COXETER GROUPS AND ASPHERICAL MANIFOLDS

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1. Contractible manifolds and aspherical manifolds.

Suppose that M^n is a compact, contractible n-manifold with boundary. These assumptions imply that the boundary of M has the homology of an (n-1)-sphere; however, they do not imply that it is simply connected. If $n \geq 3$, then for M^n to be homeomorphic to a disk it is obviously necessary that ∂M be simply connected. For $n \geq 5$ this condition is also sufficient (cf. [5], [13], [18], [19]). On the other hand, for $n \geq 4$, there exist examples of such M^n with non-simply connected boundary (cf. [11], [12], [14]). In fact, if $n \geq 6$, then the fundamental group of the boundary can be any group G satisfying $H_1(G) = 0 = H_2(G)$ (cf. [9]).

A non-compact space W is <u>simply connected at</u> ∞ if every neighborhood of ∞ (i.e., every complement of a compact set) contains a simply connected neighborhood of ∞. Suppose W is a locally compact, second countable, Hausdorff space with one end (i.e., it is connected at ∞). Then W can be written as an increasing union of compact sets $W = \bigcup_{i=1}^{\infty} C_i$ where $C_1 \subset C_2 \subset \subset \ldots$, and where each $W - C_i$ is connected. The space W is <u>semi-stable</u> if the inverse sequence

$$\pi_1 (W-C_1) \leftarrow \pi_1 (W-C_2) \leftarrow \cdots$$

satisfies the Mittag-Leffler Condition, i.e., if there exists a subsequence of epimorphisms. (This condition is independent of the choice of C_i.) If W is semi-stable, then the isomorphism class of the inverse limit $\pi_1^{\infty}(W) = \varprojlim \pi_1(W-C_i)$ is independent of all choices (including base points). The space W is simply connected at ∞ if and only if it is semi-stable and $\pi_1^{\infty}(W)$ is trivial (cf. [6], [7], [17]).

*Partially supported by NSF grant MCS-8108814(A01).

Now suppose that W^n is an open contractible n-manifold. If $n \geq 3$, then for W^n to be homeomorphic to $\mathbb{R}^n$ it is obviously necessary that it be simply connected at ∞. For $n \geq 4$ this condition is also sufficient (cf. [5],[20]). If W^n is the interior of a compact manifold M^n, then, clearly, W^n is semi-stable and $\pi_1^\infty(W) \cong \pi_1(\partial M)$. Hence, in view of our previous remarks, there exist examples which are not simply connected at ∞, for any $n \geq 4$. (For $n = 3$, there is a well-known example of Whitehead [22] of an open contractible 3-manifold which is not simply connected at ∞.)

A space is <u>aspherical</u> if its universal cover is contractible.

Aspherical manifolds arise naturally in a variety of geometric contexts. In such contexts the proof that the manifold is aspherical usually consists of a direct identification of its universal cover with Euclidean space. As examples we have: 1) the universal cover of a Riemann surface of genus > 0 is either the plane or the interior of the disk, more generally, 2) if M^n is any complete manifold of non-positive sectional curvature then the exponential map $\exp : T_xM \to M$ (at any point $x \in M$) is a covering projection; hence, the universal cover is diffeomorphic to $T_xM \cong \mathbb{R}^n$, and 3) if G is any Lie group with maximal compact subgroup K and if $\Gamma \subset G$ is any torsion-free discrete subgroup, then the universal cover of the manifold $\Gamma\backslash G/K$ is G/K which is diffeomorphic to Euclidean space. On the basis of such examples some people believed the following well-known conjecture (cf. [7], [8; p. 423]).

CONJECTURE. <u>The universal cover of any closed aspherical manifold is homeomorphic to Euclidean space</u>.

Of course, the issue here is not the existence of exotic contractible manifolds (they exist), but rather the existence of exotic contractible manifolds which simultaneously admit a group of covering transformations with compact quotient. Some positive results (i.e., non-existence results) have been obtained, e.g., in [6], [7], [10].

This paper, which is an expanded version of my lecture, is basically an exposition of some of the results of [3]. We shall discuss a method of [3] of using the theory of Coxeter groups to construct a large number of new examples of closed aspherical manifolds. Although the construction is quite classical, its full potential had not been realized previously. The most striking consequence of the construction is the existence of counterexamples to the above conjecture in each dimension ≥ 4.

In Section 2 of this paper we give some background material on Coxeter groups. In Section 3 we explain the construction in dimension two, where it reduces to the classical theory of groups generated by reflections on simply connected complete Riemann surfaces of constant curvature. The main results are explained in Section 4 where we consider the same construction in higher dimensions. In Section 5 we discuss a modification of the construction which gives many further examples. This modification is used in Section 6 to prove a result (the only new result in this paper) concerning the Novikov Conjecture. Finally, in Section 7 we discuss a conjecture concerning Euler characteristics of even-dimensional closed aspherical manifolds.

2. Coxeter groups.

In this section we review some standard material on Coxeter groups. For the complete details, see [2].

Let Ω be a finite graph (i.e., a 1-dimensional finite simplicial complex), with vertex set V and edge set E and let $m : E \to \mathbb{Z}$ be a function which assigns to each edge an integer ≥ 2. For each pair $(v,w) \in V \times V$ put

$$m(v,w) = \begin{cases} 1 & ; \text{ if } v = w \\ m(\{v,w\}); & \text{ if } \{v,w\} \in E \\ \infty & ; \text{ otherwise.} \end{cases}$$

These data give a presentation of a group:

$$\Gamma = \langle V ; (vw)^{m(v,w)} = 1 \rangle, \quad (v,w) \in V \times V.$$

Let $(e_v)_{v \in V}$ be the standard basis for the vector space $\mathbb{R}^V$. Define a symmetric bilinear form B on $\mathbb{R}^V$ by

$$B(e_v, e_w) = -\cos(\pi/m(v,w))$$

(where π/∞ is interpreted as 0). For each $v \in V$, let σ_v denote the linear reflection on $\mathbb{R}^V$ defined by $\sigma_v(x) = x - 2B(e_v, x)e_v$ and let $\bar{\Gamma}$ be the subgroup of $GL(\mathbb{R}^V)$ generated by $(\sigma_v)_{v \in V}$. (Note that $\bar{\Gamma}$ leaves the form B invariant.)

