NONPOSITIVE CURVATURE OF BLOW-UPS

M. DAVIS, T. JANUSZKIEWICZ, AND R. SCOTT

0. Introduction

Consider the following situation: $M_{\mathbb{C}}$ is a complex manifold of complex dimension n, and $D_{\mathbb{C}}$ is a union of smooth complex codimension-one submanifolds (i.e., $D_{\mathbb{C}}$ is a smooth divisor). Examples of this situation include: (1) arrangements of projective hyperplanes in $\mathbb{CP}^n$, as well as various blow-ups of such arrangements along intersections of hyperplanes, (2) nonsingular toric varieties (where $D_{\mathbb{C}}$ is the complement of the $(\mathbb{C}^*)^n$-orbit), and (3) certain compactifications of point-configurations in $\mathbb{CP}^1$ (where $D_{\mathbb{C}}$ is the complement of the nondegenerate configurations). In such examples there is often a "real version" of $(M_{\mathbb{C}}, D_{\mathbb{C}})$ which we will denote by (M, D). By this we mean that M is the fixed point set of a smooth involution on $M_{\mathbb{C}}$ which is locally isomorphic to complex conjugation on $\mathbb{C}^n$ and that $D = D_{\mathbb{C}} \cap M$. Thus, M is a smooth n-manifold and D is a union of codimension-one smooth submanifolds. Our primary interest in this paper is the geometry and topology of the pair (M, D). The examples in which we are interested will have the features discussed in (A), (B), and (C) below.

(A) *Cellulations by polytopes.* The divisor D cuts M into regions, called *chambers*, which are combinatorially equivalent to convex polytopes. In this case, we say D gives a *cellulation of* M. In addition, D will be locally isomorphic to an arrangement of hyperplanes. (If D has the last property, then each dual cell in M will be a "zonotope".)

(B) *The associated cellulation by cubes.* If the above cellulation is by simple polytopes (an n-dimensional polytope is *simple* if n-edges meet at each vertex), then so is its dual cellulation. Moreover, these cellulations will have a common subdivision by cubes.

(C) *Nonpositive curvature and asphericity.* Any cubical cell complex K has a natural piecewise-Euclidean structure in which each cell is identified with a regular Euclidean cube of some fixed size (say, of edge length 1). This then defines a "length metric" on each component of K: the length of a linear path in some cell is its Euclidean length, and the distance between two points in K is the infimum of the lengths of all piecewise linear paths between them. It follows that any two points in the same path component of K can be connected by a geodesic segment (a *geodesic segment* is the image of an interval under an isometric embedding). By

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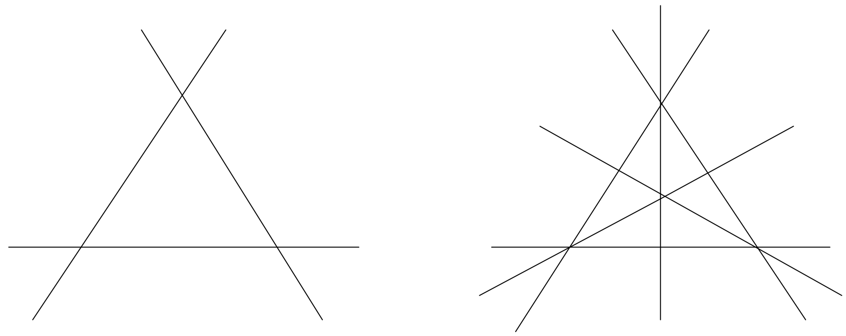
comparing small geodesic triangles in K with the corresponding triangles of the same edge lengths in the Euclidean plane, it then makes sense to ask if K has nonpositive curvature in the sense of Aleksandrov. Gromov has shown that for a cubical complex K, there is a simple combinatorial condition that is necessary and sufficient for this to be the case. The condition is that the link of each vertex in K is a "flag complex" (see Section 1.6). Gromov has also shown that the universal cover of a nonpositively curved space is contractible. Thus, if the link of each vertex in a cubical complex K is a flag complex, then K is aspherical. In other words, it is a $K(\pi, 1)$-complex. (For a more detailed discussion of this material, see [G] or [D3].) One of the main points of this paper is to show that for many of the examples in which we are interested, Gromov's condition holds. Thus, these manifolds admit piecewise Euclidean metrics of nonpositive curvature, and hence, are aspherical spaces.

0.1. Real subspace arrangements. Let V be a finite dimensional real vector space. Let $H_1, \dots, H_m$ be a collection of linear hyperplanes in V, and let $\mathcal{H}$ be the collection of all possible intersections of these hyperplanes. We call $\mathcal{H}$ the *hyperplane arrangement* generated by $\{H_1, \dots, H_m\}$. Let $\mathbb{S}(V)$ denote the sphere in V and $\mathbb{P}(V)$ the projective space of V. For $1 \leq i \leq m$, $\mathbb{P}(H_i)$ is a projective subspace of codimension-one in $\mathbb{P}(V)$; thus $D = \mathbb{P}(H_1) \cup \dots \cup \mathbb{P}(H_m)$ is a divisor in $\mathbb{P}(V)$. The arrangement $\mathcal{H}$ is *essential* if $H_1 \cap \dots \cap H_m = 0$. We suppose that this is the case. Then the H_i cut V into polyhedral cones and $\mathbb{S}(V)$ into spherical polytopes. Since these spherical polytopes occur in pairs, consisting of a polytope and its image under the antipodal map, we get a similar description of $\mathbb{P}(V)$. Thus, D gives a cellulation of $\mathbb{P}(V)$ as in paragraph (A) above. The arrangement $\mathcal{H}$ is *simplicial* if each cell in this cellulation of $\mathbb{P}(V)$ is a simplex. Next we give some examples of simplicial arrangements.

Example 0.1.1. *The coordinate hyperplane arrangement* (also called the *Boolean arrangement*). Let $V = \mathbb{R}^{n+1}$ and let H_i, $i = 1, \dots, n+1$, be the hyperplane defined by $x_i = 0$. Then $\{H_i\}$ generates the *coordinate hyperplane arrangement.* The divisor $\mathbb{P}(H_1) \cup \dots \cup \mathbb{P}(H_{n+1})$ cuts $\mathbb{P}(V)$ (otherwise, known as $\mathbb{RP}^n$) into 2^n n-simplices. The picture for $n = 2$ is given in Figure 1.

Example 0.1.2. *The braid arrangement.* Let V be the hyperplane in $\mathbb{R}^{n+2}$ defined by $\sum x_i = 0$. For $1 \leq i < j \leq n+2$, let H_{ij} denote the hyperplane in V defined by $x_i = x_j$. Then $\mathcal{H} = \{H_{ij}\}$ generates the *braid arrangement.* The symmetric group S_{n+2} acts on V by permuting the coordinates. The transposition (ij) acts as orthogonal reflection across H_{ij}. The union of the H_{ij} cuts V into simplicial cones, each of which is a fundamental domain for the action of S_{n+2}. It follows that the union of the $\mathbb{P}(H_{ij})$ cuts $\mathbb{P}(V)$ apart into $(n+2)!/2$ simplices. The picture for $n = 2$ is also given in Figure 1.

Example 0.1.3. *Reflection arrangements.* More generally, suppose that W is a finite group of linear transformations of V generated by orthogonal reflections. Then



Coordinate arrangement   Braid arrangement

FIGURE 1.

W is a Coxeter group. Further, assume that the representation of W on V is *essential* in the sense that the fixed subspace is $\{0\}$. Then the arrangement $\mathcal{H}$ of all reflection hyperplanes is essential and simplicial. This is a generalization of the previous two examples: if the Coxeter diagram of W is •–•···•–•, then we have the braid arrangement; if it is • •···• •, then we have the coordinate hyperplane arrangement.

Blow-ups of projective arrangements. Let A be a linear subspace of V and $\mathbb{P}(A)$ the corresponding projective subspace of $\mathbb{P}(V)$. To *blow-up* $\mathbb{P}(V)$ *along* $\mathbb{P}(A)$, remove $\mathbb{P}(A)$ and replace it by the projective space bundle associated to the normal bundle of $\mathbb{P}(A)$ in $\mathbb{P}(V)$. The projective space bundle (now a codimension-one submanifold of the blow-up) is called the *exceptional divisor*. If A is a hyperplane, then blowing up along $\mathbb{P}(A)$ does not alter $\mathbb{P}(V)$. However, if A is of codimension greater than one, then the blow-up along $\mathbb{P}(A)$ is a new manifold denoted by $\mathbb{P}(V)_{\#\mathbb{P}(A)}$. Now suppose that $\mathcal{H}$ is an arrangement of hyperplanes in V and that A is some intersection of hyperplanes in $\mathcal{H}$. Then for each subspace $B \in \mathcal{H}$ such that $B \not\subset A$, the blow-up $\mathbb{P}(B)_{\#\mathbb{P}(B\cap A)}$ can be identified with a smooth submanifold of $\mathbb{P}(V)_{\#\mathbb{P}(A)}$. This submanifold is called the *proper transform* of $\mathbb{P}(B)$. The union of the codimension-one proper transforms of the $\mathbb{P}(B)$, $B \in \mathcal{H}$, and the exceptional divisor give a new divisor $D_\#$ in $\mathbb{P}(V)_{\#\mathbb{P}(A)}$. We can then continue in this fashion, blowing up along iterated proper transforms of elements in $\mathcal{H}$. We denote such an iterated blow-up by $\mathbb{P}(V)_\#$ with corresponding divisor $D_\#$.

If the arrangement $\mathcal{H}$ is essential, then the divisor D gives a cellulation of $\mathbb{P}(V)$ by polytopes and $D_\#$ gives a cellulation of $\mathbb{P}(V)_{\#\mathbb{P}(A)}$. These two cellulations (of different manifolds) are related as follows. The cellulation of $\mathbb{P}(V)_{\#\mathbb{P}(A)}$ is obtained by truncating each chamber of $\mathbb{P}(V)$ which meets $\mathbb{P}(A)$ in a face (of the same dimension as $\mathbb{P}(A)$) and then gluing this truncated chamber to the antipodal chamber across $\mathbb{P}(A)$. For example, consider the coordinate hyperplane arrangement in $\mathbb{RP}^2$ in Figure 1. It gives a cellulation of $\mathbb{RP}^2$ by 4 triangles (any two of which share the same three vertices). Blow up one of these vertices. The resulting 2-manifold is $\mathbb{RP}^2\#\mathbb{RP}^2$, which is a Klein bottle. The truncation of each triangle is a square, so we obtain a cellulation of the Klein bottle by 4 squares. Similarly, we could have blown up all three vertices. The result would be the non-orientable surface with Euler characteristic -2 cellulated by 4 hexagons.

In [DP1], De Concini and Procesi define the notion of a "building set" for a hyperplane arrangement $\mathcal{H}$. Roughly speaking, this is a collection of subspaces in $\mathcal{H}$ for which: (1) the iterated blow-up along proper transforms of elements in this collection does not depend on the order of the blow-ups (for subspaces of any given dimension), and (2) the resulting divisor $D_\#$ in $\mathbb{P}(V)_\#$ has normal crossings. In particular, they show that given any subspace arrangement, there is a maximum building set as well as a minimum one. For a hyperplane arrangement, the maximum building set consists of all intersections of hyperplanes, while the minimum building set consists of the "irreducible" intersections (see Section 3.1). Given a hyperplane arrangement $\mathcal{H}$, we denote the corresponding minimal blow-up $\mathbb{P}(V)_{\#min}$ and the corresponding maximal blow-up $\mathbb{P}(V)_{\#max}$.

For the purposes of obtaining manifolds that satisfy conditions (A)-(C) above, we introduce the weaker notion of a "partial building set" for $\mathcal{H}$, dropping the requirement that $D_\#$ have normal crossings. In Section 3, we will give necessary and sufficient conditions on a partial building set guaranteeing that the iterated blow-up $\mathbb{P}(V)_\#$ will be cellulated by simple polytopes (in which case the dual cellulation will be by simple zonotopes). Hence, by (B) and (C) above, these blow-ups will have natural cubical subdivisions and natural piecewise Euclidean metrics. In particular, any building set will satisfy these conditions (the normal crossing condition (2) is equivalent to the dual zonotopes all being cubes). We shall prove the following two theorems in Section 4 and Section 5.

Theorem 0.1.4. *Let $\mathcal{H}$ be an essential hyperplane arrangement in V. Then the maximal blow-up $\mathbb{P}(V)_{\#max}$ has a nonpositively curved, cubical structure in the sense of paragraph (C) above. In particular, $\mathbb{P}(V)_{\#max}$ is aspherical.*

Theorem 0.1.5. *Suppose $\mathcal{H}$ is a simplicial hyperplane arrangement in V. Then the following statements are equivalent.*
(i) The arrangement does not admit a decomposition into 3 or more irreducible factors.
(ii) $\mathbb{P}(V)_{\#min}$ is aspherical.
(iii) The natural cubical structure on $\mathbb{P}(V)_{\#min}$ is nonpositively curved.

Before discussing more specific examples of these blow-ups, we need to define two special polytopes, the permutohedron and the associahedron.

Pick a point in the complement of the n-dimensional braid arrangement and consider its orbit under the symmetric group S_{n+1}. The convex hull of such an orbit is an n-dimensional *permutohedron.* A 2-dimensional permutohedron is a hexagon. Alternatively, start with an n-simplex and truncate all of its faces of codimension $\geq$

2. Thus, the maximal blow-up of any simplicial projective hyperplane arrangement is cellulated by permutohedra.

Consider the Coxeter diagram of A_{n+1}: •–• · · · •–•. Its nodes correspond to the codimension-one faces of an n-simplex, and proper subsets of the set of nodes correspond to proper faces of the simplex. Truncate the faces of the simplex which correspond to connected subdiagrams of A_{n+1}. The resulting n-dimensional polytope is the *associahedron*. For example, a 2-dimensional associahedron is a pentagon. These polytopes were described almost 35 years ago by Stasheff [St] in connection with the higher associativity properties of H-spaces.

Example 0.1.6. *The maximal blow-up of the coordinate hyperplane arrangement in* $\mathbb{RP}^n$. This is cellulated by 2^n permutohedra, all of which meet at any given vertex. There is a dual cellulation by "big cubes" (zonotopes) as in paragraph (A), and a common subdivision of the two cellulations by "small cubes" as in paragraph (B). The $n = 2$ picture is shown in Figure 2 with a permutohedron (lightly shaded), big cube (in grey), and small cube (in black). In the figure, the blow-up is obtained by removing the interiors of the 3 white squares and identifying antipodal points on their boundaries.

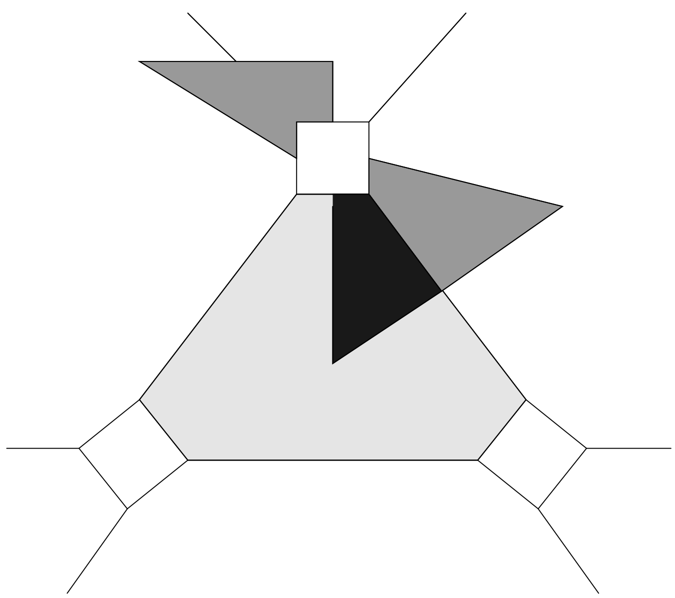


FIGURE 2.

Example 0.1.7. *The minimal blow-up of the braid arrangement in* $\mathbb{RP}^n$. This n-manifold, denoted $\mathbb{P}_{\#min}$ is cellulated by $(n+2)!/2$ associahedra, 2^n of which meet at any vertex. (See [Ka1], section 4.) Again, there is a dual cellulation by big cubes. For example, when $n = 2$, $\mathbb{P}_{\#min}$ is the blow-up of $\mathbb{RP}^2$ at 4 points (the triple points in Figure 1); hence, it is the nonorientable surface of Euler characteristic -3, cellulated by 12 pentagons (See Figure 3. Again, opposite points on the boundary of each of the white hexagons are identified.)

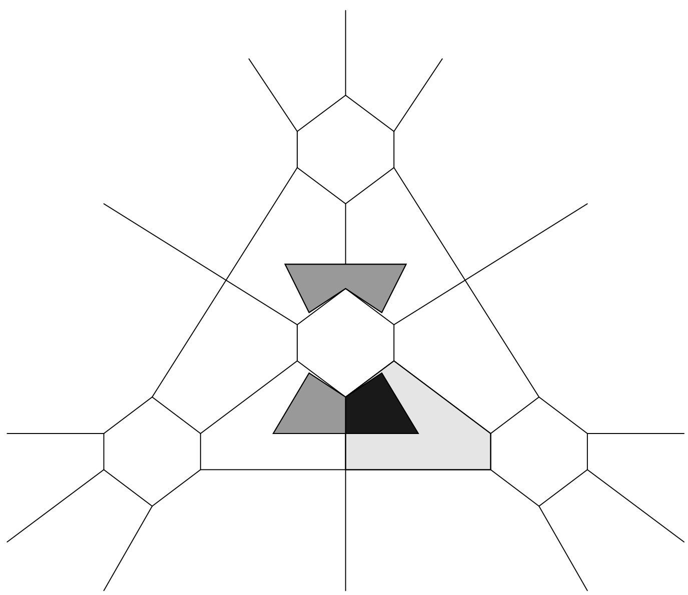


FIGURE 3.

Example 0.1.8. *Oriented blow-ups and complements of real arrangements.* As an application of *partial* building sets, we consider the $K(\pi, 1)$-problem for complements of real codimension-2 arrangements. Instead of replacing $\mathbb{P}(A)$ in $\mathbb{P}(V)$ with the associated projective-space bundle of its normal bundle, one could just as well replace it with the associated sphere bundle. Topologically, this operation is equivalent to removing an open tubular neighborhood of $\mathbb{P}(A)$, producing a manifold with boundary. As before, we can iterate this blow-up procedure, obtaining a manifold with corners which is cellulated by convex cells. We denote this "oriented" blow-up of $\mathbb{P}(V)$ with respect to a partial building set by $\mathbb{P}(V)_{\odot}$. If E is the set of all codimension ≥ 2 subspaces in the partial building set, then it follows that $\mathbb{P}(V)_{\odot}$ and the complement $\mathbb{P}(V) - E$ are homotopy equivalent. Moreover, we will show in Section 3.5 that $\mathbb{P}(V)_\#$ is nonpositively curved if and only if $\mathbb{P}(V)_\odot$ is nonpositively curved. Using these facts, we prove the following theorem in Section 4.4, settling a conjecture of Khovanov [Kh].

