most of the theorems of this paper, can be reformulated in a more general context. First, in many of our results it should be possible to remove the hypothesis that "n be greater than or equal to k" with only the obvious modifications of the conclusions. Secondly, it is our firm belief that appropriate versions of the Covering Homotopy Theorem, the Classification Theorem and of the results about the cohomology of the orbit strata of actions on homology spheres are all true for smooth G-actions, where G is an arbitrary compact connected Lie group. There are several difficulties in carrying out this program, which we indicate below.

If $\psi: G \longrightarrow O(m)$ is a representation, then, just as before, we can define a G-action modeled on ψ to be a smooth G-action which has the same orbit types and slice representations as does ψ. It should be possible to prove regularity theorems, similar to 12.1, to the effect that "generic" G-actions are precisely those which are modeled on some linear representation ψ (the Hsiangs have made considerable progress in this direction).

In view of Theorem 1.1, an action modeled on ψ always has a local semialgebraic model $\mathbb{R}^m/\psi$. The Covering Homotopy Theorem is almost certainly true for actions modeled on ψ, although its proof will probably require an analysis of the structure of the local model $\mathbb{R}^m/\psi$. One of the primary problems is to study the action of the centralizer of ψ on the orbit space $\mathbb{R}^m/\psi$, in the hope of proving some analog of Theorem 1.7. In what sense (if any) is this action a "linear" action on $\mathbb{R}^m/\psi$ and do all "linear" isomorphisms of $\mathbb{R}^m/\psi$ come from the action of the centralizer? Also, a theory relating equivariant jet extensions and equivariant vector fields to jet extensions and vector fields on the orbit space should be developed. (Bierstone in [Bi2] has recently used the idea of lifting vector fields on $\mathbb{R}^m/\psi$ to equivariant vector fields on $\mathbb{R}^m$ to prove the Covering Isotopy Theorem for G-actions with finite isotropy subgroups.)

The culmination of this program would be to prove an analog of the Classification Theorem for G-actions modeled on ψ. Proving this would involve giving a description of the possible equivariant normal bundles and formulating the corresponding notion of normal system, as well as describing the strata of $\mathbb{R}^m/\psi$. If such a theorem is true, then results similar to those in sections 13 and 14, about the cohomology of the strata of the orbit space of an action on a homology sphere, should also be available.

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ABSTRACT

Let $G_n = O(n)$, $U(n)$, or $Sp(n)$. A smooth G_n-action is said to be modeled on $k\rho_n$ if it has the same orbit types and slice representations as does the linear action $k\rho_n: G_n \times M(n,k) \longrightarrow M(n,k)$, where $M(n,k)$ denotes the real vector space of $(n \times k)$-matrices with coefficients in the appropriate division ring (of real, complex or quaternion numbers). The orbit space of such an action inherits a natural functional structure; and as a functionally structured space such an orbit space is locally isomorphic to the product of euclidean space with a space of positive semidefinite hermitian matrices. Taking into account this local structure, we prove a smooth version of Palais' Covering Homotopy Theorem for G_n-actions, modeled on $k\rho_n$, $n \geq k$. (The assumption $n \geq k$ is made throughout this paper.)

This result is used to classify compact G_n-manifolds, modeled on $k\rho_n$, over a given orbit space B. Suppose that M is a G_n-manifold modeled on $k\rho_n$, that $M/G_n \cong B$, and that the principal orbit bundle of M is trivial. Let $H_+(k)$ denote the space of positive semidefinite $(k \times k)$ hermitian matrices (with coefficients in the appropriate division ring). Then there is a strata-preserving morphism of functionally structured spaces $f: B \longrightarrow H_+(k)$ such that $M \cong f^*M(n,k) = \{(b,x) \in B \times M(n,k) \mid f(b) = x^* \cdot x\}$, where x^* denotes the conjugate transpose of x. Furthermore, if the identification of the orbit space with B is fixed, then, up to an equivariant diffeomorphism inducing the identity on B, such G_n-manifolds are class-

ified by homotopy classes of strata-preserving morphisms $f: B \longrightarrow H_+(k)$ of functionally structured spaces (modulo a choice of a homotopy class of trivialization of the principal orbit bundle). A similar result is true when the principal orbit bundle is nontrivial. Only one part of the Classification Theorem is not obvious: the existence of a morphism $f: B \longrightarrow H_+(k)$ such that $M \cong f^* M(n,k)$. The construction of f is based on an explicit description of the strata of $H_+(k)$ and on the fact that a "normal system" can be chosen for any compact G_n-manifold which is modeled on $k\rho_n$. (Roughly, a "normal system" is a collection consisting of an invariant tubular neighborhood for each orbit type together with maps which show how these tubular neighborhoods "match up" on the common intersections.) The theory of normal systems can also be used to obtain information about $\pi_0 \operatorname{Diff}_1^{G_n}(M)$, the group of isotopy classes of equivariant self-diffeomorphisms of M which cover the identity on M/G_n.

The Classification Theorem is applied to the study of actions $G_n \times \Sigma \longrightarrow \Sigma$, modeled on $k\rho_n$, where Σ is required to be a homology sphere. First of all, if $n \geq k$, we show that the principal orbit bundle of Σ is trivial. Let $B = \Sigma/G_n$ and let $B^{(i)}$ denote the image of the orbits of type G_n/G_{n-i} in B. If $f: B \longrightarrow H_+(k)$ is a classifying map, then $f|_{B^{(i)}}: B^{(i)} \longrightarrow H_+(k)^{(i)}$ induces a cohomology isomorphism for each $i > 0$ (with $\mathbb{Z}$ coefficients in the $U(n)$ and $Sp(n)$ cases and $\mathbb{Z}_{(2)}$ coefficients in the $O(n)$ case). In the unitary and symplectic cases the converse is true: if $f|_{B^{(i)}}$ induces a cohomology isomorphism for $i > 0$ and if B^0 is a homology sphere, then $f^*M(n,k)$ is a homology sphere. A partial converse is true in the orthogonal case. Finally, these results are used in a detailed study of G_n-actions, modeled on $3\rho_n$, on homology $(3n-1)$-spheres.