Suppose that v,w are distinct elements of V, that P is the plane spanned by e_v and e_w and that $m = m(v,w)$. The restriction of B to P is positive semi-definite and it is positive definite if and only if $m \neq \infty$. Moreover, if m is finite, then $\sigma_v|P$ and $\sigma_w|P$ are the orthogonal reflections through the lines orthogonal to e_v and to e_w, respectively, and these lines make an angle of π/m. Hence, $\sigma_v\sigma_w|P$ is a rotation through and angle of $2\pi/m$ and $\sigma_v|P$ and $\sigma_w|P$ generate a dihedral group of order 2m. Since $\sigma_v\sigma_w$ fixes $P^\perp$, it follows that $\sigma_v\sigma_w$ has order m. If $m = \infty$, then one can easily show that $\sigma_v\sigma_w$ has order m.

It follows that the map $v \to \sigma_v$ extends to an isomorphism $\Gamma \to \bar{\Gamma}$, called the _canonical representation_ of Γ. The construction of this representation shows that a) the natural map $i : V \to \Gamma$ is an injection, and that b) the order of $i(v)i(w)$ is equal to $m(v,w)$ (rather than just dividing $m(v,w)$). Henceforth, we identify V with i(V). The pair (Γ,V) is a _Coxeter system_ and Γ is a _Coxeter group_. The graph Ω together with the labelling of its edges is the _associated labelled graph_. It follows from property b) above that the correspondence between labelled graphs and isomorphism classes of Coxeter systems is bijective.

There is another way to record the same information as is contained in the associated labelled graph. Let Ω' be the graph with the same vertex set V as Ω but with edge set E' obtained by first deleting the elements of E labelled 2 and then adding edges for each unordered pair of distinct vertices $\{v,w\}$ not in E (i.e., with $m(v,w) = \infty$). As a notational simplification the edges labelled 3 are usually left unmarked. The graph Ω' together with the labelling of its edges is called the Coxeter diagram of (Γ,V). A Coxeter system is irreducible if its Coxeter diagram is connected.

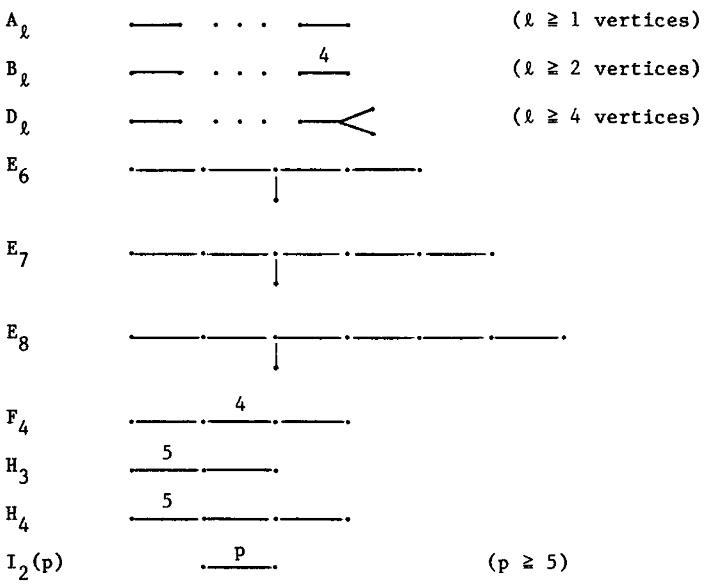
Suppose that (Γ,V) is a Coxeter system. For any subset S of V denote by Γ_S the subgroup generated by S. (It turns out that the pair (Γ_S,S) is also a Coxeter system.) If the Coxeter diagram of (Γ,V) has k components with vertex sets $V_1,\ldots,V_k$, then $\Gamma = \Gamma_{V_1} \times \ldots \times \Gamma_{V_k}$.

Finite Coxeter groups. A Coxeter group Γ is finite if and only if the form B is positive definite. Suppose that this is the case. Let C be the simplicial cone in $\mathbb{R}^V$ defined by the equations: $B(e_v,x) \geq 0$, $v \in V$. Thus, $(e_v)_{v\in V}$ is the set of inward pointing unit normals to the "panels" (i.e., codimension one faces) of C. Moreover, C is a closed fundamental domain for Γ on $\mathbb{R}^V$ in the sense that it intersects each Γ-orbit in exactly one point. (It follows that the orbit space $\mathbb{R}^V/\Gamma$ is homeomorphic to C.)

Still supposing that Γ is finite, we have that order $(vw) = m(v,w) < \infty$ for each pair (v,w) of vertices. Hence, the associated graph Ω is the 1-skeleton of the simplex with vertex set V. On the other hand, it turns out, that each component of the Coxeter diagram Ω' is a tree. The well-known list of Coxeter diagrams of irreducible Coxeter systems of finite Coxeter groups is given below. The list contains one infinite family $I_2(p)$ in dimension 2 (where $I_2(p)$ denotes the dihedral group of order $2p$), three families A_ℓ, B_ℓ, D_ℓ in each dimension ℓ (with a few restrictions in low dimensions), and 6 additional groups.

Many of these groups have other convenient descriptions. For example, A_ℓ is the symmetric group on $\ell + 1$ symbols, while A_3, B_3, H_3 are the full groups of motions of regular solids, namely, the tetrahedron, the octahedron, and the icosahedron, respectively.

Coxeter Diagrams of Irreducible Finite Coxeter Groups



3. The construction in dimension two.

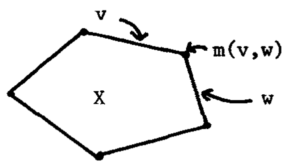
In dimension two all our constructions reduce to well-known classical results. We shall now review these results.

Let X be a polygon. We shall find it convenient to work with the graph Ω which is the dual of ∂X. Thus, if V denotes the vertex set of Ω and E the edge set, then

$$V = \{\text{edges of } \partial X\}$$

$$E = \{\{v,w\} \mid v,w \in V,\ v \neq w,\ v \cap w \neq \emptyset\}.$$

Equivalently, E is the set of vertices of X. Choose a labelling $m : E \to \{2,3,\ldots\}$. Thus, we label the vertices of X by integers ≥ 2.