Theorem 0.1.9. *Let E be a W-stable union of codimension-2 subspaces in a real reflection group arrangement. Then the natural cubical metric on the oriented blow-up is nonpositively curved; hence, $\mathbb{P}(V) - E$ is aspherical.*

0.2. Toric varieties. Consider the action on $\mathbb{CP}^N$ of the algebraic torus $H_{\mathbb{C}}^N = (\mathbb{C}^*)^{N+1}/(\mathbb{C}^*)$. By a *nonsingular toric variety*, we shall mean a complex submanifold $M_{\mathbb{C}} \subset \mathbb{CP}^N$ that is the closure of an orbit of an n-dimensional algebraic subtorus $H_{\mathbb{C}}^n \subset T^N$ (where the orbit has trivial isotropy subgroup). Such an $M_{\mathbb{C}}$ is a union of $H_{\mathbb{C}}^n$-orbits. Let $T^n = (S^1)^n$ be the compact subtorus of $H_{\mathbb{C}}^n$. Both the algebraic and compact torus act on $M_{\mathbb{C}}$. There is a "moment map" $\mu : M_{\mathbb{C}} \to \mathbb{R}^n$ which serves as a quotient map for the T^n-action in the following sense ([Jur],[A],[GS]); the image is a simple convex polytope P, μ is constant on T^n-orbits, and the induced map $\overline{\mu} : M_{\mathbb{C}}/T^n \to P$ is a homeomorphism. Moreover, the preimage of a k-dimensional face of P is a complex k-dimensional submanifold of $M_{\mathbb{C}}$. In particular, the preimage of the boundary of P, called the *toric divisor* and denoted $D_{\mathbb{C}}$, is a union of complex codimension-one submanifolds.

Let $H_{\mathbb{R}}^N$ and $H_{\mathbb{R}}^n$ be the real parts of $H_{\mathbb{C}}^N$ and $H_{\mathbb{C}}^n$, respectively. Let $J = T^n \cap H_{\mathbb{R}}^n$, the maximal compact subgroup of $H_{\mathbb{R}}^n$ (so $J \cong (Z_2)^n$ and $H_{\mathbb{R}}^n \cong J \times (\mathbb{R}_+)^n$, where $\mathbb{R}_+$ denotes the positive real numbers). Then $H_{\mathbb{R}}^n$ acts on $M_{\mathbb{C}}$ and we let M denote the closure of a generic orbit. Each of the 2^n components of such an orbit is mapped homeomorphically by μ onto the interior of P, and the restriction $\mu : M \to P$ is a quotient map for the J-action. It follows that M is cellulated by 2^n copies of P, and since the J-action is locally standard (Section 2.2), the complement of the generic orbit is a divisor with normal crossings. As in paragraph (A) there is a dual cellulation of M by big cubes and a common subdivision by small cubes. The resulting piecewise-Euclidean metric is nonpositively curved if and only if the boundary complex of P is dual to a flag complex. The following theorem is proved in Section 2.

Theorem 0.2.1. *Let $M_{\mathbb{C}}$ be a smooth (projective) complex toric variety with real part M and moment polytope P. Then the following statements are equivalent.*
(i) M is aspherical.
(ii) The boundary complex of P is dual to a flag complex.
(iii) The dual cubical cellulation of M is nonpositively curved.

Example 0.2.2. *The closure of a generic torus orbit in a generalized flag manifold.* Suppose $G_{\mathbb{C}}$ is a complex semisimple Lie group of rank n with maximal torus $H_{\mathbb{C}}$ $(\cong (\mathbb{C}^*)^n)$ and maximal compact subgroup U. Let $H_{\mathbb{R}}$ be a maximal $\mathbb{R}$-split subtorus of $H_{\mathbb{C}}$, let $G_{\mathbb{R}}$ be the real Lie group corresponding to $H_{\mathbb{R}}$, and let $K = U \cap G_{\mathbb{R}}$. Then $T = H_{\mathbb{C}} \cap U$ and $J = H_{\mathbb{R}} \cap K$ are isomorphic to $(S^1)^n$ and $(\mathbb{Z}_2)^n$, respectively. Let $B_{\mathbb{C}}$ be the Borel subgroup of $G_{\mathbb{C}}$ corresponding to $H_{\mathbb{C}}$, and let

$B_{\mathbb{R}} = G_{\mathbb{R}} \cap B_{\mathbb{C}}$. Then $G_{\mathbb{C}}/B_{\mathbb{C}} = U/T$ and $G_{\mathbb{R}}/B_{\mathbb{R}} = K/J$ are both compact manifolds.

There are natural left actions of $H_{\mathbb{C}}$ on $G_{\mathbb{C}}/B_{\mathbb{C}}$ and $H_{\mathbb{R}}$ on $G_{\mathbb{R}}/B_{\mathbb{R}}$. Moreover, there are projective embeddings into $\mathbb{CP}^N$ and $\mathbb{RP}^N$, resp., such that these torus actions are restrictions of actions of subgroups of the diagonal (see, for example, [FH]). Let $M_{\mathbb{C}}$ (resp., M) be the closure of a generic $H_{\mathbb{C}}$-orbit (resp., $H_{\mathbb{R}}$-orbit). Then $M_{\mathbb{C}}$ is a nonsingular toric variety with real part M. The image of the moment map can be identified with a "generalized permutohedron", i.e., the convex hull of an orbit of the Weyl group action on $\mathbb{R}^n$. Since the boundary complex of a generalized permutohedron is a flag complex (c.f., Lemma 1.6.4), M is always nonpositively curved.

For example, if $G_{\mathbb{C}} = SL(n+1, \mathbb{C})$, then $H_{\mathbb{C}}$ is the set of diagonal matrices. Let $B_{\mathbb{C}}$ denote the Borel subgroup corresponding to $H_{\mathbb{C}}$. Then $M_{\mathbb{C}} = G_{\mathbb{C}}/B_{\mathbb{C}}$ (resp., $M = G_{\mathbb{R}}/B_{\mathbb{R}}$) is the variety of full flags in $\mathbb{C}^{n+1}$ (resp., in $\mathbb{R}^{n+1}$). The corresponding Weyl group orbit is the generic one, and the convex hull of an orbit is the n-dimensional permutohedron. Thus, M is cellulated by 2^n copies of the permutohedron. Since the dual polytope of the permutohedron is a flag complex, M is aspherical. These manifolds M are interesting for a number of reasons. For example, they support a "Toda flow" which has recently been studied by Kodama and Ye [KY].

0.3. Point configurations and Chow quotients. The minimal blow-up of the braid arrangement in $\mathbb{P}^n$ (Example 0.1.7) has been shown by Kapranov in [Ka1] and [Ka2] to coincide with some other spaces of classical importance. These include (1) certain compactifications of $(n+3)$-point configurations in $\mathbb{RP}^1$, and (2) the "Chow quotient" of the Grassmannian $G(2, n+3)$ of 2-planes in $\mathbb{R}^{n+3}$ by the diagonal $(\mathbb{R}^*)^{n+3}$-action. We give a rough summary here.

Let U denote the set of generic $GL(2,\mathbb{R})$-orbits in $(\mathbb{RP}^1)^{n+3}$ (where the action is diagonal). Let U' be the set of generic $(\mathbb{R}^*)^{n+3}$-orbits in $G(2, n+3)$. There is a bijection $U \to U'$ (the Gelfand-MacPherson correspondence [GM]) given as follows. The $GL(2,\mathbb{R})$-orbit of the point $([x_1, y_1], \dots, [x_{n+3}, y_{n+3}])$ in $(\mathbb{RP}^1)^{n+3}$ corresponds to the $(\mathbb{R}^*)^{n+3}$-orbit of the 2-plane spanned by $x = (x_1, \dots, x_{n+3})$ and $y = (y_1, \dots, y_{n+3})$. The set U (respectively, U') is naturally realized as a subset of the Chow variety $\mathcal{C}$ (resp., $\mathcal{C}'$) which parameterizes generic $GL(2,\mathbb{R})$-orbit closures in $(\mathbb{RP}^1)^{n+3}$ (resp., generic $(\mathbb{R}^*)^{n+3}$-orbit closures in $G(2, n+3)$). The closures of U and U' in these Chow varieties are called *Chow quotients*, and denoted $(\mathbb{RP}^1)^{n+3}//GL(2,\mathbb{R})$ and $G(2,n+3)//(\mathbb{R}^*)^{n+3}$.

Theorem 0.3.1. (Kapranov [Ka1] and [Ka2]) *For all $n \geq 1$, the following varieties are isomorphic.*
(i) The minimal blow-up of the braid arrangement in $\mathbb{RP}^n$.
(ii) The Chow quotient $G(2,n+3)//(\mathbb{R}^*)^{n+3}$.
(iii) The Chow quotient $(\mathbb{RP}^1)^{n+3}//GL(2,\mathbb{R})$.
(iv) The real points of the Grothendieck-Knudsen moduli space for stable $(n+3)$-pointed curves of genus 0.

0.4. Blowing up zonotopal cell complexes and Gromov's Möbius band hyperbolization procedure. A *zonotope* is a convex polytope whose poset of faces is dual to the face poset of a hyperplane arrangement (the precise definition is given in Section 1.4). A key property of zonotopes is that they are centrally symmetric. As we explained in 0.1, a projective hyperplane arrangement in $\mathbb{RP}^n$ divides it into cells. The dual cells are zonotopes. Blowing up subspaces corresponds to a process of truncating cells, and the dual cells become simpler zonotopes. In fact, one can describe this process directly in terms of the dual cellulation by zonotopes. For example, in the braid arrangement of Figure 1, the dual cell to a triple point is a hexagon P. To blow-up the triple point, one removes the interior of P and replaces it by the Möbius band $(\partial P \times [-1,1])/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on ∂P via the central symmetry and on $[-1,1]$ via $t \mapsto -t$. It turns out that the best way to view the blowing-up procedure is as an operation of the above type on the dual cell structure. This is the point of view we take in Section 3. One reason for doing this is that the construction can then be generalized to "zonotopal cell complexes".

Another reason for taking this viewpoint is that it generalizes the "Möbius band hyperbolization procedure" introduced by Gromov ([G], Section 3.4): given a cubical cell complex K, Gromov described a functorial procedure for constructing a new cubical cell complex $h(K)$ which is nonpositively curved. One of our initial observations when we began the research for this paper was that the manifold obtained by performing Gromov's hyperbolization to the boundary of the $(n+1)$-dimensional cube and then dividing by the central symmetry is the same as the closure of a generic orbit in the flag manifold $SL(n+1,\mathbb{R})/B$ (Example 0.2.2). Furthermore, it follows from the theory of toric varieties that this generic orbit closure coincides with the maximal blow-up of the coordinate arrangement in $\mathbb{RP}^n$ (Example 0.1.6). (Proofs of these coincidences will be given in Section 4.2.) The constructions in Section 3 show that these are not just coincidences. The dual cellulation of $\mathbb{RP}^n$ corresponding to the coordinate arrangement is the boundary of the $(n+1)$-cube divided by the central symmetry, and taking the maximal blow-up is the same as applying Gromov's Möbius band procedure.

A cubical cell complex is a special case of a more general type of cell complex in which all of the cells are combinatorially equivalent to zonotopes. A natural categorical framework for all of the real blow-ups in this paper is that of zonotopal cell complexes. In this setting, the blow-up procedure can be described as a functor which, given a zonotopal cell complex K and a suitable collection $\mathcal{M}$ of cells, produces another zonotopal cell complex $K_{\#\mathcal{M}}$. In particular, if $\mathcal{M}$ is the set of all cells of K, then $K_{\#\mathcal{M}}$ is called the *maximal blow-up*, and if K is a cubical cell complex, this maximal blow-up is precisely Gromov's hyperbolization $h(K)$. We will show in Section 4 that the maximal blow-up of any zonotopal cell complex is nonpositively curved.

Acknowledgements. We would like to thank Yuji Kodama and James Stasheff for their discussions with us about the manifolds in Examples 0.1.6 and 0.1.7 which ignited our interest in the subject of this paper.

Contents

1. Nonpositively curved metrics on zonotopal cell complexes

1.1. Cells and cell complexes. Let $\mathcal{P}$ be a partially ordered set (i.e., a *poset*). A *flag* in $\mathcal{P}$ is a finite totally ordered subset of $\mathcal{P}$. The set $\mathrm{Fl}(\mathcal{P})$ consisting of all flags in $\mathcal{P}$ is itself partially ordered inclusion and is called the *order complex* (or "derived complex") for $\mathcal{P}$.

A *cell* or *polytope* P is the convex hull of a finite set of points in some real vector space. The set of faces, partially ordered by inclusion, forms a poset $\mathcal{P}$. For every face F of P, let v_F denote the barycenter of F (any point in the relative interior would suffice). Then any flag $\alpha = (F_1 < \cdots < F_k)$ in $\mathrm{Fl}(\mathcal{P})$ can be identified with the simplex with vertex set $\{v_{F_1}, \dots , v_{F_k}\}$, and this decomposition of P will be called the *barycentric subdivision*. Thus, the poset $\mathrm{Fl}(\mathcal{P})$ is precisely the partially ordered set associated to the barycentric subdivision. Two cells are *combinatorially equivalent* if their posets are isomorphic; in this case, the bijection between their sets of barycenters extends to a piecewise linear homeomorphism between the cells. We will call such a homeomorphism the *realization* of a combinatorial equivalence.

By a *cell complex* we shall mean a space K formed by gluing together cells via certain (geometric realizations of) combinatorial equivalences of their faces, together with the decomposition of K into cells. We shall also assume that different faces of the same cell are not identified; thus, each cell will be homeomorphic to its image in K. Let $\mathcal{P}(K)$ denote the associated poset of cells. Two cell complexes K and L are *combinatorially equivalent*, written $K \cong L$, if the corresponding posets are isomorphic. By passing to barycentric subdivisions, two combinatorially equivalent cell complexes are homeomorphic via a homeomorphism which restricts to a linear map on each simplex.

Convention. We will drop the word "cell" if the intersection of two cells is either empty or a single cell. (Thus, in a simplicial complex every simplex is determined by its vertex set, while this need not be the case for an arbitrary simplicial cell complex.)

A cell complex is a *cubical cell complex* if its cells are combinatorially equivalent to cubes.

The *boundary* of a cell P is the union of all proper faces of P. It is naturally a cell complex, and we denote it by ∂P. Given a convex polytope P, there is a dual polytope $\check{P}$ which has the property that $\mathcal{P}(\partial\check{P}) = \mathcal{P}(\partial P)^{op}$, where $\mathcal{P}^{op}$ denotes the *opposite poset* of $\mathcal{P}$, in which the order relation is reversed.

If F is a k-dimensional face of an n-dimensional cell P and x is a point of F, then the *link of F in P*, denoted by $\mathrm{Lk}(F, P)$ is the spherical $(n-k-1)$-cell consisting of all unit tangent vectors to P at a point $x \in F$ which are inward pointing and normal to F. Up to a linear isomorphism, the link of F is independent of the point x, so we omit x from the notation. If F is a cell in a cell complex K, then $\mathrm{Lk}(F, K)$ is the union of all $\mathrm{Lk}(F, P)$, where P is a cell of K which contains F. Since any spherical cell $\mathrm{Lk}(F, P)$ is combinatorially equivalent to a (Euclidean) cell, $\mathrm{Lk}(F, K)$ is equivalent to a cell complex.

By a *cellulation* of a manifold M, we shall mean a cell complex K together with a homeomorphism $K \to M$. One says that K is a *PL n-manifold* if for each k-cell P of K, $\mathrm{Lk}(P, K)$ is piecewise linearly homeomorphic to S^{n-k-1}. In all cases of interest to us, a stronger condition will hold: $\mathrm{Lk}(P, K)$ is combinatorially equivalent to the boundary complex of a convex $(n-k)$-cell. When this happens we will say that K is a *nice* PL-manifold. (A recent result of Mnev and Mani [MM] states that any nice PL-manifold admits a canonical smooth structure.)

If K is a nice PL-manifold, then there is a *dual cell complex* $\check{K}$ defined as follows. Let $\mathcal{P}(K)$ denote the poset of nonempty cells of K and for each $P \in \mathcal{P}(K)$, let $\mathcal{P}(K)_{\geq P}$ (resp., $\mathcal{P}(K)_{>P}$) denote the subposet of cells which contain (resp., properly contain) P. Thus, $\mathcal{P}(K)_{>P}$ is isomorphic to the poset of cells in $\mathrm{Lk}(P, K)$. Since K is nice, there is a dual cell $D(P)$ with the property that $\mathcal{P}(\partial D(P)) = \mathcal{P}(\mathrm{Lk}(P, K))^{op}$. $\check{K}$ is defined to be the union of the $D(P)$ in a natural way. To be explicit, let K' denote the barycentric subdivision of K. Then the barycentric subdivision of $D(P)$ is naturally a subcomplex of K', and K' is the union of the $D(P)$. The two cell complexes K and $\check{K}$ are combinatorially dual in the sense that $\mathcal{P}(K)$ is isomorphic to $\mathcal{P}(\check{K})^{op}$. For example, if P is a convex n-cell, then ∂P and $\partial \check{P}$ give dual cellulations of S^{n-1}.

1.2. The standard cubical subdivision of a simple cell complex. Suppose $\mathcal{P}$ is a poset and $a \in \mathcal{P}$. If $a, b \in \mathcal{P}$ and $a \leq b$, then the interval $[a, b]$ is the subposet $\mathcal{P}_{\geq a} \cap \mathcal{P}_{\leq b}$. Let $\mathcal{I}(\mathcal{P})$ denote the poset of intervals of $\mathcal{P}$ (partially ordered by inclusion).

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. Denote the power set of $\{1, \dots, n\}$ by $\mathcal{S}_n$. For each $J \in \mathcal{S}_n$, set

$$e_J = \sum_{i \in J} e_i.$$

(If $J = \emptyset$, then $e_J = 0$.) Then $\{e_J\}$ is the vertex set of the unit n-cube $[0,1]^n$. For each interval $[J_1, J_2] \in \mathcal{I}(\mathcal{S}_n)$, let $\square_{[J_1, J_2]}$ denote the face of $\square^n$ $(= [0,1]^n)$ spanned by $\{e_J\}_{J \in [J_1, J_2]}$. It is a cube of dimension $\mathrm{Card}(J_1 - J_2)$. Hence, $\mathcal{P}(\square^n)$ is isomorphic to $\mathcal{I}(\mathcal{S}_n)$ via the isomorphism $\square_{[J_1, J_2]} \leftrightarrow [J_1, J_2]$.

If $\alpha = (J_0 < \cdots < J_k)$ is a flag in $\mathcal{S}_n$, let Δ_α be the simplex spanned by $e_{J_0}, \dots, e_{J_k}$. The *standard simplicial subdivision* of $\square^n$ is the subdivision consisting of all simplices Δ_α, $\alpha \in \mathrm{Fl}(\mathcal{S}_n)$.

An n-dimensional polytope P is *simple* if exactly n codimension-one faces meet at each vertex. For example, a cube is simple, an octahedron is not. Equivalently, P is simple if $\partial \check{P}$ is a simplicial complex. A cell complex is *simple* if each cell is a simple polytope.

Let P be a simple polytope and let $\mathcal{P}$ be its poset of faces. Let $\{v_F\}_{F \in \mathcal{P}}$ be a collection of barycenters. For each flag $\alpha \in \mathrm{Fl}(\mathcal{P})$, let Δ_α be the corresponding simplex in the barycentric subdivision of P. For each interval $[F_1, F_2]$ in $\mathcal{P}$, let $\square_{[F_1, F_2]}$ be the subcomplex of the barycentric subdivision of P consisting of all simplices Δ_α, $\alpha \in \mathrm{Fl}([F_1, F_2])$. Then $\square_{[F_1, F_2]}$ is simplicially isomorphic to the standard simplicial subdivision of the cube of dimension $\dim F_2 - \dim F_1$. The subdivision of P into $\{\square_{[F_1,F_2]}\}_{[F_1,F_2]\in\mathcal{I}(\mathcal{P})}$ is called the *standard cubical subdivision of* P, denoted by $P_\square$. The subcomplexes $\square_{[F_1,F_2]}$ are called *small cubes*. We note that any combinatorial automorphism of $\mathcal{P}$ induces a simplicial automorphism of the barycentric subdivision of P and this automorphism induces an automorphism of the standard cubical subdivision (since intervals of $\mathcal{P}$ are taken to intervals by an automorphism). If K is a simple cell complex, then its *standard cubical subdivision*, denoted $K_\square$ is constructed by taking the standard cubical subdivision of each cell.