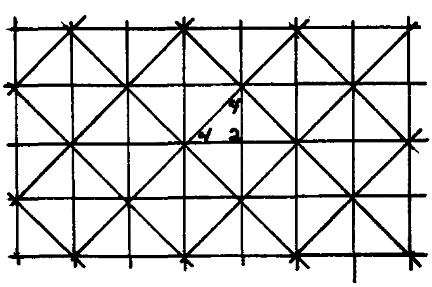


The labelled graph Ω defines a Coxeter system (Γ, V). For each x in X, let $V(x)$ denote the set of v in V such that $x \in v$. Let $\Gamma_{V(x)}$ be the subgroup generated by $V(x)$. (By convention $\Gamma_{\emptyset}$ is the trivial group.) Thus, if x belongs to the interior of X, then $\Gamma_{V(x)}$ is trivial; if x belongs to the interior of an edge v, then $\Gamma_{V(x)}$ is the cyclic group of order 2 generated by v; and if x is a vertex, then $\Gamma_{V(x)}$ is the dihedral group generated by the edges containing x.

There is an obvious method for constructing a Γ-space $\mathcal{U}$ by pasting together copies of X, one for each element of Γ. To be precise, put $\mathcal{U} = (\Gamma \times X)/\sim$ where the equivalence relation $\sim$ is defined by

$$(g,x) \sim (h,y) \iff x = y \text{ and } g^{-1}h \in \Gamma_{V(x)}.$$

Let $[g,x]$ denote the equivalence class of (g,x). There is a natural Γ-action on $\mathcal{U}$ defined by $h[g,x] = [hg,x]$. The isotropy group at $[g,x]$ is clearly $g\Gamma_{V(x)}g^{-1}$. Since each of these isotropy groups is finite, it is easy to see that the action is proper. It is also not difficult to see that $\mathcal{U}$ is a 2-manifold. (At each edge two copies of X fit together. At a vertex labelled m the picture is locally isomorphic to the canonical action of the dihedral group of order $2m$ on $\mathbb{R}^2$.)



The surface $\mathcal{U}$ is simply connected. This can be seen geometrically using the developing map. (See Remark 1 at the end of this section or [21].) It also follows from the results of the next section. There is also a direct argument using covering space theory. (Let $p : \tilde{\mathcal{U}} \to \mathcal{U}$ be the universal cover of $\mathcal{U}$, let $\tilde{X}$ be a component of $p^{-1}(X)$, and let $s : X \to \tilde{X}$ be the inverse of the homeomorphism $p|X$. Lift each involution v in V to an involution $\tilde{v}$ on $\tilde{\mathcal{U}}$ such that the fixed set of $\tilde{v}$ contains the corresponding edge of $\tilde{X}$. This defines a lift of the Γ-action to $\tilde{\mathcal{U}}$. The mapping $s: X \to \tilde{X}$, then extends to a Γ-equivariant section $\mathcal{U} \to \tilde{\mathcal{U}}$. Hence, the covering is trivial and $\mathcal{U}$ is simply connected.)

Since $\mathcal{U}$ is a simply connected surface, it is homeomorphic either to S^2 or to $\mathbb{R}^2$. The case $\mathcal{U} = S^2$ occurs if and only if Γ is finite. By the classification of finite Coxeter groups described in the previous section, this happens if and only if X is a triangle and the set of labels $\{p,q,r\}$ is either $\{2,2,r\}$, $\{2,3,3\}$, $\{2,3,4\}$, or $\{2,3,5\}$ (corresponding, respectively, to the groups $A_1 \times I_2(r), A_3, B_3$, or H_3).

The moral to be drawn from the above discussion is that apart from a few exceptional cases this construction always leads to a contractible 2-manifold $\mathcal{U}$. As we shall see in the next section, virtually the same construction works in any dimension. The surprising fact is that, under a mild restriction, the resulting manifold is also contractible.

At this point we have not yet constructed any closed aspherical manifolds. The problem is that the transformation group Γ does not act freely on $\mathcal{U}$. This can be remedied as follows. Suppose Γ is infinite and let Γ' be any torsion-free subgroup of finite index in Γ. (There are various algorithms for finding such subgroups; however, in general, none of them are very satisfactory. However, as we have seen in the previous section any Coxeter group is a subgroup of some linear group; hence, it follows from Selberg's Lemma (cf. [15]) that any Coxeter group is virtually torsion-free.) Since each Γ-isotropy group is finite, each Γ'-isotropy group is trivial; hence, $\mathcal{U} \to \mathcal{U}/\Gamma'$ is a covering projection. Since $[\Gamma:\Gamma'] < \infty$, $\mathcal{U}/\Gamma'$ is compact; hence, $\mathcal{U}/\Gamma'$ is a closed aspherical surface.

REMARK 1. Since the local picture in $\mathcal{U}$ near a vertex in X labelled m is isomorphic to the canonical action of the dihedral group of order $2m$ on $\mathbb{R}^2$, we should think of the label as specifying an interior angle of π/m at this vertex. Depending on whether the sum of this interior angles is greater than, equal to, or less than $\pi(\text{Card } V - 2)$, X can be realized as a convex polygon in, respectively, S^2, the Euclidean plane $\mathbb{R}^2$, or the hyperbolic plane H^2 with interior angles as specified by the labels. There is then a well-defined homomorphism from Γ onto $\bar{\Gamma}$, the group generated by the orthogonal reflections through the sides of this convex polygon. Using the Γ-actions, we obtain a map $\mathcal{U} \to M^2$, where M^2 denotes the appropriate choice of $S^2, \mathbb{R}^2$, or H^2. This map is easily seen to be a covering projection. Since M^2 is simply connected, this map is a homeomorphism. It follows that $\bar{\Gamma}$ is discrete and isomorphic to Γ.

The classification of finite Coxeter groups in dimension 3 can then be recovered from the facts that 1) any convex polygon in S^2 with non-obtuse interior angles is a spherical triangle and 2) the sum of the interior angles in such a triangle is $> \pi$.

REMARK 2. In higher dimensions the situation with cocompact geometric reflection groups is as follows. In the spherical case, the group Γ is a finite Coxeter group and the fundamental chamber X is a spherical simplex. In the flat case, there is also a complete classification of possible Coxeter groups (cf. [2, p. 199]); moreover, X is a product of Euclidean simplices, one for each irreducible factor of Γ. In the hyperbolic case, the Coxeter group Γ must be irreducible; however, the chamber X need not be a simplex (as we have seen already in dimension 2). If it is a simplex, then there are only a few possibilities: 9 in dimension 3, 5 in dimension 4, and none in higher dimensions (cf. Exercise 15, p. 133 in [2]). In the general hyperbolic case, the situation is as follows.

In dimension 3 there is a rich theory and complete result due to Andreev (cf. [1] or [21]). In dimensions > 3 there are a few isolated examples but no general understanding of the possibilities; while in very high dimensions (something like dimensions > 30) Vinberg has apparently proved that cocompact hyperbolic reflection groups do not exist. In summary, relatively few Coxeter groups have representations as cocompact geometric reflection groups and in these cases, at least in dimensions > 3, there are very few combinatorial types of convex polyhedra which can occur as fundamental chambers. As we shall see in the next section, if we drop our geometric requirements, then the situation reverts to its original simplicity.

4. The construction in dimension n.

Let X be a compact, contractible n-manifold with boundary and let L be a PL-triangulation of its boundary. The simplicial complex L will be used for two purposes. First, a Coxeter system will be constructed by labelling the edges of the 1-skeleton of L. Second, ∂X will be given the structure of the dual cell complex to L.

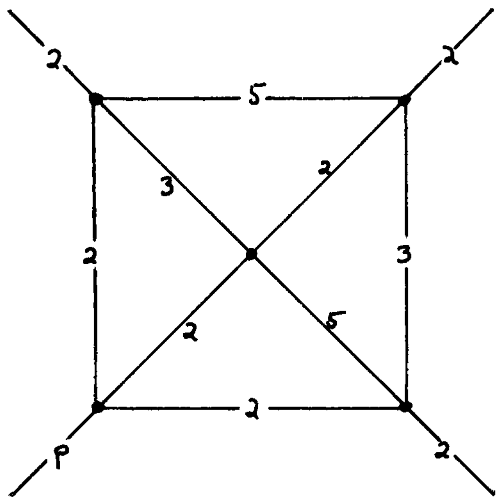
Let V be the vertex set of L, E the edge set, and Ω the 1-skeleton. There are two conditions which we want our labelling $m : E \to \{2,3,\dots\}$ to satisfy. The first condition is the following:

(*) For each simplex $S \in L$ the subgroup Γ_S, generated by S, is finite.

This means that for any $S \in L$ if we discard the edges labelled 2, then the resulting labelled graph is the Coxeter diagram of a finite Coxeter group. For example, if L is an octahedron we could label its edges as below. In general, for any simplicial complex L if we label every edge by 2, then condition (*) holds, since in this case for each $S \in L$ we will have $\Gamma_S = (\mathbb{Z}/2\mathbb{Z})^S$.

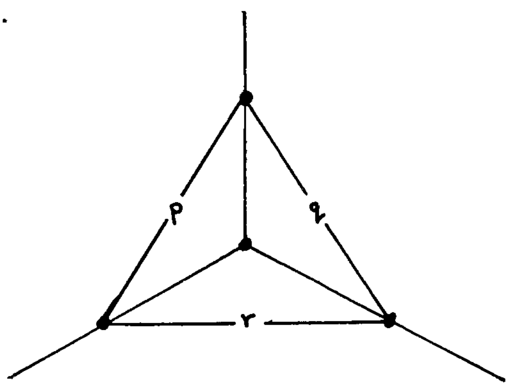
The second condition is the converse to the first:

(**) If S is a subset of V such that Γ_S is finite, then $S \in L$.



L is an octahedron with a vertex at ∞.

Since by construction an unordered pair of vertices $\{v,w\}$ belongs to E if and only if $m(v,w) < \infty$, this condition is vacuous for subsets of cardinality less than 3. Hence, condition (**) means that if Ω contains a subgraph with vertex set S which is isomorphic to the 1-skeleton of a simplex and which is not equal to the 1-skeleton of a simplex in L, then the edge labels must be such that Γ_S is infinite. For example, if L is the suspension of a triangle, then the labels p,q,r on the edges of the triangle must satisfy $p^{-1} + q^{-1} + r^{-1} \leq 1$.



L is the suspension of a triangle with a vertex at ∞.

If any subgraph of Ω which is isomorphic to the 1-skeleton of a simplex is equal to the 1-skeleton of some simplex in L, then condition (**) holds vacuously. For example, the octahedron has this property as does any polygon with more than 3 edges. More generally, if L is any simplicial complex, then its barycentric subdivision has this property (cf. Lemma 11.3 in [3]). Therefore, conditions (*) and (**) are always satisfied if we replace L by its barycentric subdivision and label each edge 2 (or in any other fashion which satisfies (*)). We now assume that we have labelled the edges of L in some fashion so that conditions (*) and (**) hold and we let (Γ, V) denote the resulting Coxeter system.

Next we cellulate ∂X as the dual cell complex. Thus, for example, if L is an octahedron, X will be a cube. For each $v \in V$, let X_v denote the dual cell of $\{v\}$ and for each simplex $S \in L$, let X_S be the dual cell of S. Thus,

$$X_S = \bigcap_{v \in S} X_v$$

(The X_v, which are faces of codimension one in X, are called the <u>panels</u> of X.) Also, for each subset S of V put

$$X_{\sigma(S)} = \bigcup_{v \in S} X_v.$$

If S is actually a simplex of L, then $X_{\sigma(S)}$ is a regular neighborhood of S in the barycentric subdivision of L. Hence,

(D) <u>If</u> $S \in L$, <u>then the union of panels</u> $X_{\sigma(S)}$ <u>is a disk of codimension zero in</u> ∂X.

For each $x \in X$ let $V(x) = \{v \in V | x \in X_v\}$ be the set of panels which contain x and let $\Gamma_{V(x)}$ be the subgroup generated $V(x)$. As before, we define a

Γ-space $\mathcal{U} = (\Gamma \times X)/\sim$, where the equivalence relation $\sim$ is defined exactly as in the previous section. The map $x \to [1,x]$ induces an embedding $X \to \mathcal{U}$ which we regard as an inclusion. Observe that a) X is a fundamental domain for Γ on $\mathcal{U}$ and that b) for each $x \in X$ the isotropy subgroup is $\Gamma_{V(x)}$. We claim that:

(1) Γ acts properly on $\mathcal{U}$.

(2) $\mathcal{U}$ is a manifold and Γ acts locally smoothly.

(3) $\mathcal{U}$ is contractible.

(4) If L $(=\partial X)$ is not simply connected, then $\mathcal{U}$ is not simply connected at ∞.

Basically, (1) is equivalent to condition (*), while (3) is equivalent to condition (**).