The proofs of the next two lemmas are straightforward unwindings of the definitions.

Lemma 1.2.1. *Let x be the barycenter of a simple polytope P. Then there is a simplicial isomorphism between* $\mathrm{Lk}(x, P_\square)$ *and the simplicial complex* $\partial\check{P}$.

Lemma 1.2.2. *Let K be a simple cell complex and x a vertex of $K_\square$. Then x is the barycenter of some cell P_x of K and*

$$\mathrm{Lk}(x, K_\square) = \partial\check{P}_x * \mathrm{Lk}(P_x, K).$$

(Here $*$ is used to denote the join of two simplicial cell complexes. See Section 3.3.)

If a simple cell complex K is a nice PL manifold, then the dual cell complex is also simple. Moreover, the standard cubical subdivisions $K_\square$ and $\check{K}_\square$ coincide.

1.3. Arrangements. Let V be a real vector space. A finite collection $\mathcal{A}$ of subspaces in V is a *subspace arrangement* if it is closed under intersections (this is slightly nonstandard, but simplifies the notation). A subspace arrangement is partially ordered by inclusion, and two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent*, written $\mathcal{A}_1 \cong \mathcal{A}_2$ if there is an isomorphism of posets $\mathcal{A}_1 \to \mathcal{A}_2$ which preserves the dimension of subspaces. An arrangement is *essential* if it contains the zero subspace. An arrangement is a *hyperplane arrangement* if every element is an intersection of codimension-one subspaces. An essential hyperplane arrangement $\mathcal{H}$ cuts V into polyhedral cones. The set of these cones, partially ordered by inclusion, is the *face poset* of $\mathcal{H}$.

Let V^* be the dual space of V and for any subspace $A \subset V$, let $A^\perp$ be the subspace of V^* defined by $\{w \in V^* | w(A) = \{0\}\}$ (the annihilator of A). For any collection of subspaces $\mathcal{A}$ in V, let $\mathcal{A}^\perp$ denote the collection of subspaces $A^\perp$, $A \in \mathcal{A}$. Thus, the posets $\mathcal{A}$ and $\mathcal{A}^\perp$ are anti-isomorphic.

1.4. Zonotopes. Associated to an essential arrangement of hyperplanes $\mathcal{H}$ in a real vector space V, there is a convex polytope $Z_\mathcal{H}$, called the "associated zonotope". Let $H_1, \dots, H_k$ be the hyperplanes in $\mathcal{H}$ and let $w_1, \dots, w_k$ be linear forms in V^* such that $H_i = w_i^{-1}(0)$.

Definition 1.4.1. The *standard zonotope $Z_\mathcal{H}$ associated to $\mathcal{H}$* (and to $w_1, \dots, w_k$) is the image of the cube $[-1,1]^k$ in $\mathbb{R}^k$ under the linear map from $\mathbb{R}^k$ to V^* which sends e_i to w_i.

That is to say, $Z_{\mathcal{H}}$ is the "Minkowski sum" (or vector sum) of the line segments $[-w_1, w_1], \dots, [-w_k, w_k]$, defined by

$$Z_{\mathcal{H}} = \{\sum t_i w_i | t_i \in [-1, 1], 1 \le i \le k\}.$$

Any polytope combinatorially equivalent to $Z_{\mathcal{H}}$ will also be called a *zonotope.*

The main property of a zonotopes is stated as the following lemma. (A proof can be found as Prop. 2.2.2, p. 54, in [BLSWZ].)

Lemma 1.4.2. *The poset of faces, $\mathcal{P}(Z_{\mathcal{H}})$, is anti-isomorphic to the face poset of $\mathcal{H}$.*

Examples 1.4.3. (i) An m-gon is a zonotope if and only if m is even.

(ii) The n-dimensional cube $[-1, 1]^n$ is the zonotope corresponding to the arrangement of coordinate hyperplanes in $\mathbb{R}^n$.

(iii) More generally, a cartesian product of zonotopes is a zonotope (since a product of hyperplane arrangements is a hyperplane arrangement).

(iv) Any face of a zonotope is a zonotope.

(v) If $\mathcal{H}$ is the braid arrangement in $\mathbb{R}^n$, then the corresponding zonotope is called the *permutohedron.*

(vi) More generally, if W is a finite reflection group on $\mathbb{R}^n$ (i.e., a finite Coxeter group), then the zonotope corresponding to the arrangement of reflection hyperplanes is called in [D3] a *Coxeter cell.* We denote it Z_W.

The dual polytope $X_{\mathcal{H}}$ to $Z_{\mathcal{H}}$ is a convex polytope in V. The poset $\mathcal{P}(\partial X_{\mathcal{H}})$ is canonically isomorphic to the poset of nonzero faces of $\mathcal{H}$, i.e., $\partial X_{\mathcal{H}}$ is combinatorially equivalent to the cellulation of the unit sphere in V cut out by the hyperplanes in $\mathcal{H}$. It follows that the order complexes $\mathrm{Fl}(\mathcal{P}(X_{\mathcal{H}}))$ and $\mathrm{Fl}(\mathcal{P}(Z_{\mathcal{H}}))$ are canonically isomorphic. This means that the barycentric subdivisions of $X_{\mathcal{H}}$ and $Z_{\mathcal{H}}$ are canonically simplicially isomorphic. In what follows, we shall often use this to identify subcomplexes of the barycentric subdivision of $X_{\mathcal{H}}$ with subcomplexes of the barycentric subdivision of $Z_{\mathcal{H}}$.

Parallel faces and subspace pieces. Let $\mathcal{H}$ be an essential hyperplane arrangment in V with associated standard zonotope Z. Let $\mathcal{C}$ $(= \mathcal{H}^{\perp})$ be the collection of subspaces dual to $\mathcal{H}$. If F is a face of Z which is not a vertex, then the affine subspace spanned by F is parallel to a unique subspace $A_F \in \mathcal{C}$. Let $\mathcal{C}_Z$ denote the set of parallel classes of faces of Z. Note that the dimension of an element of $\mathcal{C}_Z$ is well-defined as the dimension of any representative. The natural bijection $[F] \leftrightarrow A_F$ between $\mathcal{C}$ and $\mathcal{C}_Z$ induces a partial order on $\mathcal{C}_Z$. This partial order can be described as follows. Given f_1 and f_2 in $\mathcal{C}_Z$, $f_1 \le f_2$ if and only if f_1 and f_2 have representatives F_1 and F_2, respectively, such that F_1 is a face of F_2. If $\widehat{\mathcal{C}} = \mathcal{C} \cup \{0\}$ and $\widehat{\mathcal{C}}_Z = \mathcal{C}_Z \cup \{0\}$ (where 0 denotes the equivalence class of vertices in Z), then $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{C}}_Z$ are isomorphic *lattices.* Thus, any two elements $f_1, f_2 \in \widehat{\mathcal{C}}_Z$ have a greatest lower bound, $f_1 \cap f_2$, as well as a least upper bound, $f_1 + f_2$ (called the *span* of

f_1 and f_2). If $\dim(f_1 + f_2) = \dim f_1 + \dim f_2$ (equivalently, if $\dim(f_1 \cap f_2) = 0$), then we write $f_1 \oplus f_2$ for $f_1 + f_2$.

A *decomposition* of a zonotope Z is a subset $\{f_1, \ldots, f_k\}$ of $\mathcal{C}_Z$ such that for all $g \in \mathcal{C}_Z$, $f_1 \cap g, f_2 \cap g, \ldots, f_k \cap g \in \mathcal{C}_Z$ and $g = (f_1 \cap g) \oplus \cdots \oplus (f_k \cap g)$. If F_i is a representative for f_i, $1 \leq i \leq k$, this means that Z is isomorphic to the product $F_1 \times \cdots \times F_k$. Z is *irreducible* if it cannot be decomposed into more than one factor.

We shall now show that the parallel relation on $\mathcal{P}(Z)$ can be defined combinatorially. First suppose that $\dim Z = 2$. Then Z is a $2m$-gon and two edges are parallel if and only if they are opposite (i.e., if there are exactly $m-2$ edges between them). This generates an equivalence relation on the set of edges of an n-dimensional zonotope Z: two edges e and e' are parallel if and only if we can find a sequence of 2-dimensional faces $F_1, \ldots, F_m$ and pairs (e_i, e'_i) of opposite edges in F_i such that $e = e_1$, $e' = e'_m$, and $e'_{i-1} = e_i$ for $1 < i \leq m$. This combinatorial definition of parallelism can then be extended to faces as follows. Given a face F of Z, let $\mathcal{E}(F)$ denote the set of parallel classes of edges of Z which are represented by edges of F. Then F and F' are parallel if and only if $\mathcal{E}(F) = \mathcal{E}(F')$. From the fact that we can define parallelism combinatorially, we immediately deduce the following.

Lemma 1.4.4. *Parallelism is preserved by combinatorial isomorphisms of zonotopes.*

If F_1 and F_2 are parallel faces of a (standard) zonotope Z in V^*, then there is a translation of V^* which takes F_1 onto F_2. This induces a combinatorial isomorphism c_{F_1,F_2}, called the *canonical isomorphism*, between $\mathcal{P}(Z)_{\leq F_1}$ and $\mathcal{P}(Z)_{\leq F_2}$. As we shall see in the proof of the next lemma, this isomorphism can be defined combinatorially.

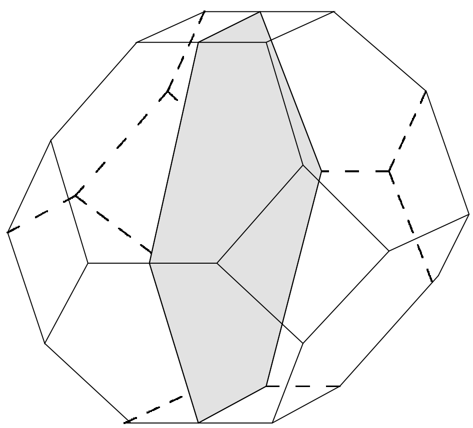
Lemma 1.4.5. *If $\phi : \mathcal{P}(Z) \to \mathcal{P}(Z')$ is an isomorphism and F_1 and F_2 are parallel faces of Z, then $c_{\phi(F_1),\phi(F_2)} = \phi \circ c_{F_1,F_2} \circ \phi^{-1}$.*

Before proving this lemma, we define "subspace pieces" in Z, the notion dual to that of parallel classes of faces. For any $f \in \mathcal{C}_Z$, let A be the corresponding element of $\mathcal{C}$. Then the collection of quotient spaces $\mathcal{C}/A = \{B/A | A \subseteq B,\ B \in \mathcal{C}\}$ is the dual (lattice) of a hyperplane arrangement in $A^{\perp}$. Let Z_f denote the corresponding zonotope in V^*/A. We have natural identifications $\mathcal{C}/A \cong \mathcal{C}_{\geq A} \cong (\mathcal{C}_Z)_{\geq f}$. Define a subposet $\mathcal{P}(Z)_{\geq f}$ of $\mathcal{P}(Z)$ by

$$\mathcal{P}(Z)_{\geq f} = \{F \in \mathcal{P}(Z) | [F] \geq f\},$$

where $[F]$ denotes the parallel class of F. Then there is a natural isomorphism of posets $\mathcal{P}(Z)_{\geq f} \cong \mathcal{P}(Z_f)$ (which takes a face F of Z to the face of Z_f corresponding to the subspace A_F/A in $\mathcal{C}/A$). This isomorphism induces an injection of order complexes $\mathrm{Fl}(\mathcal{P}(Z_f)) \to \mathrm{Fl}(\mathcal{P}(Z))$ and, hence, a simplicial embedding of barycentric subdivisions $Z'_f \to Z'$. The image P_f is called the *subspace piece dual to* f (or *dual to* A). If $\dim f = 1$, then P_f is called a *hyperplane piece*. Figure 4 shows a zonotope (the permutohedron) with one of its hyperplane pieces shaded.

Remark 1.4.6. Suppose X is the polytope dual to Z and E is the subspace in V dual to A. Then $X \cap E$ is naturally a subcomplex of the barycentric subdivision of X. That is, $E \cap \partial X$ is a subcomplex of ∂X, and $E \cap X$ is the cone on this subcomplex. Since X and Z are dual, we can identify $E \cap X$ with a subcomplex of the barycentric subdivision of Z. This subcomplex is precisely the subspace piece P_f defined above. For this reason we will also use $E \cap X$ to denote the subcomplex associated to E, relying on the context to determine whether it is a subcomplex of X' or a subcomplex of Z'. As a subcomplex of Z', $E \cap X$ intersects each of the faces in f in its barycenter.



Subspace piece in a zonotope

FIGURE 4.

If L is a line in $\mathcal{C}$, then an orientation for L induces an orientation on each edge parallel to L. This can be said combinatorially as follows. If e and e' are edges parallel to L and v and v' are vertices of e and e', respectively, then v and v' have the same orientation if they lie on the same side of the hyperplane piece dual to L.

Proof of Lemma 1.4.5. Let $n = \dim Z$. We will show by induction on n that the canonical isomorphism between two faces of Z can be defined combinatorially. This is obvious if $n = 1$. Let F_1 and F_2 be parallel faces of Z and let l be the parallel class of an edge of F_1. Then $\mathcal{P}(Z)_{\geq l}$ is isomorphic to the poset of faces of the zonotope Z_l of dimension $n-1$. Moreover, the subposets $(\mathcal{P}(Z)_{\geq l})_{\leq F_1}$ and $(\mathcal{P}(Z)_{\geq l})_{\leq F_2}$ correspond to parallel faces of this zonotope. By induction, the canonical isomorphism can be combinatorially defined on $(\mathcal{P}(Z)_{\geq l})_{\leq F_1}$. Hence, it can be defined on all faces in $\mathcal{P}(Z)_{\leq F_1}$ of dimension > 0. Now suppose that v is a vertex of F_1. We may as well assume that v is a vertex of an edge e in l and that

l is oriented. Then c_{F_1,F_2} takes e to an edge e' of F_2 parallel to e. Let v' be the vertex of e' with the same orientation as v. Then define c_{F_1,F_2} on v to be v'. □

Central symmetry of a zonotope. It follows from the definition that multiplication by -1 stabilizes any (standard) zonotope in V^*. Hence, it gives a well-defined combinatorial symmetry $a \in \mathrm{Aut}(\mathcal{P}(Z))$. The next lemma shows that the involution a can be defined combinatorially.

Lemma 1.4.7. *The involution a lies in the center of* $\mathrm{Aut}(\mathcal{P}(Z))$.

In view of this, a will be called the *central symmetry* of Z.

Proof. It suffices to give a combinatorial definition of a. Furthermore, it suffices to give the definition on the vertex set of Z. We shall do this by induction on the dimension n of the zonotope. If $n = 1$, then Z is the line segment and a switches its two vertices. Now suppose $n > 1$ and a combinatorial definition has been given for all zonotopes of dimension $n - 1$. Let v be a vertex of an n-dimensional zonotope Z. Choose an edge e containing v and let l be the parallel class of e. Then the intersection of e and the hyperplane piece $P_l = \mathrm{Im}(Z_l \to Z)$ is a vertex w of Z_l. By induction, the central symmetry a_l of Z_l is defined. Let $a(e)$ be the edge of Z which is dual to P_l and which intersects P_l in $a_l(w)$. Then $a(e)$ has two vertices v' and v''. One of these, say v' lies on the opposite side of P_l from v. Define $a(v) = v'$. Since hyperplane pieces are defined combinatorially (Lemma 1.4.4), this definition is combinatorial and clearly agrees with the linear definition. The lemma follows. □

1.5. Zonotopal cell complexes. A cell complex K is *zonotopal* if each cell is (combinatorially isomorphic to) a zonotope. For example, any cubical cell complex is zonotopal and simple. We will call the cells of a cubical complex K "big cubes" to distinguish them from the "small cubes" of the standard subdivision $K_\square$.

A Cartesian product of zonotopal cell complexes is a zonotopal cell complex. Any subcomplex of a zonotopal cell complex is zonotopal.

Example 1.5.1. (i) If Z is an n-dimensional zonotope, then ∂Z is a zonotopal complex homeomorphic to S^{n-1}.

(ii) The central involution $a : \partial Z \to \partial Z$ freely permutes the cells of ∂Z. Hence the orbit space

$$\mathbb{P}(Z) = \partial Z / a$$

is a zonotopal cell complex, homeomorphic to $\mathbb{RP}^{n-1}$.

(iii) The *blow-up of Z at its center*, denoted by $Z_\#$, is the quotient of $\partial Z \times [-1, 1]$ by the involution $\widehat{a}$ defined by $\widehat{a}(z, t) = (a(z), -t)$, i.e.,

$$Z_\# = \partial Z \times_{\mathbb{Z}_2} [-1, 1].$$

In other words, $Z_\#$ is the total space of the canonical interval bundle over $\mathbb{P}(Z)$. $Z_\#$ is naturally a zonotopal complex: the cells are restrictions of the interval bundle to cells of $\mathbb{P}(Z)$, so a cell of $Z_\#$ is a zonotope of the form $F \times [-1,1]$. We also note that the barycentric subdivision of $\mathbb{P}(Z)$ is naturally identified with a subcomplex of the barycentric subdivision of $Z_\#$. This subcomplex is called the zero-section or *exceptional divisor*. The zero section is a "hyperplane" in the sense that any cell $F \times [-1,1]$ of $Z_\#$ intersects the zero section in the hyperplane piece $F \times 0$.

Example 1.5.2. (*The dual cell complex and its blow-up.*) Suppose, as in Section 1.4, that Z is the zonotope associated to an essential hyperplane arrangement in $\mathbb{R}^n$ and that X is the dual polytope. Then ∂X is isomorphic to the cellulation of S^{n-1} cut out by the hyperplanes, and

$$\mathbb{P}(X) = \partial X/a$$

is the dual cell complex to $\mathbb{P}(Z)$. Let $X_\#$ denote the total space of the canonical interval bundle over $\mathbb{P}(X)$:

$$X_\# = \partial X \times_{\mathbb{Z}_2} [-1,1].$$

The cells of $X_\#$ are restrictions of the interval bundle to the cells of $\mathbb{P}(X)$. Hence, the n-cells of $X_\#$ are in one-to-one correspondence with the $(n-1)$-cells of $\mathbb{P}(X)$. The process of constructing $X_\#$ from X can be thought of as follows. Subdivide X as the cone on ∂X so that each cell C in ∂X determines a cone whose vertex is the center of X. Truncating the vertex of each such cone, we obtain a cell of the form $C \times [0,1]$. A cell of $X_\#$ is obtained by gluing the two cells $C \times [0,1]$ and $-C \times [0,1]$ by identifying $C \times 0$ and $-C \times 0$ via the antipodal map.

Example 1.5.3. (*Modified Coxeter complexes.*) There is a zonotopal complex associated to any Coxeter system (W,S) (as in [Bo], W is the Coxeter group and S is a distinguished set of generators). The details of the following construction can be found in [CD2] or [D3]. For each subset T of S, let W_T denote the subgroup generated by T. Let $\mathcal{S}^f = \{T | T \subseteq S \text{ and } \operatorname{Card} W_T < \infty\}$ and let $W\mathcal{S}^f$ be the set of all cosets of the form wW_T, with $w \in W$ and $T \in \mathcal{S}^f$, partially ordered by inclusion. The subposet $(W\mathcal{S}^f)_{\leq wW_T}$ is isomorphic to the poset of faces of the Coxeter cell associated to (W_T, T). Thus, the geometric realization Σ of $W\mathcal{S}^f$ is naturally a zonotopal complex. One of the principal features of this construction is that W acts simply transitively on the vertex set of Σ. This implies that the links of any two vertices in Σ are isomorphic. Other examples of zonotopal complexes can be constructed by the following two operations: (1) taking a subcomplex of Σ and (2) taking the quotient by a torsion-free subgroup Γ of W.