Proof of (1) and (2). To say that the cells of a polyhedron intersect in general position means that the dual polyhedron is a simplicial complex. Hence, the panels of X intersect in general position. This means that for each $x \in X$ we can find a neighborhood U_x of x in X of the form $\mathbb{R}^m \times C^{V(x)}$ where $\mathbb{R}^m$ is a neighborhood of x in $X_{V(x)}$ and where $C^{V(x)}$ is the standard simplicial cone in $\mathbb{R}^{V(x)}$. Let $W_x = \Gamma_{V(x)}U_x$ be the corresponding neighborhood of x in $\mathcal{U}$. Recall that an action of a discrete group is proper if and only if (i) the orbit space is Hausdorff, (ii) each isotropy group is finite and (iii) each point has a neighborhood which is invariant under the isotropy subgroup and which is disjoint from all other translates of itself.
Since $\mathcal{U}/\Gamma \cong X$, (i) holds. Condition (*) implies (ii). Also, gW_x is a neighborhood of $[g,x]$ satisfying (iii). Hence, Γ acts properly. Since $\Gamma_{V(x)}$ is a finite Coxeter group, a fundamental chamber for its canonical action on $\mathbb{R}^{V(x)}$ is $C^{V(x)}$. Hence, $\Gamma_{V(x)}C^{V(x)} \cong \mathbb{R}^{V(x)}$ and $W_x = \Gamma_{V(x)}U_x \cong \mathbb{R}^m \times \mathbb{R}^{V(x)}$. This shows that $\mathcal{U}$ is locally Euclidean and that the action is locally linear, proving (2).
Proof of (3) and (4). For any $g \in \Gamma$ let $\ell(g)$ denote its word length with respect to the generating set V. Let V^g denote the set of "reflections" through the panels of gX (i.e., $V^g = gVg^{-1}$.) Put

$$C(g) = \{w \in V^g \mid \ell(wg) < \ell(g)\} \quad \text{and}$$
$$B(g) = \{v \in V \mid \ell(gv) < \ell(g)\} = g^{-1}C(g)g.$$

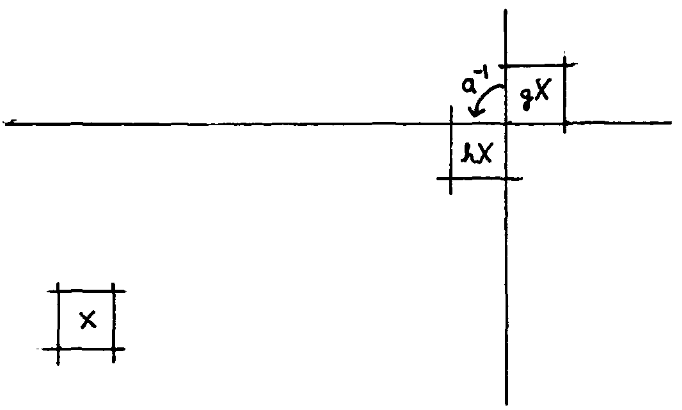
($C(g)$ is the set of reflections across panels of gX such that the reflected image of gX is closer to X than is gX. $B(g)$ is the set of reflections across these same panels after they have been translated back to X.) Also, put

$$\delta(gX) = gX_{\sigma(B(g))},$$

i.e., $\delta(gX)$ is the union of those panels of gX which are indexed by $C(g)$.

LEMMA A (cf. [3, Lemma 7.12] or [16, p. 108]). *For any* $g \in \Gamma$, *the subgroup* $\Gamma_{B(g)}$ *is finite*.

Sketch of Proof. Finite Coxeter groups are distinguished from infinite Coxeter groups by the fact that each finite one has a unique element of longest length. It is not hard to see that $\Gamma_{B(g)}$ has such an element. Explicitly, let h be the (unique) element of shortest length in the coset $g\Gamma_{B(g)}$. Then it follows from Exercises 3 and 22, p. 43, in [2] that $a = gh^{-1}$ is the element of longest length in $\Gamma_{B(g)}$. Q.E.D.



Next order the elements of Γ,

$$g_1, g_2, \dots .$$

so that $\ell(g_{i+1}) \geq \ell(g_i)$. Since 1 is the unique element of length 0, $g_1 = 1$. For each integer $m \geq 1$, put

$$X_m = g_m X, \qquad \delta X_m = \delta(g_m X),$$

$$T_m = \bigcup_{i=1}^{m} X_i .$$

The next lemma asserts that the chambers intersect as one would expect them to. (For a proof see [3, Lemma 8.2].)

LEMMA B. <u>For each integer</u> $m \geq 2$, $X_m \cap T_{m-1} = \delta X_m$.

Thus, T_m is obtained from T_{m-1} by pasting on a copy of X along a certain union of panels.

By Lemma A, for each $g \in \Gamma$ the group $\Gamma_{B(g)}$ is finite. Condition (**) implies that $B(g) \in L$. Statement (D) then implies that the union of panels $X_{\sigma(B(g))}$ is a disk of codimension zero in ∂X for each $g \in \Gamma$. Since $\delta X_m = g_m X_{\sigma(B(g_m))}$, this means that δX_m is a disk of codimension zero in ∂X_m. Hence T_m is the boundary connected sum of m copies of X. Since X is contractible, so is T_m. Since $\mathcal{U} = \bigcup_{m=1}^{\infty} T_m$, $\mathcal{U}$ is contractible, which proves 3).

Since $\mathcal{U}$ is formed by successively pasting on copies of X (which are contractible) to T_m along disks, $\mathcal{U} - T_m$ is homotopy equivalent to ∂T_m. Since ∂T_m is the connected sum of m copies of ∂X, we have (provided $\dim X \geq 3$) that $\pi_1(\partial T_m)$ is the free product of m copies of $\pi_1(\partial X)$.

Moreover, the map $\pi_1(\mathcal{U}-T_{m+1}) \cong \pi_1(\partial T_{m+1}) \to \pi_1(\mathcal{U}-T_m) \cong \pi_1(\partial T_m)$ induced by inclusion can clearly be identified with the projection onto the first m factors of the free product. In particular, this map is onto for each $m \geq 1$. Thus, $\mathcal{U}$ is semi-stable and the inverse limit $\pi_1^\infty(\mathcal{U})$ is the "projective free product" of an infinite number of copies of $\pi_1(\partial X)$. Hence, if $\pi_1(\partial X)$ is not trivial, then this inverse limit is not trivial (or even finitely generated). This proves (4).

REMARK 1. As we have previously remarked, Γ always contains torsion-free subgroups of finite index and any such subgroup Γ' leads to a closed aspherical manifold $\mathcal{U}/\Gamma'$. In view of the fact that ∂X may be non-simply connected whenever $\dim X \geq 4$, statement (4) implies that the conjecture of Section 1 is false in dimensions ≥ 4.

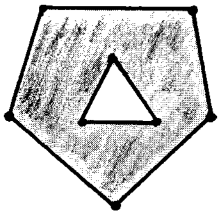
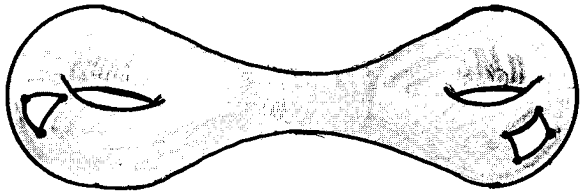
REMARK 2. In the special case where (Γ,V) is obtained by labelling each edge of a graph by 2, there is any easy construction of a torsion-free subgroup Γ'. Let (H,V) be the Coxeter system defined by setting $m(v,w) = 2$ for each pair $\{v,w\}$ of distinct vertices in V. Thus, H is the finite Coxeter group $(\mathbb{Z}/2\mathbb{Z})^V$. There is a natural epimorphism $\varphi : \Gamma \to H$ which is the identity on V. Let Γ' be the kernel of φ. If S is any subset of V such that Γ_S is finite, then $\Gamma_S \cong ((\mathbb{Z}/2\mathbb{Z})^S = H_S$; hence, for any such S, $\varphi|\Gamma_S$ is an isomorphism onto H_S. Since any finite subgroup of Γ is contained in some isotropy group and is consequently conjugate to a subgroup of some Γ_S, we have that Γ' is torsion-free. (Incidentally, Γ' is the commutator subgroup.) If $\mathcal{U} = (\Gamma\times X)/\sim$, then there is an alternative description of the quotient $\mathcal{U}/\Gamma'$. Namely, $\mathcal{U}/\Gamma' \cong (H\times X)/\sim'$ where $(g,x) \sim' (h,x) \Leftrightarrow g^{-1}h \in H_{V(x)}$. Thus, H acts a reflection group on $\mathcal{U}/\Gamma'$ with quotient X.

REMARK 3. The construction of this section suggests several questions concerning fundamental groups at ∞. First of all, R. Geoghegan has conjectured that if the universal cover of a finite complex has one end, then it must be semi-stable. After seeing the above construction, H. Sah and R. Schultz both asked if the fundamental group at ∞ of the universal cover of a closed aspherical manifold can ever be non-trivial and finitely generated (i.e., can the universal cover be stable at ∞ without being simply connected at ∞).

REMARK 4. There is some flexibility in the main construction of this section. First of all, for $\mathcal{U}$ to be contractible it is not necessary that each face of X be a cell. All that the proof requires is that X be contractible and that each proper face be acyclic. (For example, we can change a panel of X by taking connected sun with a homology sphere.) Secondly and more interestingly, it is not necessary for the simplicial complex L to be a PL triangulation of a homology sphere. All that is required is that L have the homology of an (n-1)-sphere and that it be a polyhedral homology manifold (i.e., the link of each k-simplex must have the homology of an (n-k-2)-sphere). Any such L can then be "dualized" (i.e., "resolved") to produce a contractible n-manifold X with contractible faces. This X can then be used to produce a contractible $\mathcal{U}$ as before. If one is interested in constructing proper, locally smooth actions of a discrete group generated by "reflections" on a contractible manifold with compact quotient, then there is no further flexibility. That is to say, with the above two provisos, <u>every cocompact reflection group on a contractible manifold can be constructed as above</u> (cf. [3]). However, as we shall see in the next section, there is quite a bit more flexibility if we are only interested in producing more examples of aspherical manifolds.

5. A generalization.

We begin this section by considering a modification of our construction in dimension 2. Rather than starting with the underlying space of X a 2-disk, let it be any compact surface with nonempty boundary. Take a polygonal subdivision of the boundary and label the vertices by integers ≥ 2. For example, X could be one of the "orbifolds" pictured below with random labels on the vertices. The polygonal subdivision of the boundary together with the labelling

defines a Coxeter system (Γ,V). As before, paste together copies of X to obtain a surface $\mathcal{U}$ $(=(\Gamma\times X)/\sim)$ with Γ-action. The surface $\mathcal{U}$ will no longer be contractible, but it will be aspherical provided X is not a 2-disk. (After all, almost every surface is aspherical.) If Γ' is any closed torsion-free subgroup of finite index in Γ, then $\mathcal{U}/\Gamma'$ will be a closed aspherical surface.

Next let us try to make the same modification in an arbitrary dimension. Let X be a compact aspherical n-manifold with boundary. (The boundary need not be aspherical.) Let L be a triangulation of ∂X. After possibly replacing L by its barycentric subdivision, we can find a labelling of its edges so that the resulting Coxeter system (Γ,V) satisfies conditions (*) and (**). As in Section 4, there results an n-manifold $\mathcal{U}$ with proper Γ-action. The proof of Claim (3) in Section 4 shows that $\mathcal{U}$ is homeomorphic to the infinite boundary connected sum of copies of X. Since X is aspherical, so is $\mathcal{U}$ (the wedge of two aspherical spaces is again aspherical). The fundamental group of $\mathcal{U}$ is an infinite free product of copies of $\pi_1(X)$. If Γ' is a torsion-free subgroup of finite index in Γ, then $\mathcal{U}/\Gamma'$ is a closed aspherical manifold; its fundamental group is, of course, an extension of Γ' by $\pi_1(\mathcal{U})$.

REMARK. One can imagine situations where it would be convenient if one could double a compact aspherical manifold along its boundary and obtain a closed aspherical manifold as the result. However, the doubled manifold is not aspherical unless the boundary is aspherical and its fundamental group injects into the fundamental group of the original manifold. The method described above can be viewed as a fancy method of doubling so that the result will be aspherical.

In [21] Thurston considers the above construction for certain compact aspherical 3-manifolds. (In fact, the discussion in [21] inspired the results of Sections 4 and 5.) Thurston shows that with a few more hypotheses the 3-dimensional orbifold can be given a hyperbolic structure. This allows him to "double" certain hyperbolic 3-manifolds along their boundaries (or actually sub-surfaces of their boundaries). This "orbifold trick" plays an important technical role in the proof of his famous theorem on atoroidal Haken 3-manifolds.

6. An observation concerning the Novikov Conjecture.

A group is geometrically finite if it is the fundamental group of an aspherical finite complex (or equivalently, if it is the fundamental group of an aspherical compact manifold with boundary).