Example 1.5.4. (*Salvetti complexes.*) Let $\mathcal{H}$ be a hyperplane arrangement in a real vector space V and $Z_\mathcal{H}$ the corresponding zonotope. Salvetti [Sal] defined a zonotopal cell complex, which we will denote $\mathrm{Sal}_\mathcal{H}$, such that (a) each cell in $\mathrm{Sal}_\mathcal{H}$ is combinatorially equivalent to a face of $Z_\mathcal{H}$ and (b) $\mathrm{Sal}_\mathcal{H}$ is a deformation retract of the complement of the union of the complexified hyperplanes in $V \otimes \mathbb{C}$. To define $\mathrm{Sal}_\mathcal{H}$ we first construct its poset $\mathcal{S}_\mathcal{H}$ of cells. $\mathcal{S}_\mathcal{H}$ is the set of all pairs of the form (F, v) where F is a face of $Z_\mathcal{H}$ and v is a vertex of F. The partial order on $\mathcal{S}_\mathcal{H}$ is described as follows. For each subspace E in $\mathcal{H}$, let $\mathcal{H}_{\geq E}$ be the (usually nonessential) hyperplane arrangement consisting of all subspaces containing E. An *E-sector* is a chamber for $\mathcal{H}_{\geq E}$. For example, if $E = 0$, then an E-sector is a chamber for $\mathcal{H}$; if E is a hyperplane, then an E-sector is a half space. In general, an E-sector is an intersection of half-spaces. If F is a face of $Z_{\mathcal{H}}$, then let E_F denote its dual subspace in $\mathcal{H}$. If v is a vertex of $Z_{\mathcal{H}}$, then C_v denotes its dual chamber. For each $(F, v) \in \mathcal{S}_{\mathcal{H}}$, we define $\mathrm{Sec}(F, v)$ to be the E_F-sector which contains C_v. (If $F = v$ is a vertex, then put $E_F = V$ and $\mathrm{Sec}(F, v) = V$.) The partial order on $\mathcal{S}_{\mathcal{H}}$ is now defined as follows: $(F', v') \leq (F, v)$ if and only if $F' \leq F$ and $\mathrm{Sec}(F, v) \subseteq \mathrm{Sec}(F', v')$.

It is straightforward to see that given $(F, v) \in \mathcal{S}_{\mathcal{H}}$ and a face F' of F, there is a unique vertex v' of F' such that $(F', v') \leq (F, v)$. It follows that $(\mathcal{S}_{\mathcal{H}})_{\leq (F,v)} \cong \mathcal{P}(Z_{\mathcal{H}})_{\leq F}$. In other words each subposet $(\mathcal{S}_{\mathcal{H}})_{\leq (F,v)}$ is isomorphic to the poset of faces of a polytope (in fact, a zonotope).

We call such a poset an *abstract cell complex*. It has a geometric realization as a cell complex where, up to combinatorial equivalence, the cells are determined by the poset. In the case at hand, the geometric realization of $\mathcal{S}_{\mathcal{H}}$ is the *Salvetti complex* $\mathrm{Sal}_{\mathcal{H}}$. Salvetti [Sal] proved that the complex hyperplane complement is homotopy equivalent to $\mathrm{Sal}_{\mathcal{H}}$. One way to see this is to construct a cover of the hyperplane complement in $V \otimes \mathbb{C}$ by open convex sets indexed by the elements of $\mathcal{S}_{\mathcal{H}}$ so that the nerve of the covering is the order complex of $\mathcal{S}_{\mathcal{H}}$. See [Sal] and [CD2].)

In [De] Deligne proved that if $\mathcal{H}$ is a simplicial arrangement, then the complex hyperplane complement is aspherical. In other words, if $Z_{\mathcal{H}}$ is simple, then $\mathrm{Sal}_{\mathcal{H}}$ is aspherical. For example, if $\dim V = 1$, then $Z_{\mathcal{H}}$ is the interval $[-1, 1]$ and $\mathrm{Sal}_{\mathcal{H}}$ is the boundary of a digon (which is homeomorphic to a circle). Hence, if $\mathcal{H}$ is a product of one-dimensional arrangments (i.e., if $\mathcal{H}$ is the coordinate hyperplane arrangement in $\mathbb{R}^n$), then $\mathrm{Sal}_{\mathcal{H}}$ is an n-torus, cellulated by 2^n n-cubes. On the other hand, it follows from Remark 2.3.2 in [CD2], that if $Z_{\mathcal{H}}$ is simple but not a cube, then the cubical structure, described below, on the zonotopal cell complex $\mathrm{Sal}_{\mathcal{H}}$ will not be nonpositively curved.

Classes of parallel faces and subspaces in zonotopal cell complexes. The equivalence relation of parallelism on the set of faces of a zonotope extends naturally to zonotopal cell complexes. That is, two cells F and F' of a zonotopal cell complex K are *parallel* if there exists a sequence of cells $Z_1, \dots, Z_k$ and pairs of parallel faces $F_i, F_i' \subset Z_i$ such that $F = F_1$, $F' = F_k'$, and $F_{i+1} = F_i'$, for $1 \leq i \leq k - 1$. Note any such sequence determines an isomorphism $F \to F'$, but this isomorphism is no longer canonical since it depends on the "path" $Z_1, \dots, Z_k$. Given a zonotopal cell complex K, we let $\mathcal{C}_K$ denote the set of equivalence classes of parallel faces. It is clear that any class has a well-defined dimension.

For any parallel class in $\mathcal{C}_K$, there is a dual notion of an immersed "subspace" in the barycentric subdivision of K. On any given cell Z of K, such a subspace will restrict to a union of subspace pieces in the barycentric subdivision of Z. Let f be a parallel class in $\mathcal{C}_K$. We shall define (a) a subcomplex $\widehat{E}_f$ of the barycentric subdivision of K, (b) a connected zonotopal complex E_f, and (c) an "immersion" $i : E_f \to K$ such that the image of i is $\widehat{E}_f$ (generically, i will be an injection). We will say that $i : E_f \to K$ is a *(codimension-k) subspace* of K (a *hyperplane* if $k = 1$).

Let f be a (k-dimensional) parallel class of cells in K. Let $\mathcal{K}_f$ denote the set of pairs (Z, P), where Z is a cell in K and P is a subspace piece in Z corresponding to a parallel class $g \in \mathcal{C}_Z$ with $g \subset f$. (Restricting f to Z will, in general, be a union of parallel classes, and g is one of them.) We will say that (Z_1, P_1) is a *face* of (Z_2, P_2), written $(Z_1, P_1) \le (Z_2, P_2)$, if Z_1 is a face of Z_2 and $P_1 = P_2$. The subcomplex $\widehat{E}_f$ is defined by

$$\widehat{E}_f = \bigcup_{(Z,P)\in\mathcal{K}_f} P.$$

Note that it might happen that $\mathcal{K}_f$ contains two pairs of the form (Z, P_1) and (Z, P_2) where $P_1 \neq P_2$. In this case $P_1 \cap P_2$ is also a subspace piece in Z. To define E_f, we desingularize $\widehat{E}_f$ by removing these extraneous intersections. First form the disjoint union of the P, where $(Z, P) \in \mathcal{K}_f$. Then glue P_1 to P_2 along a common face P_0 if and only if there is a (Z_0, P_0) in $\mathcal{K}_f$ that is a common face of (Z_1, P_1) and (Z_2, P_2). Let $i : E_f \to \widehat{E}_f \subset K$ be the natural map.

The codimension-0 subspaces of a zonotopal complex K are precisely the connected components of K. Hence, a cell in a zonotopal cell complex is a codimension-0 subspace piece of itself. A subspace is *proper* if its codimension is positive.

The next lemma follows immediately from the definitions.

Lemma 1.5.5. *(Isomorphic links.) Let $i : E_f \to K$ be a subspace in a zonotopal complex K and let $(Z, P) \in \mathcal{K}_f$. Then $\mathrm{Lk}(P, E_f)$ is naturally identified with $\mathrm{Lk}(Z, K)$.*

Corollary 1.5.6. *If a zonotopal complex is a PL-manifold, then so is any subspace.*

Remark 1.5.7. If each cell of a zonotopal complex K is simple, then the image $\widehat{E}_f$ of any subspace is a subcomplex of the standard cubical subdivision of K.

1.6. Nonpositively curved cubical complexes. A metric space X is *nonpositively curved* (in the sense of Aleksandrov) if (a) any two points of X can be joined by a geodesic segment and (b) any sufficiently small triangle in X is "thinner" than the corresponding comparison triangle in $\mathbb{R}^2$ (in other words, for every point $x \in X$ there exists a small ϵ-ball B about x such that any geodesic triangle in B satisfies the CAT(0)-inequality of [G], p. 107). If X is nonpositively curved, then any two points in its universal cover $\widetilde{X}$ can be connected by a unique geodesic segment. It follows that $\widetilde{X}$ is contractible and hence, that X is aspherical (see Section 4 of [G]).

Examples of such nonpositively curved X are given by polyhedra which admit certain piecewise Euclidean metrics. For example, if K is a cubical cell complex, then there is an obvious way to give K a metric in which each cell is isometric to a Euclidean cube of unit edge length. Any piecewise-linear path in K is given a length by adding the (Euclidean) lengths of the segments of the path in each cell. The distance between two points in K is then defined to be the infimum of the lengths of all piecewise-linear paths joining the two points.

It turns out that there is a simple combinatorial condition on links of cells which is necessary and sufficient for a cubical complex K to be nonpositively curved. Since the link of a face in a cube is a simplex, the link of any cell in a cubical cell complex is a simplicial cell complex. To formulate the condition on these links, we need the definition of a "flag complex". We will say that a subcomplex of a cell complex L is an *empty simplex* if it is isomorphic to the 1-skeleton of a simplex but does not span a simplex of L.

Definition 1.6.1. A simplicial cell complex L is a *flag complex* if it is a simplicial complex and any finite set $\{v_0, \dots, v_m\}$ of pairwise joinable vertices spans a simplex in L (two vertices are "joinable" if they span a simplex). Equivalently, L is a flag complex if and only if it is a simplicial complex and contains no empty simplices of dimension ≥ 2.

Example 1.6.2. (i) An m-gon is a flag complex if and only if $m > 3$.

(ii) The order complex of any poset is a flag complex. In particular, the barycentric subdivision of any cell complex (in the sense of Section 1.1) is a flag complex.

In the following lemma we collect some standard properties of flag complexes.

Lemma 1.6.3. *(i) A join of two simplicial complexes K and L is a flag complex if and only if both K and L are flag complexes.*
(ii) The link of any simplex in a flag complex is a flag complex.
(iii) A full subcomplex of a flag complex is a flag complex (a subcomplex $L \subset K$ is full *if L contains all simplices of K which are spanned by vertices in L.)*

Another class of flag complexes is provided by the following lemma.

Lemma 1.6.4. ([Bro], p. 29) *Let $\mathcal{H}$ be a simplicial arrangement and $X_{\mathcal{H}}$ the simplicial polytope defined in Section 1.4. Then $\partial X_{\mathcal{H}}$ is a flag complex.*

We now state the fundamental result on nonpositive curvature for cubical complexes.

Lemma 1.6.5. (Gromov [G], p. 122) *The natural piecewise Euclidean metric on a (connected) cubical complex is nonpositively curved if and only if the link of every vertex is a flag complex.*

Corollary 1.6.6. *Suppose that K is a simple cell complex and that $K_{\square}$ is its standard cubical subdivision. Then the natural piecewise Euclidean metric $d_{\square}$ on $K_{\square}$ is nonpositively-curved if and only if the following two conditions are satisfied:*
(i) For each vertex v of K, $\mathrm{Lk}(v, K)$ is a flag complex.
(ii) For each cell P in K, $\partial\check{P}$ (the boundary of the dual simplicial polytope) is a flag complex.

Proof. By Lemma 1.2.2, any vertex x of $K_\Box$ is the barycenter of a unique cell P_x in K and

$$\mathrm{Lk}(x, K_\Box) = \mathrm{Lk}(x, P_x) * \mathrm{Lk}(P_x, K) \cong \partial \check{P}_x * \mathrm{Lk}(P_x, K).$$

By Gromov's Lemma (Lemma 1.6.5), $K_\Box$ is nonpositively curved if and only if $\mathrm{Lk}(x, K_\Box)$ is a flag complex for each vertex x. Suppose this is the case. Taking x to be a vertex of K we see that (i) holds. By Lemma 1.6.3(ii), $\mathrm{Lk}(P, K)$ is a flag complex for any cell P in K (since $\mathrm{Lk}(P, K)$ can be identified with a link in $\mathrm{Lk}(v, K)$ for any vertex v of P). Therefore, taking x to be the barycenter of a cell P, we see by Lemma 1.6.3(i), that (ii) holds. Conversely, if (i) and (ii) hold, then $\mathrm{Lk}(x, K_\Box)$ is a flag complex since both $\partial \check{P}_x$ and $\mathrm{Lk}(P_x, K)$ are. □

By Lemma 1.6.4, condition (ii) is superfluous in the case of a simple zonotopal cell complex. This gives the following corollary.

Corollary 1.6.7. *Suppose that K is a simple zonotopal cell complex. Then $d_\Box$ is nonpositively curved if and only if the link of every vertex in K is a flag complex.*

Corollary 1.6.8. *Supppose that a simple cell complex K is an n-dimensional nice PL-manifold and that its dual cell complex $\check{K}$ is zonotopal. Then $d_\Box$ is nonpositively curved if and only if for each n-cell P of K, $\partial \check{P}$ is a flag complex.*

1.7. Totally geodesic immersions. If K is a simple zonotopal complex and $i : E \to K$ is an immersed subspace, then E is a simple zonotopal complex and i maps the standard cubical subdivision of E onto a subcomplex of the standard cubical subdivision of K.

We can abstract this situation as follows. Suppose K' and K are cubical complexes. A map $i : K' \to K$ is an *immersion* if
(i) For each cell P in K', $i(P)$ is cell of K and the restriction $P \to i(P)$ is a combinatorial isomorphism.
(ii) For each vertex v of K', the induced simplicial map $\mathrm{Lk}(i) : \mathrm{Lk}(v, K') \to \mathrm{Lk}(i(v), K)$ is a simplicial embedding.

Next equip K' and K with their natural cubical metrics. Then the immersion $i : K' \to K$ is *totally geodesic* if each point x in K' has a geodesically convex neighborhood U_x such that $i|_{U_x}$ is an isometric embedding.

Proposition 1.7.1. *An immersion $i : K' \to K$ is totally geodesic if and only if for each vertex v of K', the embedding $\mathrm{Lk}(i)$ maps $\mathrm{Lk}(v, K')$ onto a full subcomplex of $\mathrm{Lk}(i(v), K)$.*

Proof. Given a point x in a (locally finite) piecewise Euclidean cell complex X, let C_x denote the cone on the piecewise spherical polyhedron $\mathrm{Lk}(x, X)$. Thus, $C_x = ([0, \infty) \times \mathrm{Lk}(x, X))/\sim$ where the equivalence relation identifies all points with first coordinate 0. The distance d between points (r_1, θ_1) and (r_2, θ_2) in C_x

is given by the usual formula for Euclidean distance in polar coordinates,

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta$$

where $\theta = \min\{\pi, d(\theta_1, \theta_2)\}$ and $d(\theta_1, \theta_2)$ is the distance in $\mathrm{Lk}(x, X)$. It follows from this formula that a geodesic segment from (r_1, θ_1) to (r_2, θ_2) goes through the origin (= the cone point) if and only if $\theta = \pi$. Now suppose that X' is a subcomplex of X containing x, and that C'_x is the cone on $\mathrm{Lk}(x, X')$. Then C'_x is a convex subset of C_x if and only if the following condition holds: $(*)$ given any two points θ_1, θ_2 in $\mathrm{Lk}(x, X')$, and a geodesic segment γ in $\mathrm{Lk}(x, X)$ between them, then if the length $l(\gamma)$ of γ is $< \pi$, γ is contained in $\mathrm{Lk}(x, X)$.

Since the metric on X is such that a small ball about X is isometric to a small ball about the origin in C_x, for X' to be totally geodesic in X, we need to check $(*)$ at each point $x \in X'$. In the case at hand, $X' = K'$ and $X = K$ are cubical complexes. The link of a k-face in an $(n+k)$-cube is an "all right" $(n-1)$-simplex. (This means that it is isometric to the spherical simplex spanned by the standard basis in $\mathbb{R}^n$.) Similarly, links in cubical complexes are all right simplicial cell complexes in the sense that each simplex is all right. Hence, the following lemma verifies condition $(*)$ and completes the proof of the proposition. □

Lemma 1.7.2. *Let L' be a full subcomplex of an all right simplicial cell complex L. Let θ_1, θ_2 be points in L', and let γ be a geodesic segment in L connecting them. Then if $l(\gamma) < \pi$, then γ lies in L'.*

Proof. For $i = 1, 2$, θ_i belongs to the relative interior of some simplex in L'. Let V_i denote the vertex set of this simplex. Let V be the set of vertices v such that γ intersects the open star of v in L. (The open star is the same as the open ball of radius $\pi/2$ about v.) Then $V = V_1 \cup V_2$. (If $v \in V - (V_1 \cup V_2)$, then, by the argument on p. 122 of [G], γ must intersect the ball of radius $\pi/2$ about v in a segment of length π, contradicting the assumption that $l(\gamma) < \pi$.) Obviously, γ is contained in the full subcomplex of L spanned by $V_1 \cup V_2$. But since L' is a full subcomplex and $V_1 \cup V_2 \subset L'$, this complex (and hence γ) lies in L'. □

Corollary 1.7.3. *Suppose that K is a simple zonotopal cell complex and that $i : E \to K$ is a subspace as in Section 1.5. Taking standard cubical subdivisions, we get an immersion $i : E_\square \to K_\square$. This immersion is totally geodesic.*

Proof. Let v be a vertex of $E_\square$. Let Z be the cell of K determined by the vertex $i(v) \in K_\square$ (Lemma 1.2.2). The link of $i(v)$ in K is naturally identified with the join $\mathrm{Lk}(i(v), Z) * \mathrm{Lk}(Z, K)$. By Lemma 1.5.5, the image of $\mathrm{Lk}(v, E_\square)$ is $\emptyset * \mathrm{Lk}(Z, K)$ which is a full subcomplex. The corollary then follows from Proposition 1.7.1. □

Remark 1.7.4. Suppose K_2 is nonpositively curved and that $i : K_1 \to K_2$ is a totally geodesic immersion. Then

(i) K_1 is nonpositively curved.
(ii) If K_2 is simply connected (and hence CAT(0)), then any totally geodesic immersion is an embedding onto a geodesically convex subspace.
(iii) Let $\widetilde{K}_1$ and $\widetilde{K}_2$ denote the universal covers of K_1 and K_2, respectively. The immersion $\widetilde{K}_1 \to K_1 \to K_2$ lifts to an immersion $\widetilde{i} : \widetilde{K}_1 \to \widetilde{K}_2$ and by (ii) this must be an embedding.
(iv) It follows that the induced map $i_* : \pi_1(K_1) \to \pi_1(K_2)$ is a monomorphism.
(v) When K_2 is a 3-manifold and K_1 is a surface, condition (iv) means that K_1 is an immersed incompressible surface. For nonpositively curved cubical structures on 3-manifolds, this fact has been examined by Aitchison and Rubinstein in [AR].

1.8. Splitting fundamental groups of nonpositively curved zonotopal cell complexes. In [Ca], Cappell defined $\mathcal{G}_0$ to be the smallest class of groups that contains the trivial group and that is closed under amalgamated products and HNN-extensions. This class includes fundamental groups of surfaces of genus > 0, as well as the fundamental groups of Haken 3-manifolds. (Cappell proved that the Novikov Conjecture holds for groups in $\mathcal{G}_0$.)

Theorem 1.8.1. *Suppose K is a connected, finite, simple zonotopal cell complex. Suppose further that $K_\square$ is nonpositively curved and that each hyperplane in K is embedded. Then $\pi_1(K)$ is in Cappell's class $\mathcal{G}_0$.*

Proof. Let H be any hyperplane in K. Cut K open along H and denote the result by $\widehat{K}_\square$. Since, locally, H is a convex subspace, $\widehat{K}_\square$ is also a nonpositively curved cubical complex. If H' is any other hyperplane in K, then any component of its image in $\widehat{K}_\square$ is again a totally geodesic subspace, which we continue to call a "hyperplane".

If $\widehat{K}_\square$ has two connected components (i.e., if H separates), then $\pi_1(K)$ is an amalgamated product. If H is two-sided and nonseparating, then $\pi_1(K)$ is an HNN-extension. If H is one-sided, then, letting H_0 denote its normal S^0-bundle, we have that $\pi_1(H_0)$ is a subgroup of index two in $\pi_1(H)$ and that $\pi_1(K)$ is the amalgamated product of $\pi_1(\widehat{K}_\square)$ and $\pi_1(H)$ along $\pi_1(H_0)$. We continue this process, cutting along a hyperplane in $\widehat{K}_\square$. The process terminates after we have cut along the images of all the original hyperplanes in K, after which all components, as well as all hyperplanes, have become contractible. (Such a component is the closed star of the barycenter of a maximal cell of K.) Hence, $\pi_1(K)$ is in $\mathcal{G}_0$. □

Remark 1.8.2. The above theorem applies to all of the nonpositively curved blow-ups of projectivized hyperplane arrangements which we consider in Section 4 since in these cases the hyperplanes are indeed embedded.

Further properties of groups acting on simply-connected, nonpositively curved cubical cell complexes can be found in [NR] and [Sag].

2. Toric varieties

2.1. Toric manifolds. As an application of the previous section (and a as warm up for the next), we consider the question of asphericity for some manifolds related to toric varieties. Since we are only interested in the topology of nonsingular examples, the generality of "toric manifolds" and "small covers" studied in [DJ1] seems to be the most appropriate context.

Let $M_{\mathbb{C}}$ be a smooth $2n$-manifold on which an n-dimensional torus $T^n = (S^1)^n$ acts with quotient homeomorphic to a convex polytope P. The quotient map is further assumed to be locally modeled on the quotient map of the standard T^n-action on $\mathbb{C}^n$, i.e., the map $\mathbb{C}^n \to (\mathbb{R}_{\geq 0})^n$ which takes $(z_1, \dots, z_n)$ to $(|z_1|^2, \dots, |z_n|^2)$. In other words, any point $m \in M_{\mathbb{C}}$ has a T^n-stable neighborhood which is equivariantly homeomorphic (up to an automorphism of T^n) to a T^n-stable neighborhood of $\mathbb{C}^n$. It follows that the polytope P must be simple. Such a manifold $M_{\mathbb{C}}$ will be called a *toric manifold.* We denote the quotient map $M_{\mathbb{C}} \to P$ by μ.

Example 2.1.1. Consider an algebraic action of $(\mathbb{C}^*)^n$ on $\mathbb{CP}^N$. It is known that such an action lifts to a linear action on $\mathbb{C}^{N+1}$ and any such action is conjugate to a diagonal action. We assume then, without loss of generality, that the action on $\mathbb{CP}^N$ is given by

$$t \cdot [z_0, \dots, z_N] = [t^{m_0} z_0, \dots, t^{m_N} z_N],$$

where $m_i = (m_i(1), \dots, m_i(n)) \in \mathbb{Z}^n$ and $t^{m_i} = t_1^{m_i(1)} \cdots t_n^{m_i(n)}$ for $0 \leq i \leq N$. Let M be the closure of the orbit of $[1, 1, \dots, 1]$. Under suitable conditions on the action (see for example [Od] and the references therein), M is a projective toric variety. If P is the convex hull (in $\mathbb{R}^n$) of the weights $\{m_0, \dots, m_N\}$, then there is a natural "moment map" $M \to P$ given by

$$[z_0, \dots, z_N] \mapsto \frac{\sum |z_i| m_i}{\sum |z_i|},$$

which is a quotient map for the action of the compact torus $T^n \subset (\mathbb{C}^*)^n$ ([Jur]). The vertices of P correspond to T^n-fixed points and if M is smooth (hence, a Kähler manifold) the exponential map at these fixed points gives a T^n-equivariant local homeomorphism with $\mathbb{C}^n$. Thus, a smooth projective toric variety is a toric manifold. (In general, $M_{\mathbb{C}}$ need not be a complex manifold. For example, $\mathbb{CP}^2 \# \mathbb{CP}^2$ can be realized as a toric manifold, but does not admit a complex structure.)

Toric manifolds have an elementary topological description as a quotient of $P \times T$ by an equivalence relation defined over proper faces of P. Let $\mathcal{F}$ be the set of codimension-one faces of P. A map $\lambda : \mathcal{F} \to \mathbb{Z}^n$ is a *characteristic function* for P if, for every vertex v of P, the set of vectors $\{\lambda(F) | F \in \mathcal{F},\ v \in F\}$ is a basis for $\mathbb{Z}^n$. Given λ, we define an equivalence relation $\sim$ on $P \times T$ as follows. Each element $\lambda(F)$ of $\mathbb{Z}^n$ determines a one-dimensional subtorus of $T^n = \mathbb{R}^n / \mathbb{Z}^n$; we set $(x, s) \sim (x, t)$ if $x \in F$ and st^{-1} is in the subtorus determined by $\lambda(F)$. The action of T^n on itself descends to an action on the quotient $(P \times T)/\sim$. A proof of the following can be found in Section 1.5 of [DJ1].

Proposition 2.1.2. *Let M be a toric manifold over P. Then there is a characteristic function λ for P and a T^n-equivariant homeomorphism $M \to (P \times T)/\sim$, where $\sim$ is the equivalence relation determined by λ.*

Conversely, $(P \times T)/\sim$ is equivariantly homeomorphic to a toric manifold for any λ.

2.2. Small covers. Let $(\mathbb{Z}_2)^n$ be the subgroup $\{(\pm 1, \dots, \pm 1)\}$ in T^n, and let M be a smooth n-manifold with a $(\mathbb{Z}_2)^n$-action and quotient space homeomorphic to a convex polytope P. If, in addition, $M \to P$ is locally modeled on the quotient map of the standard $(\mathbb{Z}_2)^n$-action on $\mathbb{R}^n$, then M is called a *small cover* of P.

Example 2.2.1. In Example 2.1.1, replace $(\mathbb{C}^*)^n$ with the split torus $(\mathbb{R}^*)^n$ and $\mathbb{CP}^N$ with $\mathbb{RP}^N$. Let M be the corresponding $(\mathbb{R}^*)^n$-orbit closure. The restriction of the moment map μ to $M \subset \mathbb{RP}^N \subset \mathbb{CP}^N$ still maps onto P (the convex hull of the weights) and is a quotient map for the $(\mathbb{Z}_2)^n$-action. Under suitable conditions on the action, M is a nonsingular real projective toric variety and a small cover of P.

More generally, if $M_{\mathbb{C}}$ is a toric manifold over P, then by Proposition 2.1.2, there is a section $P \to M_{\mathbb{C}}$ for the quotient map, and this section can be taken to be smooth. Let M be the $(\mathbb{Z}_2)^n$-orbit of this section. It is a small cover of P and any two such small covers are equivariantly diffeomorphic (a consequence of Proposition 2.2.2, below). M will be called the small cover associated to $M_{\mathbb{C}}$.

Again there is an elementary topological description of a small cover. Let $\mathcal{F}$ be the set of codimension-one faces of P. A map $\lambda : \mathcal{F} \to (\mathbb{Z}_2)^n$ is a *real characteristic function* for P if, for every vertex v, the set $\{\lambda(F) | F \in \mathcal{F},\ v \in F\}$ is a basis for $(\mathbb{Z}_2)^n$. Let $\sim$ be the equivalence relation on $P \times (\mathbb{Z}_2)^n$ defined by $(x, s) \sim (x, t)$ if $x \in F$ and $s \equiv t \bmod \lambda(F)$. The $(\mathbb{Z}_2)^n$-action is compatible with this equivalence relation, hence descends to the quotient $P \times (\mathbb{Z}_2)^n/\sim$.

Proposition 2.2.2. *Let M be a small cover of P. Then there exists a real characteristic function λ and an equivariant homeomorphism $M \to P \times (\mathbb{Z}_2)^n/\sim$ where $\sim$ is the equivalence relation defined by λ.*

We first address the nonpositive curvature question for small covers. It follows from the proposition above that M is homeomorphic to a simple cell complex K all of whose cells are isomorphic to faces of P. This cell complex has a natural subdivision $K_\square$ into small cubes. Since the $(\mathbb{Z}_2)^n$-action on M is modeled on the standard action on $\mathbb{R}^n$, the dual cell complex to K is zonotopal, the zonotopes being *big* cubes. Since the vertices of this zonotopal complex are dual to the maximal cells (which are all copies of P), Corollary 1.6.7 then implies the following.

Proposition 2.2.3. *The natural piecewise Euclidean cubical metric on a small cover of P is nonpositively curved if and only if the boundary of P is dual to a flag complex.*

It follows from this proposition that if ∂P is dual to a flag complex, then M is aspherical. The proof of the converse follows from the main result of [D1]. We recall the basic construction in the case of the "right-angled" Coxeter group associated to P. Let (W, S) be the Coxeter system defined as follows. There is one generator, which we denote by s_F, for each codimension one face $F \in \mathcal{F}$. The relations are $(s_F)^2 = 1$ for all F, and $s_F s_E = s_E s_F$ whenever E and F intersect (necessarily in a codimension-two face). The pair (W, S) is called the right-angled Coxeter system associated to the 1-skeleton of the simplicial polytope dual to P. Once again, we define an equivalence relation, this time on $P \times W$. We set $(x, s) \sim (x, t)$ if st^{-1} is in the subgroup generated by $\{s_F | x \in F\}$.

Lemma 2.2.4. *The quotient space* $\widetilde{M_{\mathbb{R}}} = (P \times W)/\sim$ *is the universal cover of every small cover of* P.

Proof. The construction $\widetilde{M_{\mathbb{R}}}$ is precisely the space $\mathcal{U}(W, X)$ of [D1] with $P = X$. The group W acts on $\widetilde{M_{\mathbb{R}}}$ with quotient space P (Section 13 of [D1]). The characteristic function λ for $M_{\mathbb{R}}$ determines a surjective homomorphism $W \to (\mathbb{Z}_2)^n$ given on generators by $s_F \to \lambda(F)$. The kernel K acts freely on $\widetilde{M_{\mathbb{R}}}$ with quotient M. That $\widetilde{M_{\mathbb{R}}}$ is simply-connected follows from Corollary 10.2 of [D1]. □

Combining the previous proposition with the main result of [D1] applied to the universal cover $M_{\mathbb{R}}$ gives the following.

Theorem 2.2.5. *Let M be a small cover of P. Then the following statements are equivalent.*
(i) M is aspherical.
(ii) The boundary of P is dual to a flag complex.
(iii) The natural piecewise Euclidean metric on the dual cubical cellulation of M is nonpositively curved.

Proof. (ii)⇔(iii) is Proposition 2.2.3, so it suffices to prove (i)⇒(ii). The main theorem of [D1] states that $\widetilde{M_{\mathbb{R}}}$ will be contractible if and only if for every subset $\mathcal{S}$ of $\mathcal{F}$ which generates a finite subgroup of W, the intersection $\cap_{F \in \mathcal{S}} F$ is nonempty. This is equivalent to the condition that ∂P be dual to a flag complex, since the subsets $\mathcal{S}$ which generate finite subgroups are precisely those with the property that any two elements are adjacent codimension-one faces. □

Remark 2.2.6. It is immediate from the construction of the universal cover $\widetilde{M_{\mathbb{R}}}$ that it is homeomorphic to a simple cell complex all of whose cells are faces of P. In fact, the associated poset is anti-isomorphic to the poset $W\mathcal{S}^f$ defined in Example 1.5.3. Thus, $\widetilde{M_{\mathbb{R}}}$ has the structure of a zonotopal complex (the zonotopes are big cubes) and the link of every vertex is isomorphic to the dual of ∂P.

3. Blowing up zonotopal cell complexes

Suppose we are given an essential hyperplane arrangement in $\mathbb{R}^{n+1}$. It cuts the sphere S^n into convex spherical polytopes (which are combinatorially isomorphic to convex polytopes in affine space). Dividing by the antipodal map, we get a cell structure on $\mathbb{RP}^n$, which is isomorphic to the cell complex $\mathbb{P}(X)$ described in Example 1.5.2. The dual cells are zonotopes (because the local picture is that of a hyperplane arrangement in $\mathbb{R}^n$), and the dual cell complex is the zonotopal cell complex $\mathbb{P}(Z)$ described in Example 1.5.1 The projective subspaces of the projectivized arrangement in $\mathbb{RP}^n$ are actually subspaces of the zonotopal cell complex $\mathbb{P}(Z)$ in the sense of Section 1.5.

In [DP1], De Concini and Procesi introduce compactifications of complements of subspace arrangements in which the union of subspaces in the arrangement is replaced by a divisor with normal crossings. In the De Concini-Procesi procedure, certain subspaces of the arrangement are blown up. In the case of a projectivized hyperplane arrangement, the effect of their procedure on $\mathbb{P}(X)$ is to truncate some of the faces of some of the n-cells, obtaining a new cell complex $\mathbb{P}(X)_\#$ with the same number of n-cells. The dual cell complex $\mathbb{P}(Z)_\#$ is a new zonotopal cell complex with the same vertex set as $\mathbb{P}(Z)$ but with certain cells "blown up" as in Example 1.5.1. The new divisor is a union of subspaces in $\mathbb{P}(Z)_\#$. The fact that the divisor has normal crossings translates to the statement that the zonotopal cells of $\mathbb{P}(Z)_\#$ are actually (big) cubes.

3.1. Building sets and minefields. The fundamental notion for the construction in [DP1] is that of a "building set". If $\mathcal{H}$ is a hyperplane arrangement in a vector space V with dual collection $\mathcal{C} = \mathcal{H}^\perp$, then a building set is a subset of $\mathcal{C}$ with respect to which all subspaces in $\mathcal{C}$ have natural direct sum decompositions. Given the correspondence between $\mathcal{C}$ and parallel classes of faces in the associated zonotope $Z_\mathcal{H}$, we can give a similar definition for zonotopes and extend it to zonotopal cell complexes. The following should be compared with the definitions and properties in [DP1].

Let Z be a zonotope, and let $\mathcal{C}_Z$ denote the set of parallel classes of faces of Z as defined in Section 1.4. Recall that a subset $\{f_1, \dots, f_k\}$ of $\mathcal{C}_Z$ is called a decomposition of f if, for every $g \leq f$, $g = (f_1 \cap g) \oplus \cdots \oplus (f_k \cap g)$. A parallel class f is called *irreducible* if it admits no decomposition with more than one factor, and we denote by $\mathcal{I}_Z$ the set of all irreducible elements of $\mathcal{C}_Z$. A subset of a decomposition is called a *partial decomposition*.

Definition 3.1.1. A subset $\mathcal{G}_Z$ of $\mathcal{C}_Z$ is a *(partial) building set for* Z if for any $f \in \mathcal{C}_Z$, the set of maximal elements of $(\mathcal{G}_Z)_{\leq f}$ is a (partial) decomposition of f.

Of course, any partial decomposition $\mathcal{F}$ of f can be extended to a decomposition $\{f_1, \dots, f_k\}$ of f. Although there may be more than one possible way to extend $\mathcal{F}$, there is a unique one with a minimal number of elements (namely, adjoin to $\mathcal{F}$ the sum of the missing f_i's). We call this the *minimal decomposition containing* $\mathcal{F}$.

Given a partial building set $\mathcal{G}_Z$ for Z, the minimal decomposition of f containing the maximal elements of $(\mathcal{G}_Z)_{\leq f}$ is called the *$\mathcal{G}_Z$-decomposition of f*.

Example 3.1.2. (i) The set $\mathcal{I}_Z$ of irreducible elements is a building set for Z. Moreover, any building set must contain all of the irreducibles. For this reason, we call $\mathcal{I}_Z$ the *minimal building set* for Z.
(ii) The set $\mathcal{C}_Z$ is a building set for Z, called the *maximal building set* for Z.

The next lemma provides examples of partial building sets.

Lemma 3.1.3. *Let $\mathcal{A}$ be any subset of $\mathcal{C}_Z$ and let $\mathcal{G}_Z$ be a building set for Z. Let $\mathcal{G}_{\geq \mathcal{A}}$ denote the set of all elements g in $\mathcal{G}_Z$ such that $g \geq a$ for some $a \in \mathcal{A}$. Then $\mathcal{G}_{\geq \mathcal{A}}$ is a partial building set. In particular, $(\mathcal{C}_Z)_{\geq \mathcal{A}}$ is a partial building set (and partial building sets of this form will be called "maximal partial building sets").*

Proof. It suffices to show that, for any $f \in \mathcal{C}_Z$, the maximal elements of $(\mathcal{G}_{\geq \mathcal{A}})_{\leq f}$ are all maximal in $(\mathcal{G}_Z)_{\leq f}$. But this is clear since any element g of $(\mathcal{G}_Z)_{\leq f}$ which is greater than or equal to an element in $(\mathcal{G}_{\geq \mathcal{A}})$ is necessarily greater than or equal to an element of $\mathcal{A}$; hence, $g \in (\mathcal{G}_{\geq \mathcal{A}})_{\leq f}$. ☐

More generally, let $\mathcal{C}_K$ be the set of parallel classes of cells in any zonotopal cell complex K. If $\mathcal{G}_K$ is a subset of $\mathcal{C}_K$, then for any subcomplex K', there is a corresponding subset $\mathcal{G}_{K'}$ of $\mathcal{C}_{K'}$ defined by

$$\mathcal{G}_{K'} = \{f' \in \mathcal{C}_{K'} | f' \text{ is parallel to some } f \in \mathcal{G}_K\},$$

which we call the *restriction of $\mathcal{G}_K$ to K'* (note that the parallel relation in $\mathcal{C}_{K'}$ might be finer than the parallel relation in $\mathcal{C}_K$). In particular, if Z is a zonotope and F is any face, then the restriction of a (partial) building set $\mathcal{G}_Z$ to F is a (partial) building set. This leads us to the general definition.

Definition 3.1.4. Let K be any zonotopal cell complex. Then a subset $\mathcal{G}_K$ of $\mathcal{C}_K$ is a *(partial) building set* if and only if for every cell Z in K, its restriction $\mathcal{G}_Z$ is a (partial) building set for $\mathcal{C}_Z$.

The following lemma is a straightforward consequence of the definition of a partial building set.