If M^n is a manifold with boundary with fundamental group π, then there is a surgery map $\sigma : [(M,\partial M);(G/TOP,*)] \to L_n(\pi)$. The map σ need not be a homomorphism; however, it is if we replace M by $M \times D^i$, $i > 0$, and ∂M by $\partial(M\times D^i)$.

The Novikov Conjecture for a geometrically finite group π asserts that if M^n is any aspherical compact manifold with fundamental group π, then $\sigma : [M\times I,\partial(M\times I),(G/TOP,*)] \to L_{n+1}(\pi)$ becomes a monomorphism after tensoring with the rationals. A stronger version of this asserts that σ is a monomorphism. (See [4] for a discussion of these conjectures.)

PROPOSITION. If the Novikov Conjecture (resp. the strong version of the Novikov Conjecture) holds for the fundamental group of every closed aspherical manifold, then it holds for every geometrically finite group.*

*The fact that the construction of the previous section could be used to prove this proposition first came up during a conversation with John Morgan.

Proof. Let $(X,\partial X)$ be an aspherical compact manifold with fundamental group π. Triangulate ∂X, take the barycentric subdivision, and label each edge by 2 to form a Coxeter system (Γ,V). Let $H = (\mathbb{Z}/2\mathbb{Z})^V$, $\Gamma' = \ker(\Gamma \to H)$, $\mathcal{U} = (\Gamma\times X)/\sim$, and $\mathcal{U}' = \mathcal{U}/\Gamma' = (H\times X)/\sim'$, be as in Remark 2 of Section 4. There is a commutative diagram

$$\begin{array}{ccc} [(X,\partial X),\ (G/TOP,*)] & \xrightarrow{\sigma} & L_n(\pi) \\ \downarrow \Lambda^* & & \downarrow i_* \\ [\mathcal{U}',\ G/TOP] & \xrightarrow{\sigma} & L_n(\pi') \end{array}$$

where $\Lambda : \mathcal{U}' \to X/\partial X$ is the map which collapses everything outside X to a point, where $\pi' = \pi_1(\mathcal{U}')$, and where i_* is the map induced by the inclusion $\pi = \pi_1(X) \hookrightarrow \pi_1(\mathcal{U}') = \pi'$. The proposition follows easily from the next claim. (If σ is not a homomorphism, then replace X by $X \times D^4$, ∂X by $\partial(X\times D^4)$, $\mathcal{U}'$ by $\mathcal{U}' \times D^4$ and $[\mathcal{U}',G/TOP]$ by $[(\mathcal{U}'\times D^4, \mathcal{U}'\times S^3); (G/TOP,*)]$.)

CLAIM. The map $\Lambda^* : [(X,\partial X),(G/TOP,*)] \to [\mathcal{U}',G/TOP]$ is a monomorphism.

Proof of Claim. We first proved the corresponding statement in cohomology. The argument is based on the existence of an "alternation map" (cf. Section 9 in [3]). If α is a singular chain in X, then we can "alternate" it to form the chain

$$A(\alpha) = \sum_{h\in H} (-1)^{\ell(h)} h \cdot \alpha$$

in $\mathcal{U}'$. The map $\alpha \to A(\alpha)$ clearly vanishes on $C_*(\partial X)$; hence, there is a chain map $A : C_*(X,\partial X) \to C_*(\mathcal{U}')$. This induces a homomorphism $A^* : H^*(\mathcal{U}') \to H^*(X,\partial X)$ which is a splitting for $\Lambda^* : H^*(X,\partial X) \to H^*(\mathcal{U}')$. Hence, Λ^* is a split monomorphism on cohomology (with arbitrary coefficients). Since for any space Y, $[Y,G/TOP] \otimes \mathbb{Q} \cong \Sigma H^{4*}(Y;\mathbb{Q})$, this is enough to prove that Λ^* is rationally a monomorphism on $[(X,\partial X),(G/TOP,*)]$. (Hence, the proposition holds for the weak version of the Novikov Conjecture.) According to Sullivan's calculation of the homotopy type of G/TOP, showing that Λ^* is a monomorphism is equivalent to showing that it is a monomorphism on $H^*(\ ,\mathbb{Z}_{(2)})$ and on $KO^*(\ ) \otimes \mathbb{Z}[\frac{1}{2}]$. We have already proved it for ordinary cohomology with arbitrary coefficients. To prove the corresponding statement for an extraordinary cohomology theory we first need to make some small modifications.

Let R be a ring in which $|H|$ is invertible (i.e., in which 2 is invertible). Put

$$\hat{A} = |H|^{-1} \sum_{h \in H} (-1)^{\ell(h)} h \in R[H].$$

For any $R[H]$-module M define a submodule $M^{Alt} = \{z \in M \mid vz = -z, \forall v \in V\}$. It is easily checked that

(a) $\hat{A}^2 = \hat{A}$

(b) $M^{Alt} = \text{Image}(\hat{A}: M \to M)$.

For an arbitrary cohomology theory it is not clear how to split Λ^*. However, suppose $\mathcal{H}^*(\ )$ is a cohomology theory with value in R-modules and that 2 is invertible in R (e.g. $\mathcal{H}^*(\ ) = KO^*(\ ) \otimes \mathbb{Z}[\frac{1}{2}]$). Let $\Sigma \subset \mathcal{U}$ be the singular set. By excision,

(c) $\mathcal{H}^*(\mathcal{U}', \Sigma) \cong \sum_{h \in H} \mathcal{H}^*(X, \partial X)$ from which it follows that

(d) $\mathcal{H}^*(\mathcal{U}', \Sigma)^{Alt} \cong \mathcal{H}^*(X, \partial X)$.

Using (a), (b), and the sequence of the pair $(\mathcal{U}', \Sigma)$, we find that $\mathcal{H}^*(\mathcal{U}')^{Alt} \cong \mathcal{H}^*(\mathcal{U}', \Sigma)^{Alt}$. Hence, there is a map

$$\hat{A}^* : \mathcal{H}^*(\mathcal{U}') \to \mathcal{H}^*(\mathcal{U}')^{Alt} = \mathcal{H}^*(X, \partial X)$$

which splits Λ^*. This proves the claim and consequently, the proposition.

7. The rational Euler characteristic and some conjectures.

Associated to any orbifold there is a rational number, called its "Euler characteristic," which is multiplicative with respect to orbifold coverings (cf. [21]). If the orbifold is cellulated so that each stratum is a subcomplex, then this is defined as the alternating sum of the number of cells in each dimension where each cell is given a weight of the inverse of the order of its isotropy group.