Lemma 3.1.5. *Let $\mathcal{G}_K$ be any building set for K, and let $\mathcal{A}$ be any collection of parallel classes in $\mathcal{C}_K$. Let $\mathcal{G}_{\geq \mathcal{A}}$ be the set of all parallel classes $f \in \mathcal{C}_K$ such that $f \geq a$ for some $a \in \mathcal{A}$. Then $\mathcal{G}_{\geq \mathcal{A}}$ is a partial building set.*

We will need to distinguish between parallel classes in a building set and the cells in these classes. We introduce the notion of a "minefield" as the collection of cells in a building set (the point being that these cells are precisely the ones which will be "blown-up" in the next subsection). A subset $\mathcal{M}$ of $\mathcal{P}(K)$ is a *(partial) minefield for K* if it has the following properties

(a) If $Z \in \mathcal{M}$, then $\mathcal{M}$ contains all cells in $\mathcal{P}(K)$ which are parallel to Z.
(b) The set of parallel classes in $\mathcal{M}$ are a (partial) building set for K.

If $\mathcal{G}$ is a (partial) building set for K, then there is an associated (partial) minefield $\mathcal{M}$, consisting of all cells whose parallel classes lie in $\mathcal{G}$. A *partially mined* zonotopal complex is a pair $(K, \mathcal{M})$ where $\mathcal{M}$ is a partial minefield for K. It is *mined* if $\mathcal{M}$ is a minefield. If K' is a subcomplex of K, then let $\mathcal{M}_{K'}$ be the set of all cells of K' which lie in $\mathcal{M}$.

It is clear that for any partially mined zonotopal cell complex $(K, \mathcal{M})$ and any subcomplex K' of K, the pair $(K', \mathcal{M}_{K'})$ is a partially mined zonotopal cell complex, called a *subcomplex* of $(K, \mathcal{M})$. The *product* $(K_1, \mathcal{M}_1) \times (K_2, \mathcal{M}_2)$ of two partially mined zonotopal cell complexes is the partially mined zonotopal cell complex $(K_1 \times K_2, \mathcal{M}_1 \times \mathcal{M}_2)$, where $\mathcal{M}_1 \times \mathcal{M}_2$ consists of all products $Z_1 \times Z_2$ where $Z_1 \in \mathcal{M}_1$ and $Z_2 \in \mathcal{M}_2$. The following lemma is clear.

Lemma 3.1.6. *Let Z be a zonotope and let $\mathcal{M}$ be the partial minefield corresponding to a partial building set $\mathcal{G}_Z$. Let $\mathcal{F}$ be the $\mathcal{G}_Z$-decomposition of Z. Fix a vertex v of Z. Then for each $f \in \mathcal{F}$, there is a unique $Z_f \in f$ which has v as a vertex. Moreover, if $(Z_f, \mathcal{M}_f)$ is the subcomplex of $(Z, \mathcal{M})$ corresponding to Z_f, then there is a natural identification*

$$(Z, \mathcal{M}) = \prod_{f \in \mathcal{F}} (Z_f, \mathcal{M}_f).$$

By a *morphism* of zonotopal cell complexes $\phi : K_1 \to K_2$, we shall mean an isomorphism onto a subcomplex. Equivalently, a *morphism* $\phi : \mathcal{P}(K_1) \to \mathcal{P}(K_2)$ is an order-preserving injection such that for every $Z \in \mathcal{P}(K_1)$, ϕ restricts to an isomorphism $\mathcal{P}(K_1)_{\leq Z} \cong \mathcal{P}(K_2)_{\leq \phi(Z)}$.

Similarly, a *morphism* $\phi : (K_1, \mathcal{M}_1) \to (K_2, \mathcal{M}_2)$ is an isomorphism onto a subcomplex of $(K_2, \mathcal{M}_2)$.

3.2. Blowing up zonotopal cell complexes. Let $\mathcal{Z}$ be the category of zonotopal cell complexes, and let $\widetilde{\mathcal{Z}}$ be the category of mined zonotopal cell complexes (with morphisms defined above). We denote by $\mathcal{Z}_n$ and $\widetilde{\mathcal{Z}}_n$ the full subcategories consisting of complexes of dimension $\leq n$.

We are going to define a functor from $\widetilde{\mathcal{Z}}$ to $\mathcal{Z}$ which we denote by $(K, \mathcal{M}) \to K_{\#\mathcal{M}}$ and call the *blow-up of K along $\mathcal{M}$*. (We shall write simply $K_{\#}$ when there is no ambiguity.) Roughly speaking, we shall blow-up (as in Example 1.5.1) the cells in $\mathcal{M}$ and no others.

The construction will also satisfy the following two properties.

(P1) (*Identity property.*) If $\mathcal{M} = \emptyset$, then $K_{\#\mathcal{M}} = K$.

(P2) (*Naturality with respect to products.*) If $(K, \mathcal{M}) = (K_1, \mathcal{M}_1) \times (K_2, \mathcal{M}_2)$, then $K_{\#\mathcal{M}} = (K_1)_{\#\mathcal{M}_1} \times (K_2)_{\#\mathcal{M}_2}$.

The definition of this construction is inductive, i.e., we define it on the subcategories $\widetilde{\mathcal{Z}}_n$ by induction on n. (The following is fairly close to the exposition of the "Möbius band hyperbolization procedure" on pages 334-335 of [CD3].) If $\dim K \leq 1$, then $K_{\#\mathcal{M}} = K$. (Hence, the construction leaves the 1-skeleton alone.) In other words, on $\widetilde{\mathcal{Z}}_1$, the functor is the natural projection onto $\mathcal{Z}_1$. Now suppose by induction that, for $n > 1$, the functor has been defined on $\widetilde{\mathcal{Z}}_{n-1}$ and that it satisfies (P1) and (P2) for mined complexes in $\widetilde{\mathcal{Z}}_{n-1}$. Let $(K, \mathcal{M})$ be a partially mined complex of dimension n. The $(n-1)$-skeleton of $K_{\#\mathcal{M}}$ must then be defined to be the blow-up of the $(n-1)$-skeleton of K, i.e.,

$$(K_{\#\mathcal{M}})^{(n-1)} = (K^{(n-1)})_{\#\mathcal{M}_{K^{(n-1)}}}.$$

Suppose Z is an n-cell in K. Let $\mathcal{G}_Z$ be the partial building set associated to $\mathcal{M}_Z$ (the restriction of $\mathcal{M}$ to Z).

Case 1. $Z \notin \mathcal{M}$. Let

$$(Z, \mathcal{M}_Z) = \prod_{f \in \mathcal{F}} (Z_f, \mathcal{M}_f)$$

be the decomposition given in Lemma 3.1.6. The boundary ∂Z decomposes as

$$\bigcup_{f \in \mathcal{F}} \partial_f Z$$

where

$$\partial_f Z = \partial Z_f \times \prod_{g \in \mathcal{F} - \{f\}} Z_g.$$

Since (P2) holds for complexes of dimension $n-1$,

$$(\partial_f Z)_{\#\mathcal{M}} = (\partial Z_f)_{\#\mathcal{M}_f} \times \prod_{g \in \mathcal{F} - \{f\}} (Z_g)_{\#\mathcal{M}_g}.$$

In other words, $(\partial Z)_{\#\mathcal{M}}$ is naturally identified with a subcomplex of the product

$$\prod_{f \in \mathcal{F}} (Z_f)_{\#\mathcal{M}_f}.$$

Hence, we define $Z_\# = Z_{\#\mathcal{M}}$ to be

$$\prod_{f \in \mathcal{F}} (Z_f)_{\#\mathcal{M}_f}.$$

Case 2. $Z \in \mathcal{M}$. Consider the central symmetry $a : \partial Z \to \partial Z$. It induces an involution on any parallel class of faces in Z; hence, it takes $\mathcal{M}_{\partial Z}$ to itself. That is to say, $a : (\partial Z, \mathcal{M}_{\partial Z}) \to (\partial Z, \mathcal{M}_{\partial Z})$ is an automorphism of mined complexes. By induction $(\partial Z)_\#$ is defined and by functoriality, we get an involution $a_\# : (\partial Z)_\# \to (\partial Z)_\#$. As in Example 1.5.1, we shall define $Z_\#$ to be the canonical interval-bundle over $(\partial Z)_\# / a_\#$. In other words, $Z_\#$ is the quotient space of $(\partial Z)_\# \times [-1, 1]$ by the involution defined by $(z, t) \mapsto (a_\#(z), -t)$. We note as before, that $(\partial Z)_\#$ is canonically identified with the subcomplex $(\partial Z)_\# \times_{\mathbb{Z}_2} \{\pm 1\}$ of $Z_\#$.

This shows how to perform the construction on each n-cell. To do it on an n-dimensional complex K, we glue each blown up n-cell $Z_\#$ to $K_\#^{(n-1)}$ via the natural identification of $(\partial Z)_\#$ with a subcomplex of $K_\#^{(n-1)}$. We must also say how to extend morphisms across the n-cells. In Case 1, this follows since (by Lemma 3.1.6) any morphism of a product of mined zonotopes preserves the $\mathcal{G}_Z$-decompositions. In Case 2, we take the product of the morphism on $(\partial Z)_\#$ with the identity map on $[-1,1]$. Since, by Lemma 1.4.7, a lies in the center of $\mathrm{Aut}(\mathcal{P}(Z))$, this map is equivariant with respect to the involution. Hence, it induces a morphism on $Z_\#$.

Remark 3.2.1. There is a functorial "blow-down" map $\pi : K_\# \to K$ defined as follows. We define π by defining its restriction to $Z_\#$ for every cell Z in K. If $Z \in \mathcal{M}$, we can assume by induction that $\pi : (\partial Z)_\# \to \partial Z$ is defined. Identifying Z with the cone on ∂Z $(= \partial Z \times [0,1]/\sim$ where $(z,0) \sim (z',0))$, we define π on $Z_\# = (\partial Z)_\# \times_{\mathbb{Z}_2} [-1,1]$ by $\pi([z,t]) = [\pi(z), |t|]$. If $Z \notin \mathcal{M}$, then Z is a product $\Pi_f Z_f$ of cells in $\mathcal{M}$ and we take π to be the product $Z_\# = \Pi_f (Z_f)_\# \to Z = \Pi_f Z_f$ of the corresponding blow-down maps.

The locus in K consisting of points which have more than one preimage in $K_\#$ is called the *center of the blow-up.* It is the union of the images of the subspaces $E_f \to K$ (Section 1.5) as f runs over all parallel classes in $\mathcal{G}$. The preimage in $K_\#$ of the center is a union of hyperplanes $H_f \to K_\#$, again indexed by $f \in \mathcal{G}$. This union of hyperplanes is called the *exceptional divisor of the blow-up.*

3.3. Links in blow-ups. It is immediate from the definition of the blow-up functor that the vertices of K correspond bijectively with the vertices of the blow-up $K_{\#\mathcal{M}}$. Moreover, it turns out that the link of a vertex in $K_{\#\mathcal{M}}$ is a subdivision of the link of the corresponding vertex in K. An example is shown in Figure 5. The two figures on the left are neighborhoods of the corresponding vertex in K and $K_\#$. A corner of a zonotope is shown intersecting each neighborhood in a small cube. The figures on the right represent the links of the vertices (the second one being a subdivision of the first). We next describe the functorial setting for the description of the link of a vertex in a blow-up $K_{\#\mathcal{M}}$. It is more or less analogous to the construction of blow-ups.

Suppose σ_1 and σ_2 are two convex polytopes in general position in some affine space of dimension greater than $\dim \sigma_1 + \dim \sigma_2$. Then their convex hull is denoted $\sigma_1 * \sigma_2$ and called the *join* of σ_1 and σ_2. The combinatorial type of $\sigma_1 * \sigma_2$ does not depend on the affine space or the embedding. For example, the join of a j-simplex and a k-simplex is a $(j+k+1)$-simplex.

A collection of disjoint faces $\{\sigma_1, \dots, \sigma_k\}$ of a convex cell σ is a *join decomposition* if σ is the join $\sigma_1 * \cdots * \sigma_k$. A cell σ is *join-irreducible* if it does not admit a nontrivial join decomposition. A subset of a join decomposition for σ is called a *partial join decomposition* for σ.

Definition 3.3.1. Let L be a cell complex. A subset $\mathcal{S}$ of $\mathcal{P}(L)$ is a *(partial) subdivision set* for L if for every $\sigma \in \mathcal{P}(L)$, the set of maximal elements of $\mathcal{S}_{\leq \sigma}$ is a (partial) join decomposition of σ.

Example 3.3.2. The fundamental example of a partial subdivision set is the one which is induced on the link of a vertex in a zonotopal cell complex by a partial minefield. Let $(K, \mathcal{M})$ be a partially mined zonotopal cell complex, v a vertex of K, and $L = \mathrm{Lk}(v, K)$. Then the induced subset $\mathcal{S} = \{\mathrm{Lk}(v, Z) | Z \in \mathcal{M}$ and $v \in Z\} \subseteq \mathcal{P}(L)$ is a partial subdivision set for L.

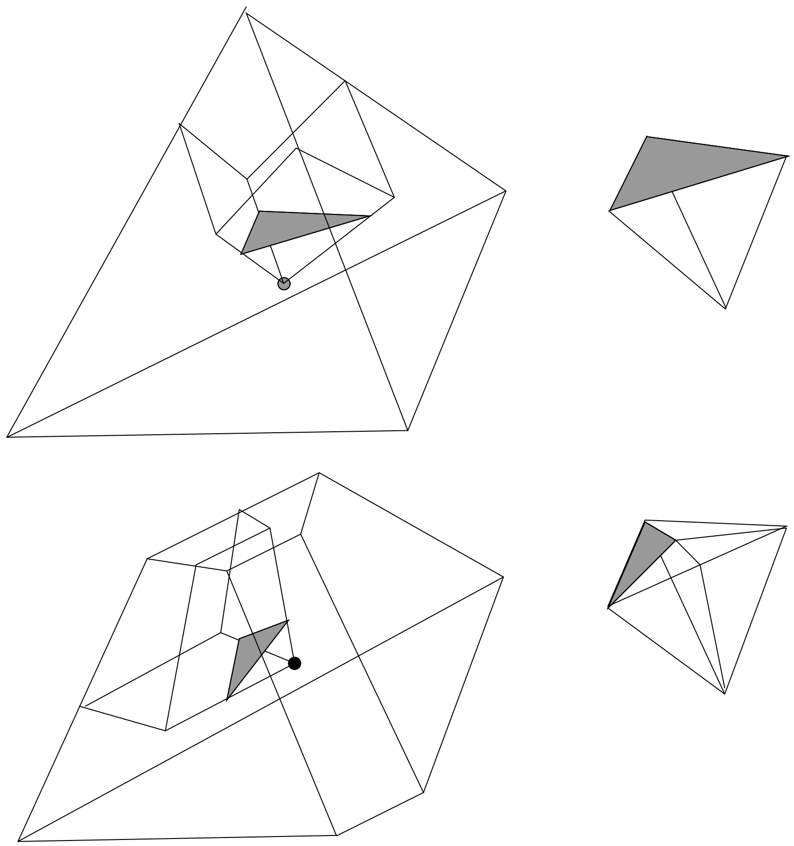


FIGURE 5.

Example 3.3.3. Given any subset $\mathcal{T} \subset \mathcal{P}(L)$ and any subdivision set $\mathcal{S}$, let $\mathcal{S}_{\geq \mathcal{T}}$ be the set of all $\sigma \in \mathcal{S}$ such that there exists $\tau \in \mathcal{T}$ with $\sigma \geq \tau$. Then $\mathcal{S}_{\geq \mathcal{T}}$ is a partial subdivision set for L. In particular, if $\mathcal{C}_{\geq \mathcal{A}}$ is a maximal partial building set (as in Lemma 3.1.3) for a zonotopal cell complex K with corresponding partial minefield $\mathcal{M}$, then the induced partial subdivision set on $\mathrm{Lk}(v, K)$ (Example 3.3.2) is $\mathcal{P}(L)_{\geq \mathcal{T}}$ where $\mathcal{T}$ is the set $\{\mathrm{Lk}(v, Z) | [Z] \in \mathcal{C}_{\geq \mathcal{A}},\ v \in Z\}$.

As in Section 3.1, any partial join decomposition $\mathcal{J}$ can be extended to a join decomposition with a minimal number of elements. We call this extension the *minimal join decomposition containing* $\mathcal{J}$.

Given a partial subdivision set $\mathcal{S}$ and a cell σ in L, the *$\mathcal{S}$-decomposition of σ* is the minimal join decomposition of σ containing the maximal elements of $\mathcal{S}_{\leq\sigma}$. If $\{\sigma_1, \dots, \sigma_k\}$ is the $\mathcal{S}$-join decomposition of a cell σ, then

$$\mathcal{S}_{\leq\sigma} = \coprod_{i=1}^{k} \mathcal{S}_i,$$

where $\mathcal{S}_i = \mathcal{S}_{\leq\sigma_i}$, $1 \leq i \leq k$ (note at most one of the $\mathcal{S}_i$ will be empty).

Given a pair $(L, \mathcal{S})$ where $\mathcal{S}$ is a partial subdivision set on a cell complex L, there is a functorial subdivision $L_{\oslash\mathcal{S}}$ of L such that:

(i) If $\mathcal{S} = \emptyset$, then $L_{\oslash\mathcal{S}} = L$, and

(ii) If $(L, \mathcal{S}) = (L_1 * L_2, \mathcal{S}_1 \coprod \mathcal{S}_2)$, then $L_{\oslash\mathcal{S}} = (L_1)_{\oslash\mathcal{S}_1} * (L_2)_{\oslash\mathcal{S}_2}$.

The definition of $L_{\oslash\mathcal{S}}$ is again by induction on dimension. The vertex set of $L_{\oslash\mathcal{S}}$ is equal to the vertex set of L. We suppose a subdivision $L^{(n-1)}{}_{\oslash\mathcal{S}}$ of the $(n-1)$-skeleton has been constructed. Let σ be an n-cell, and let $\{\sigma_1, \dots, \sigma_k\}$ be the $\mathcal{S}$-join decomposition of σ. If this decomposition has more than one factor, then we define $\sigma_{\oslash\mathcal{S}}$ to be

$$\sigma_{\oslash\mathcal{S}} = (\sigma_1)_{\oslash\mathcal{S}_1} * \cdots (\sigma_k)_{\oslash\mathcal{S}_k}.$$

If this decomposition has only one factor, then $\sigma \in \mathcal{S}$. In this case, the boundary subdivision $\partial\sigma_{\oslash\mathcal{S}}$ has been defined, and we introduce a barycenter in the interior of σ and cone off the subdivision of the boundary.

Remark 3.3.4. If L is the boundary complex of the dual of a convex polytope P, then $L_{\oslash\mathcal{S}}$ is the boundary complex of the dual of a polytope obtained by "truncating" certain faces of P. Namely, one truncates exactly those faces which are dual to cells in $\mathcal{S}$.

Example 3.3.5. The set $\mathcal{S}$ of all (positive dimensional) cells of L forms a subdivision set, called the *maximal subdivision set*, and it follows from the description that $L_{\oslash\mathcal{S}}$ is the barycentric subdivision of L. In particular, if L is the boundary of an n-simplex, then it follows from the previous remark that $L_{\oslash\mathcal{S}}$ is the boundary complex of the polytope dual to the n-dimensional permutohedron (i.e., the permutohedron is the full truncation of the simplex).

Combinatorics. The combinatorics of the subdivision $L_{\oslash\mathcal{S}}$ can be described as follows. The cells of $L_{\oslash\mathcal{S}}$ correspond bijectively with collections $\Sigma = \{F_1, \dots, F_k\}$ where F_i $(1 \leq i \leq k)$ is a flag of the form

$$\sigma_0 < \cdots < \sigma_j,$$

where $\sigma_i \in \mathcal{S}$ for all $i > 0$ and where the set consisting of maximal elements of the flags $F_1, \dots, F_k$ is the $\mathcal{S}$-join decomposition of a cell in L. The partial ordering is the obvious one: $\Sigma \leq \Sigma'$ if and only if for every $F \in \Sigma$ there exists an $E \in \Sigma'$ with $F \leq E$. The cell corresponding to Σ is isomorphic to the join of the maximal elements of the flags in Σ. Hence, we have the following.

Lemma 3.3.6. *The subdivision $L_{\oslash \mathcal{S}}$ is a simplicial cell complex if and only if $\mathcal{S}$ contains every join-irreducible element of $\mathcal{P}(L)$ which is not a simplex (i.e., which is not a vertex).*

Recall that a cell complex L is a complex if the intersection of any two cells is either empty or a single cell. Thus, any cell in a complex is determined by its vertex set. Clearly any subdivision of a complex is again a complex. Suppose $\mathcal{S}$ is a partial subdivision set for a complex L, and $\mathcal{S}$ satisfies the conditions of Lemma 3.3.6. Then $L_{\oslash \mathcal{S}}$ is a simplicial complex. Let $\widetilde{\mathcal{S}}$ be the union of $\mathcal{S}$ and the set of vertices of L. Then the vertices of $L_{\oslash \mathcal{S}}$ correspond to singleton flags which are precisely the elements of $\widetilde{\mathcal{S}}$, and the simplices of $L_{\oslash \mathcal{S}}$ correspond to certain subsets of $\widetilde{\mathcal{S}}$.

Definition 3.3.7. A subset $S \subset \widetilde{\mathcal{S}}$ is *nested* if every subset of S whose elements are pairwise noncomparable is a join decomposition of a cell not in $\mathcal{S}$ (two elements σ_1 and σ_2 in a poset are *noncomparable* if $\sigma_1 \not\leq \sigma_2$ and $\sigma_2 \not\leq \sigma_1$).

Remark 3.3.8. The notion of nested subsets for indexing strata in compactifications of configuration spaces was introduced by Fulton and MacPherson in [FM]. De Concini and Procesi generalized this notion to study the combinatorics of their compactifications of complements of subspace arrangements [DP1].

The following proposition follows immediately from the combinatorial description of the cells of $L_{\oslash \mathcal{S}}$ in terms of collections of flags. Note that the hypothesis that L be a complex (rather than just a cell complex) cannot be weakened.

Proposition 3.3.9. *Assume that L is a complex and that $\widetilde{\mathcal{S}}$ contains every join irreducible element of $\mathcal{P}(L)$, then the simplices of $L_{\oslash \mathcal{S}}$ are in one-to-one correspondence with nested subsets of $\widetilde{\mathcal{S}}$. In particular, $L_{\oslash \mathcal{S}}$ is a flag complex if and only if every subset of $\widetilde{\mathcal{S}}$ consisting of pairwise nested elements is nested.*

As a special case, note that if $\mathcal{S} = \mathcal{P}(L)$, then the cells of $L_{\oslash \mathcal{S}}$ correspond to flags in $\mathcal{P}(L)$. Thus, $\mathcal{P}(L_{\oslash \mathcal{S}}) = \mathrm{Fl}(\mathcal{P}(L))$ and $L_{\oslash \mathcal{S}}$ is the barycentric subdivision of L. More generally, suppose $\mathcal{S}$ is a partial decomposition set of the form $\mathcal{P}(L)_{\geq \mathcal{T}}$ (as in Example 3.3.3) where $\mathcal{T}$ is a subset of $\mathcal{P}(L)$ consisting of cells of positive dimension (subdividing vertices has no effect). Then the $\mathcal{S}$-join decomposition of any cell σ is $\{\sigma\}$; hence, $L_{\oslash \mathcal{S}}$ is an iterated stellar subdivision of L.

Lemma 3.3.10. *Let $\mathcal{T}$ be any subset of $\mathcal{P}(L)$ (with no vertices) and let $\mathcal{S} = \mathcal{P}(L)_{\geq \mathcal{T}}$. Assume further that L is a complex. Then $L_{\oslash \mathcal{S}}$ is a flag complex if and only if $\widetilde{\mathcal{S}}$ contains every join irreducible cell of $\mathcal{P}(L)$ and $\mathcal{T}$ contains an edge of every empty simplex in L.*

Proof. Any subset S of $\widetilde{\mathcal{S}}$ consisting of pairwise nested elements must be of the form $S = \{v_0, \dots, v_j, \sigma_1, \dots, \sigma_k\}$ where $\sigma_1 < \cdots < \sigma_k$ is a flag of cells in $\mathcal{S}$ and $v_0, \dots, v_j$ are pairwise joinable vertices of σ_1 whose edges are not in $\mathcal{T}$. If $\mathcal{T}$ contains an edge of every empty simplex in L, then $v_0 * \cdots * v_j$ is a simplex in σ_1

and S is nested. Conversely, if $v_0, \dots, v_j$ are vertices of an empty simplex with no edge in $\mathcal{T}$ (and therefore no edge in $\mathcal{S}$), then $\{v_0, \dots, v_j\}$ is not nested. □

Links of vertices in blow-ups. The significance of the partial subdivision construction above is that it describes the link of a vertex in a blow-up $K_{\#\mathcal{M}}$ in terms of the link of the vertex in K (note that the vertex sets of K and $K_\#$ are the same). Let Z be an n-dimensional cell in K and let v be a vertex of Z. Let

$$Z = \prod_{f \in \mathcal{F}} Z_f$$

be the decomposition given in Lemma 3.1.6, so v is a vertex of each Z_f. On the level of links, we have by property (P2),

$$\begin{aligned} \mathrm{Lk}(v, Z_\#) &= \mathrm{Lk}(v, (\prod Z_f)_\#) \\ &= \mathrm{Lk}(v, \prod (Z_f)_\#) \\ &= *_{f \in \mathcal{F}} \mathrm{Lk}(v, (Z_f)_\#) \end{aligned}$$

If there is more than one factor in the above join decomposition (*Case 1*), then each Z_f has dimension less than n. By induction, this determines $\mathrm{Lk}(v, Z_\#)$. If the join decomposition has only one factor (*Case 2*), then the blow-up construction on Z introduces a new one-dimensional cell I_Z (corresponding to the restriction of the canonical interval bundle over $\mathbb{P}(Z)_\#$ to the single point $\overline{v}$). It follows from the blow-up construction that

$$\mathrm{Lk}(v, Z_\#) = \mathrm{Lk}(v, (\partial Z)_\#) * \mathrm{Lk}(v, I_Z).$$

In other words, $\mathrm{Lk}(v, Z_\#)$ is the cone on $\mathrm{Lk}(v, (\partial Z)_\#)$. Hence, the subdivision construction and this description of the links in the blow-up coincide. We state this as the following lemma.

Lemma 3.3.11. *Let $(K, \mathcal{M})$ be a mined zonotopal cell complex, and let v be a vertex of K. Let $\mathcal{S}$ be the induced partial subdivision set $\mathcal{S} = \{\mathrm{Lk}(v, Z) | Z \in \mathcal{M}$ and $v \in Z\}$ for $\mathrm{Lk}(v, K)$. Then*

$$\mathrm{Lk}(v, K_{\#\mathcal{M}}) = \mathrm{Lk}(v, K)_{\oslash \mathcal{S}}.$$

Combining this with Lemma 3.3.6 we have the following corollary.

Corollary 3.3.12. *Let $(K, \mathcal{M})$ be a partially mined zonotopal cell complex. Then $K_{\#\mathcal{M}}$ is simple if and only if $\mathcal{M}$ satisfies the condition:*
(S) $\mathcal{M}$ contains every irreducible cell Z in K which is not simple.

Example 3.3.13. (i) Let $\mathcal{M}$ be any minefield on K (not just a partial minefield). Then $K_{\#\mathcal{M}}$ is a cubical cell complex (in particular, the cells are simple).
(ii) Let K be a zonotopal cell complex and let $\mathcal{A}$ be any subset of $\mathcal{C}_K$ which contains the parallel class of every irreducible, non-simple cell. Then $\mathcal{G} = \mathcal{C}_{\geq \mathcal{A}}$ (see Lemma 3.1.3) is a partial building set, and $K_{\#\mathcal{G}}$ is simple.

3.4. Comparison with the DeConcini-Procesi procedure. Let $\mathcal{H}$ be an essential hyperplane arrangement in an n-dimensional real vector space V, $\mathcal{C}$ ($= \mathcal{H}^{\perp}$) the dual arrangement in V^*, and Z and X the corresponding zonotope and its dual. Recall that $\mathcal{C}$ and $\mathcal{C}_Z$ are naturally identified. A *partial building set* for $\mathcal{C}$ is defined analogously to Definition 3.1.1, i.e., it corresponds to a partial building set for $\mathcal{C}_Z$. Let $\mathcal{G}$ be a partial building set for $\mathcal{C}$. Following [DP1], put

$$\widehat{V} = V - \bigcup_{A \in \mathcal{G}} A^{\perp}$$

and let

$$\pi : \widehat{V} \to V \times \prod_{A \in \mathcal{G}} \mathbb{P}(V/A^{\perp})$$

be the natural map. Then $V_{\#}$, the *blow-up of* V *with respect to* $\mathcal{G}$, is defined to be the closure of $\pi(\widehat{V})$ in the product. The blow-up of $\mathbb{P}(V)$, denoted by $\mathbb{P}(V)_{\#}$ is similarly defined except that one replaces V by $\mathbb{P}(V)$ in the product. The closure of the image of $X \cap \widehat{V}$ in $V_{\#}$ is denoted by $X_{\#}$.

Let $s : V \to X$ be radial projection. That is, $s(v) = v$ if $v \in X$ and if $v \notin X$ then $s(v)$ is the point where the line segment from 0 to v intersects ∂X. Since s preserves subspaces through the origin, it induces a map $s_{\#} : V_{\#} \to X_{\#}$. The following lemma is then clear.

Lemma 3.4.1. *$s_{\#} : V_{\#} \to X_{\#}$ is a deformation retraction.*

We recall that $\mathbb{P}(X) = \partial X/a$ and $\mathbb{P}(Z) = \partial Z/a$ are dual cell complexes. It follows from Example 1.5.2 and the fact that $\mathbb{P}(V)_{\#}$ can be obtained by a sequence of blow-ups, as in [DP1], that if σ is an open $(n-1)$-cell in $\mathbb{P}(X)$, then its closure $\overline{\sigma}$ in $\mathbb{P}(V)_{\#}$ is an $(n-1)$-cell obtained from σ by truncating certain faces. (To be precise one truncates those faces of σ which are dual to faces in the partial minefield $\mathcal{M}$.) Thus, $\mathbb{P}(V)_{\#}$ is naturally a cell complex which we denote by $\mathbb{P}(X)_{\#}$. The set of $(n-1)$-cells in $\mathbb{P}(X)_{\#}$ is naturally bijective with the set of $(n-1)$-cells in $\mathbb{P}(X)$. Let $\mathbb{P}(Z)_{\#}$ be the functorial blow-up of the zonotopal cell complex $\mathbb{P}(Z)$ along the partial minefield $\mathcal{M}$. The main result of this subsection is the following.

Theorem 3.4.2. *$\mathbb{P}(X)_{\#}$ and $\mathbb{P}(Z)_{\#}$ are dual cell complexes.*

Proof. The link of a vertex v in $\mathbb{P}(Z)_{\#}$ is obtained by subdividing $\mathrm{Lk}(v, \mathbb{P}(Z))$, as described in the previous subsection. This subdivision process is dual to the process of truncating D_v (where D_v denotes the cell in $\mathbb{P}(X)$ which is dual to v). The result follows. □

Remark 3.4.3. It follows from [DP1] that the manifold $\mathbb{P}(V)_{\#}$ ($= \mathbb{P}(X)_{\#} = \mathbb{P}(Z)_{\#}$) can be obtained from the real projective space $\mathbb{P}(V)$ by a sequence of blow-ups along smooth centers (in the sense of algebraic geometry). This shows that $\mathbb{P}(V)_{\#}$ has a natural smooth structure (in fact, a nonsingular real algebraic structure) in which the subspaces of $\mathbb{P}(Z)_{\#}$ are smooth submanifolds.

3.5. Oriented blow-ups. If Z is a zonotope, the *oriented blow-up of Z at its center* is the product $\partial Z \times [0,1]$. The entire construction of the functor $\#$ can be repeated for oriented blow-ups. In other words, there is a functor from mined zonotopal cell complexes to zonotopal cell complexes, denoted $(K, \mathcal{M}) \to K_{\odot\mathcal{M}}$, which has the properties:
(P1) $K_{\odot\emptyset} = K$
(P2) $(K_1 \times K_2)_{\odot\mathcal{M}_1\times\mathcal{M}_2} = (K_1)_{\odot\mathcal{M}_1} \times (K_2)_{\odot\mathcal{M}_2}$.

Again the construction is by induction. $K_\odot^{(1)} = K^{(1)}$ where $K^{(1)}$ is the 1-skeleton of K. Assume by induction that $K_\odot^{(n-1)}$ has been defined. For any n-cell Z, let

$$(Z, \mathcal{M}_Z) = \prod_{\mathcal{F}} (Z_f, \mathcal{M}_f)$$

be the decomposition of Lemma 3.1.6. If there is more than one factor, then each factor has dimension less than n, so define $Z_{\odot\mathcal{M}} = \prod (Z_f)_{\odot\mathcal{M}_f}$. If there is only one factor (i.e., if $Z \in \mathcal{M}$ or if $\mathcal{P}(Z) \cap \mathcal{M} = \emptyset$), then $(\partial Z)_\odot$ has been defined, and $Z_\odot$ is defined to be the product $(\partial Z)_\odot \times [0,1]$.

In both cases, $(\partial Z)_\odot$ is naturally identified with a subcomplex of $Z_\odot$. In the first case, this follows from the product structure. In the second case, $(\partial Z)_\odot$ is the subcomplex $(\partial Z)_\odot \times \{1\}$ of $Z_\odot$. For each n-cell Z, the complex $Z_\odot$ is then glued to $K_\odot^{(n-1)}$ along the subcomplex $(\partial Z)_\odot$.

Remarks 3.5.1. (i) If Z is minimal in $\mathcal{M}$, then $Z_\odot = \partial Z \times [0,1]$ while $Z_\# = (\partial Z \times [0,1])/\sim$ where the equivalence relation identifies $(z,0)$ and $(a(z),0)$. To put this another way, $Z_\odot$ is obtained from $Z_\#$ by cutting $Z_\#$ open along $\partial Z/a(=$ the image of $\partial Z \times 0)$. More generally, $K_\#$ is an identification space of $K_\odot$ formed by identifying certain points in the boundary via involutions.
(ii) The notion of a maximal oriented blow-up makes sense for any cell complex, not just one which is zonotopal.
(iii) In our definition of $K_\odot$ we have not altered the 1-skeleton of K. Dually, we have not blown up codimension-one subspaces. Instead, we could have defined the oriented blow-up construction starting with the 0-skeleton of K, in which case 1-cells in the partial minefield would be divided into two intervals. $K_\odot$ would then tend to break up into more connected components. For example, if one were to apply this procedure to the maximal minefield for ∂Z, the result would be a disjoint union of the truncation of all top dimensional cells in ∂X (= the dual of ∂Z).

Let $\widehat{K}$ be the complement of all codimension ≥ 2 immersed subspaces of K (see Section 1.5) which are dual to cells in $\mathcal{M}$. Then $K_\odot$ and $\widehat{K}$ both retract onto a common subcomplex of K. Namely, let $U_\mathcal{M}$ be the union of the interiors of all cells of K which have a face in $\mathcal{M}$. Then the following is clear from the various constructions.

Lemma 3.5.2. *There are deformation retracts $K_\odot \to K - U_\mathcal{M}$ and $\widehat{K} \to K - U_\mathcal{M}$. In particular, $K_\odot$ and the complement $\widehat{K}$ are homotopy equivalent.*

The oriented blow-up $K_\odot$ can also be obtained by "cutting open" the ordinary blow-up $K_\#$ along all of the hyperplanes in the exceptional divisor (see Remark 3.2.1 and Figure 6). In fact the identification map $q : K_\odot \to K_\#$ in the previous remark (part (i)) is just the corresponding regluing map. If $\mathcal{M}$ contains every irreducible non-simple cell of K, then $K_\#$ will be a simple zonotopal cell complex by Corollary 3.3.12, hence, it has a natural cubical subdivision into small cubes. Moreover, the exceptional divisor will be a subcomplex of this cubical subdivion $(K_\#)_\square$, so $K_\odot$ will have a natural decomposition $(K_\odot)_\square$ with respect to which the identification map $q : (K_\odot)_\square \to (K_\#)_\square$ will be a map of cubical complexes. Observing the effect on the links when a simple zonotope is cut along hyperplane pieces, we obtain the following lemma.

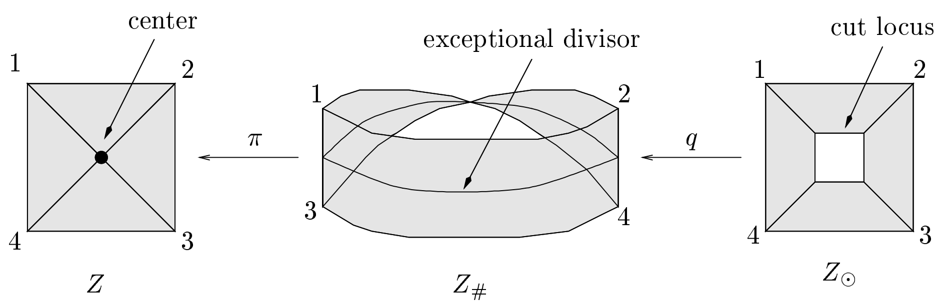


FIGURE 6.

Lemma 3.5.3. *Suppose $\mathcal{M}$ contains every irreducible, non-simple cell in K. Then for every vertex $v \in (K_\odot)_\square$, the simplicial complex* $\mathrm{Lk}(v, (K_\odot)_\square)$ *is a full subcomplex of* $\mathrm{Lk}(q(v), (K_\#)_\square)$.

3.6. "Doubling" an oriented blow-up. Suppose that K is a zonotopal cell complex, that $\mathcal{G}$ is a partial building set for K, and that $\mathcal{M}$ is the associated partial minefield. For simplicity, let us also assume that for each $g \in \mathcal{G}$ the corresponding immersed subspace of K is actually embedded. (In other words, for any $g \in \mathcal{G}$ and $Z \in \mathcal{P}(K)$ with $[Z] \geq g$, there is exactly one parallel class of faces in Z in the equivalence class of g.) Let $\mathcal{G}^{(2)}$ denote the set of subspaces in $\mathcal{G}$ of codimension ≥ 2 (i.e., $\mathcal{G}^{(2)}$ consists of those parallel classes which are represented by cells of dimension ≥ 2). We will define below a distinguished family $(\delta_g K_\odot)_{g \in \mathcal{G}^{(2)}}$ of codimension-one subcomplexes of $K_\odot$ indexed by $\mathcal{G}^{(2)}$. Roughly speaking, $\delta_g K_\odot$ is the boundary of a regular neighborhood of the subspace corresponding to g in K. (Equivalently, $\delta_g K_\odot$ is the double cover of the "exceptional divisor" corresponding to g in $K_\#$.) Once the $\delta_g K_\odot$ have been defined, it will then be possible to glue together 2^m copies of $K_\odot$, where $m = \mathrm{Card}(\mathcal{G}^{(2)})$, along the $\delta_g K_\odot$ to get a new zonotopal cell complex $DK_\odot$, called the "double" of $K_\odot$.