We suppose, as usual, that X^n is a compact n-manifold with boundary, that L is a triangulation of ∂X, that (Γ,V) is a Coxeter system obtained by labelling the edges of L, that condition (*) of Section 4 is satisfied, and that $\mathcal{U} = (\Gamma\times X)/\sim$. For each $S \in L$, let $X_S \subset \partial X$ be the dual cell of S. Let e(X) be the ordinary Euler characteristic of the underlying topological space of X minus that of ∂X. The rational Euler characteristic of X is then defined by

$$(1)\quad \chi(X) = e(X) + \sum_{S\in L} (-1)^{\dim X_S} \frac{1}{|\Gamma_S|}$$

$$= e(X) + (-1)^n \sum_{S\in L} (-1)^{\mathrm{Card}(S)} \frac{1}{|\Gamma_S|}$$

where $|\Gamma_S|$ denotes the order of Γ_S. It is then clear that if Γ' is a torsion-free subgroup of finite index in Γ, then $\chi(\mathcal{U}/\Gamma') = [\Gamma:\Gamma']\chi(X)$. Also, if $\mathcal{U}$ is contractible, then $\chi(X) = \chi(\Gamma)$, where $\chi(\Gamma)$ is the rational Euler characteristic as defined in [16].

For example, if X is a triangle and Γ is the (p,q,r)-triangle group, then

$$\chi(X) = 1 - \frac{3}{2} + \left(\frac{1}{2p} + \frac{1}{2q} + \frac{1}{2r}\right) = \frac{1}{2}((p^{-1}+q^{-1}+r^{-1})-1).$$

If M^2 is a closed surface which is aspherical, then $\chi(M^2) \leq 0$. It follows that if $N^{2k} = M_1^2 \times \ldots \times M_k^2$ is a product of closed aspherical surfaces, then $(-1)^k\chi(M^{2k}) \geq 0$.

A well-known conjecture of Hopf is the following:

CONJECTURE 1 (Hopf). *If M^{2k} is a closed manifold of non-positive sectional curvature, then $(-1)^k\chi(M^{2k}) \geq 0$.*

This was proved by Chern for 4-manifolds and by Serre [16] for local symmetric spaces. Recently, H. G. Donnelly and F. Xavier have established it under a hypothesis of pinched negative curvature.

More generally W. Thurston has asked if the following conjecture is true.

CONJECTURE 2 (Thurston). _If_ M^{2k} _is a closed aspherical manifold, then_ $(-1)^k\chi(M^{2k}) \geq 0$.

It should be pointed out that this conjecture contradicts another conjecture of Kan-Thurston which asserts that any closed manifold has the same homology as some closed aspherical manifold.

Conjecture 2 implies the following conjecture.

CONJECTURE 3. _Let_ $X^{2k}, \Gamma, \mathcal{U}$ _be as above and let_ $\chi(X)$ _denote the rational Euler characteristic. If_ $\mathcal{U}$ _is aspherical, then_ $(-1)^k\chi(X^{2k}) \geq 0$.

One might try to construct a 4-dimensional counterexample to the above conjecture as follows. Let X be a 4-cell and L some specific triangulation of S^3. Label the edges in L in some fashion so that conditions (*) and (**) hold and calculate $\chi(X)$ using (1). After making a number of such calculations and having the result invariably come out non-negative, I now believe Conjectures 2 and 3 are true. A more or less random example of such a calculation is included below.

EXAMPLE. Let J be a triangulation of S^2 and let L be the suspension of J. Label each edge in J by 2 and each edge in L-J by 3. This satisfies (*). If J is not the boundary of a tetrahedron and if each circuit of length three in J bounds a triangle then it also satisfies (**). Let a_i denote the number of i-simplices in J. Let us calculate $\chi(X)$ using

(1) $$\chi(X) = 1 + \sum_{S \in L} (-1)^{\mathrm{Card}(S)} |\Gamma_S|^{-1}$$

There are (a_0+2) vertices in L each of weight $\frac{1}{2}$; hence, the vertices contribute $-\frac{1}{2}(a_0+2)$. There are two types of edges: a_1 of type $\cdot\,\cdot$ (weight $\frac{1}{4}$) and $2a_0$ of type •—• (weight $\frac{1}{6}$); hence, the edges contribute $\frac{1}{4}a_1 + \frac{1}{3}a_0$. There are two types of triangles: a_2 of type $\cdot\cdot\cdot$ (weight $\frac{1}{8}$) and $2a_1$ of type •—•—• (weight $\frac{1}{24}$); hence the triangles contribute $-(\frac{1}{8}a_2 + \frac{1}{12}a_3)$. There are $2a_2$ tetrahedron each of type •—< (weight $\frac{1}{192}$); hence, the tetrahedra contribute $\frac{1}{96}a_2$. Thus,

$$\begin{aligned}\chi(X) &= 1 - \tfrac{1}{2}(a_0+2) + (\tfrac{1}{4}a_1 + \tfrac{1}{3}a_0) - (\tfrac{1}{8}a_2 + \tfrac{1}{12}a_1) + \tfrac{1}{96}a_2 \\ &= \tfrac{1}{6}(-a_0+a_1 - \tfrac{11}{16}a_2).\end{aligned}$$

Since $a_0 - a_1 + a_2 = 2$ and $3a_2 = 2a_1$, we can rewrite this as

$$\chi(X) = \frac{1}{6}\left(\frac{5}{24}a_1 - 2\right).$$

If this is to be ≤ 0, then we must have $a_1 \leq \frac{48}{5} < 10$. Thus, $a_1 \leq 9$. If $a_1 \leq 9$, then the equation $a_0 - \frac{1}{3}a_1 = 2$ implies that either $(a_0,a_1) = (4,6)$ or $(5,9)$. The first case can only happen if J is the boundary of a tetrahedron and the second only if J is the suspension of the boundary of a triangle. In either case (**) does not hold; while in every other case $\chi(X) \geq 0$.

The geometric picture of X is as follows. Let Y be a 3-cell with ∂Y cellulated as the dual polyhedron to J and with all dihedral angles $90°$. Combinatorially, X is $Y \times I$; however, the top and bottom faces $Y \times \{0\}$ and $Y \times \{1\}$ meet each (face of Y) $\times$ I at a dihedral angle of $60°$.

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