By definition, $\delta_g K_\odot$ will be the union of the $\delta_g Z_\odot$ where Z is a cell in K with $[Z] \geq g$, and where $\delta_g Z_\odot$ will be defined below. If g is not less that or equal to

$[Z]$, then set $\delta_g Z_\odot = \emptyset$. Suppose by induction that $\delta_g K'_\odot$ has been defined for all subcomplexes K' of dimension $< n$, and that Z is an n-cell with $[Z] \geq g$. As in the previous section, let

$$(Z, \mathcal{M}) = \prod_{f \in \mathcal{F}} (Z_f, \mathcal{M}_f)$$

be the $\mathcal{G}_Z$-decomposition of Lemma 3.1.6. If there is more than one factor, then there is exactly one element of $\mathcal{F}$, call it f', such that $g \leq f'$ (since we are assuming each subspace is embedded). In this case, set

$$\delta_g Z_\odot = \delta_g(Z_{f'}) \times \prod_{f \in \mathcal{F} - f'} Z_f.$$

If there is only one factor, then $Z \in \mathcal{M}$ and by the inductive hypothesis $\delta_g(\partial Z_\odot)$ has been defined. If $[Z] \neq g$, then set

$$\delta_g Z_\odot = \delta_g(\partial Z_\odot) \times [0,1],$$

while if $[Z] = g$, then set

$$\delta_g Z_\odot = (\partial Z_\odot) \times 0.$$

Let $J = (\mathbb{Z}_2)^{\mathcal{G}^{(2)}}$ be a product of cyclic groups of order 2, and let $\{s_g\}_{g \in \mathcal{G}^{(2)}}$ be a set of generators. Define

$$DK_\odot = (K_\odot \times J)/\sim$$

where the equivalence relation is defined by $(x, j) \sim (x', j')$ if and only if $x = x'$ and $j^{-1}j'$ belongs to the subgroup generated by $\{s_g | x \in \delta_g K_\odot\}$. The group J acts naturally on $DK_\odot$ (as a reflection group) and the orbit space is $K_\odot$.

In the case of a maximal blow-up or maximal partial blow-up (i.e., a blow-up with respect to a maximal partial building set as in Lemma 3.1.3), there is a smaller "double" of $K_\odot$. Suppose that $\dim K = n$. For each $i \in \{2, \dots, n\}$, let $\delta_i K_\odot$ denote the union of the $\delta_g K_\odot$ where g corresponds to a subspace of codimension i. Let $\{s_i\}_{2 \leq i \leq n}$ be a set of generators for $(\mathbb{Z}_2)^{n-1}$ and set

$$\widetilde{D} K_\odot = (K_\odot \times (\mathbb{Z}_2)^{n-1})/\sim$$

where the equivalence relation is defined as above. Since we are dealing with the maximal blow-up, $\delta_g K_\odot \cap \delta_{g'} K_\odot = \emptyset$ if g and g' are distinct and have the same codimension. It follows that $DK_\odot$ is a covering space of $\widetilde{D} K_\odot$.

Remark 3.6.1. If $\mathcal{G}$ is the maximal building set, then $\widetilde{D} K_\odot$ is the "cross-with-interval hyperbolization procedure" of [DJ2] applied to the cell complex K. In this case, the $(\mathbb{Z}_2)^{n-1}$-action on $\widetilde{D} K_\odot$ is discussed on page 334 of [CD3].

Remark 3.6.2. $K_\odot$ is a retract of $DK_\odot$. Hence, if $DK_\odot$ is aspherical, so is $K_\odot$.

Remark 3.6.3. If K is a smooth manifold (e.g., if $K = \mathbb{P}(Z)$), then $K_\odot$ is a manifold with corners, and the $\delta_g K_\odot$ are its codimension-one strata. Thus, $K_\odot$ has the structure of an orbifold (a right-angled "reflectofold" as in [CD3]), and $DK_\odot$ is a smooth manifold.

4. Nonpositive curvature of blow-ups

Let K be a zonotopal cell complex, and let $K_\#$ be the blow-up with respect to some partial minefield. In this section we consider the question of when $K_\#$ admits a piecewise Euclidean metric of nonpositive curvature. We treat three cases: (1) K is any zonotopal cell complex and $\mathcal{M}$ consists of all cells, (2) $K = \mathbb{P}(Z)$ where Z is a simple zonotope and $\mathcal{M}$ is the set of all irreducible cells, and (3) K is any zonotopal cell complex and $\mathcal{M}$ is a partial minefield with the property that every cell with a face in $\mathcal{M}$ is also in $\mathcal{M}$. In the first two cases, $\mathcal{M}$ is a minefield, hence the blow-ups will be zonotopal cell complexes whose cells are all (big) cubes. Passing to the standard cubical subdivision does not essentially alter the metric. On the other hand, for case (3) the resulting zonotopes need not be cubes, so here we must use the cubical subdivision (provided the zonotopes are simple) in order to get a natural piecewise Euclidean structure. Case (3) is a generalization of case (1).

4.1. Maximal blow-ups. Let K be a zonotopal cell complex, and let $\mathcal{M}$ be the minefield consisting of all cells in K. We call $K_{\#\mathcal{M}}$ the *maximal blow-up* of K. The maximal blow-up is a generalization of Gromov's "Moebius band hyperbolization" (see [CD3], for details). In particular, it is always nonpositively curved.

Theorem 4.1.1. *Let K be a zonotopal cell complex. Then the maximal blow-up $K_\#$ has a natural piecewise Euclidean (cubical) metric of nonpositive curvature.*

Proof. By Gromov's Lemma (Corollary 1.6.7), it suffices to show that $\mathrm{Lk}(v, K_\#)$ is a flag complex for every vertex v. But it follows from the explicit description of the links (Section 3.3) that $\mathrm{Lk}(v, K_\#)$ is isomorphic to the barycentric subdivision of $\mathrm{Lk}(v, K)$. The latter is a flag complex by Example 1.6.2. □

Remark 4.1.2. By Lemma 3.5.3, the link of any vertex of the maximal oriented blow-up $K_{\odot\mathcal{M}}$ is isomorphic to a full subcomplex of the link of a vertex in $K_{\#\mathcal{M}}$ and is, therefore, a flag complex. Hence, $K_{\odot\mathcal{M}}$ is nonpositively curved. Since $K_{\odot\mathcal{M}}$ is homotopy equivalent to the complement $\widehat{K}$ of all codimension-2 immersed subspaces of K (Lemma 3.5.2), it follows that $\widehat{K}$ is aspherical.

Applying this theorem to the case of hyperplane arrangements, we have the following.

Corollary 4.1.3. *Let $\mathcal{H}$ be an essential hyperplane arrangement in V, let $\mathbb{P}(V)_\#$ denote the maximal blow-up of $\mathbb{P}(V)$ as in* [DP1] *(see Section 3.4). Then $\mathbb{P}(V)_\#$ admits a metric of nonpositive curvature.*

Proof. Let Z be the zonotope associated to $\mathcal{H}$. From Section 3.4, $\mathbb{P}(Z)_\#$ is a cell decomposition for $\mathbb{P}(V)_\#$, so (1) follows from Theorem 4.1.1 applied to $K = \mathbb{P}(Z)$. □

4.2. Manifolds tiled by permutohedra. We have previously mentioned several different constructions of closed n-manifolds cellulated by n-dimensional permutohedra with precisely 2^n of the permutohedra meeting at each vertex. It turns out that these manifolds are "commensurable" in the sense that any two have a common finite-sheeted covering space. We begin by discussing their common universal covering space. We let P^n denote the n-dimensional permutohedron.

Example 4.2.1. (*The universal cover.*) This is a special case of the construction described in the paragraph preceding Lemma 2.2.4. Let S be an index set for the set of codimension-one faces of P^n and let (W, S) be the corresponding right-angled Coxeter system (generators s and t commute if and only if the corresponding faces F_s and F_t intersect). Set

$$U = (P^n \times W)/\sim$$

where $(x, w) \sim (x', w')$ if and only if $x = x'$ and $w^{-1}w'$ belongs to the subgroup generated by $\{s \in S | x \in F_s\}$. Then U is a simply-connected manifold tiled by permutohedra. Moreover, it follows from the fact that the dual of ∂P is a flag complex that U is contractible (e.g., see [D2]).

An equivalent point of view to that of the preceding paragraph is the following. P^n can be regarded as a right-angled Coxeter orbifold in the sense of [CD3] or [DJ1]. Its orbifold fundamental group is W, and its universal covering space (in the sense of orbifolds) is U. Hence, if Γ is any torsion-free subgroup of W, then U/Γ is a manifold tiled by permutohedra, and the natural map $U/\Gamma \to P^n$ is an orbifold covering. Conversely, if $M^n \to P^n$ is an orbifold covering and M^n is a manifold, then the universal covering space of M^n is U, and $\pi_1(M^n)$ can be identified with a torsion free subgroup of W. We restate this as the following.

Theorem 4.2.2. *Suppose that M^n is a manifold tiled by permutohedra, with 2^n meeting at each vertex, and that $p : M^n \to P^n$ is a map such that*

(i) the restriction of p to any n-dimensional permutohedron in M^n is a homeomorphism, and

(ii) the map p is locally isomorphic to the orbit map $\mathbb{R}^n \to \mathbb{R}^n/(\mathbb{Z}_2)^n$ (where $(\mathbb{Z}_2)^n$ acts as a linear reflection group on $\mathbb{R}^n$).

Then the universal cover of M^n is U, and $\pi_1(M^n)$ is a torsion-free subgroup of W. If M^n is compact, then $\pi_1(M^n)$ has finite index in W.

The full group of symmetries Λ of the cellulation of U by permutohedra is slightly larger than W. In fact, we have an exact sequence

$$1 \to W \to \Lambda \to G_n \to 1$$

where G_n is the group of combinatorial symmetries of P^n. It follows from the definition of P^n that the symmetric group S_{n+1} is contained in G_n. Since P^n is a zonotope, its symmetry group also contains the central symmetry. It is not hard to see that, in fact, G_n is the product $S_{n+1} \times \mathbb{Z}_2$ (the second factor being the central symmetry). Hence, there are $2n!$ copies of a fundamental domain for G_n in P^n. We also remark that (i) S_{n+1} acts on P^n as a group generated by reflections with quotient space a combinatorial cube, and (ii) the inverse image of S_{n+1} in Λ is a Coxeter group W' with diagram as in Figure 7, discussed on page 337 of [CD3].

FIGURE 7.

Theorem 4.2.2 applies to any small cover of P^n (see Section 2). Next we consider some specific small covers. First we state formally the coincidences among the examples mentioned in the Introduction (Section 0.4).

Theorem 4.2.3. *For each integer $n \geq 0$, the following three manifolds are isomorphic as manifolds tiled by permutohedra.*
(i) M_1^n, the maximal blow-up of the coordinate hyperplane arrangement in $\mathbb{RP}^n$.
(ii) M_2^n, the closure of a generic $(\mathbb{R}^*)^n$-orbit in the flag manifold $SL(n+1,\mathbb{R})/B$ (where B is the Borel subgroup of upper triangular matrices).
(iii) M_3^n, the quotient of the Möbius band hyperbolization of the boundary complex of an $(n+1)$-cube by the central involution.

Proof. The coordinate hyperplanes give $\mathbb{RP}^n$ the structure of a small cover of the n-simplex Δ^n (it is the real part of the complex toric variety $\mathbb{CP}^n \to \Delta^n$). If we blow-up such a toric variety along all $(\mathbb{R}^*)^n$-stable subvarieties (in order of increasing dimension), we obtain another toric variety over the full truncation of Δ^n, that is, over the permutohedron. Thus, $M_1^n \to P^n$ is a small cover as well as the real part of a toric variety. M_2^n is also the real part of a toric variety over P^n. An explicit computation (e.g., see [FH]) shows that M_1^n and M_2^n have the same characteristic function; hence, $M_1^n = M_2^n$. The coordinate hyperplane arrangement cuts $\mathbb{RP}^n$ into n-simplices and, hence, gives it the structure of an n-dimensional simplicial cell complex. Explicitly, this cell complex is the boundary complex of an $(n+1)$-dimensional octahedron divided by the antipodal map. Its dual zonotopal complex is the boundary of the $(n+1)$-cube divided by the antipodal map. Hence, its maximal blow-up (which in this case is Gromov's Möbius band construction) is diffeomorphic to M_1^n. More precisely, if M_3^n is the dual cell complex to this cubical manifold, then M_3^n is cellulated by permutohedra and $M_3^n = M_1^n$. □

Example 4.2.4. (*Tomei manifolds.*) Fix a set $\Omega = \{\omega_1, \dots, \omega_{n+1}\}$ of distinct real numbers and let N^n be the set of symmetric tridiagonal matrices with spectrum equal to Ω. As explained in [D2] and [T], $(\mathbb{Z}_2)^n$ acts as a reflection group on N^n, and each chamber is isomorphic to P^n. Hence, $N^n \to P^n$ is also a small cover of P^n. For $n > 1$, the manifold N^n is not diffeomorphic to the n-dimensional manifold in the previous theorem. For example N^2 is orientable while M_2^2 is not (however both have Euler characteristic -2). In fact, contrary to the assertion in the final paragraph of [DJ1], N^n is not the real part of a toric variety.

The natural action of S_{n+1} on P^n extends to an action on M_2^n and to one on N^n (as shown in [D2]). Hence, both M_2^n and N^n can be regarded as orbifold covers of the cube with orbifold fundamental group W' (whose diagram is the "comb" in Figure 7, above). Hence, $\pi_1(M_2^n)$ and $\pi_1(N^n)$ are both subgroups of W' of index $2^n(n+1)!$. Let B_{n+1} denote the semidirect product of $(\mathbb{Z}_2)^{n+1}$ with S_{n+1} where S_{n+1} acts on $(\mathbb{Z}_2)^{n+1}$ by permuting the coordinates. Since $M_2^n = M_3^n$, it follows from Proposition 4.1, p. 338 of [CD3], that $\pi_1(M_2^n)$ can be identified with the kernel of a natural surjection $W' \to B_{n+1}$. There is another natural surjection $W' \to (\mathbb{Z}_2)^{n+1} \times S_{n+1}$ (namely, (ϕ_1, ϕ_2) where ϕ_1 sends the generators on the top of the comb to the generators of $(\mathbb{Z}_2)^{n+1}$ and the others to the identity element, and where $\phi_2 : W' \to S_{n+1}$ sends the generators on the bottom of the comb to those of S_{n+1} via the obvious surjection and the others to the identity element). It follows from [D2] that $\pi_1(N^n)$ is the kernel of $(\phi_1, \phi_2) : W' \to (\mathbb{Z}_2)^{n+1} \times S_{n+1}$.

There are many different small covers of P^n. What distinguishes the Tomei manifold N^n as well as one of the manifolds M^n of Theorem 4.2.3 from the others is that they both admit an action of the symmetric group S_{n+1} inducing the reflection group action on P^n. Thus, both N^n and M^n are orbifold covers (with the same number of sheets) of the n-cube with the orbifold structure having W' as fundamental group.

Example 4.2.5. (*Maximal blow-ups.*) Suppose that $\mathcal{H}$ is a simplicial hyperplane arrangement in $\mathbb{R}^{n+1}$ and that we are considering the maximal blow-up (so that the building set $\mathcal{G}$ consists of all subspaces of $\mathcal{H}^\perp$). Let X be the corresponding simplicial polytope. Then the boundary of each n-cell in $\mathbb{P}(X)_\#$ is dual to the barycentric subdivision of the boundary of an n-simplex (see Example 3.3.5). In other words, each n-cell of $\mathbb{P}(X)_\#$ is a permutohedron. We now show that any two of these maximal blow-ups are commensurable (as well as being commensurable with manifolds in Theorem 4.2.3 and Example 4.2.4. First we need a lemma.

Lemma 4.2.6. *Let X be the simplicial polytope associated to a simplicial arrangement in $\mathbb{R}^{n+1}$ and let Δ^n be a top dimensional simplex (or chamber) in ∂X. Then there is a simplicial projection $p : \partial X \to \Delta^n$ which takes each simplex in ∂X isomorphically onto a face of Δ^n.*

Proof. Suppose that Δ' is an adjacent chamber to Δ. Let v' (resp., v) be the vertex of Δ' (resp., Δ) opposite to $\Delta' \cap \Delta$. Then there is a unique simplicial isomorphism $\Delta' \to \Delta$ which fixes $\Delta' \cap \Delta$ and which maps v' to v. Hence, if $\gamma = (\Delta_0, \dots, \Delta_m)$ is any sequence of adjacent chambers (i.e., a *gallery*) from $\Delta = \Delta_0$ to $\Delta'' = \Delta_m$, we get a simplicial isomorphism $\theta_\gamma : \Delta'' \to \Delta$ by composing the above isomorphism. In fact this isomorphism depends only on the chambers Δ'' and Δ, not on the gallery γ. The reason for this is that two galleries with the same endpoints are equivalent by a sequence of moves which involves going around a codimension-two face in a different direction. Since each codimension-two face is contained in an even number of chambers, such moves do not affect θ_γ. If Δ'' and Δ' are adjacent chambers, then the simplicial isomorphism $\Delta'' \to \Delta$ and $\Delta' \to \Delta$ must agree on $\Delta'' \cap \Delta'$. It follows that we have a well-defined simplicial map $p : \partial X \to \Delta^n$. □

Corollary 4.2.7. *The map $p : \partial X \to \Delta^n$ induces a projection $p_\# : \partial X_\# \to P^n$ which is an orbifold covering.*

Combining this with Theorem 4.2.2 we have the following.

Corollary 4.2.8. *Let $\mathbb{P}(X)_\#$ be the maximal blow-up of a projective $(n+1)$-dimensional simplicial arrangement. Then the universal cover of $\mathbb{P}(X)_\#$ is U (described in Example 4.2.1). Moreover, the maximal blow-ups of two such simplicial arrangements (of the same dimension) are commensurable.*

Proof. By the previous corollary, the universal cover of $\partial X_\#$ is U. Since $\partial X_\#$ is a 2-fold covering of $\mathbb{P}(X)_\#$, it follows that U is also the universal covering of $\mathbb{P}(X)_\#$.

Next suppose that X_1 and X_2 are polytopes corresponding to simplicial arrangements in $\mathbb{R}^{n+1}$. Then the fundamental groups of $(\partial X_1)_\#$ and $(\partial X_2)_\#$ are torsion-free subgroups Γ_1 and Γ_2 of W, and $U/(\Gamma_1 \cap \Gamma_2)$ is the desired common finite sheeted cover of both. □

Remark 4.2.9. One might speculate that any simply connected n-manifold, cellulated by permutohedra, with 2^n at each vertex is isomorphic (as a cell complex) to U. In fact, this is false for $n \geq 3$. To see this, consider the 3-dimensional version of U. At each vertex of a permutohedron P^3 we have two hexagons and one square. There is a two-dimensinonal subspace (= wall) containing each of these polygons. In the case of the hexagons, such a subspace is isomorphic to the hyperbolic plane tiled by right-angled hexagons. For the square, it is isomorphic to the standard tiling of $\mathbb{R}^2$ by squares. Now cut U open along one of the hyperbolic planes, rotate by $\pi/2$, and glue back. The resulting manifold $\widehat{U}$ is cellulated by permutohedra, and the cellulation is not isomorphic to that of U. (The reason that this does not contradict the previous discussion is that there is no map $\widehat{U} \to P^3$.)

4.3. Minimal blow-ups of simple zonotopes. The minimal blow-up of the braid arrangement described in the Introduction (Example 0.1.7) is a special case of the blow-up of a zonotopal cell complex of the form $\mathbb{P}(Z)$ along the minefield $\mathcal{M}$ consisting of all irreducible cells. If Z is a simple zonotope (i.e., if Z is the zonotope associated to any simplicial hyperplane arrangement), then the following theorem gives a simple necessary and sufficient condition for this irreducible blow-up $\mathbb{P}(Z)_{\#\mathcal{M}}$ to be nonpositively curved.

Theorem 4.3.1. *Let Z be a simple zonotope. Let $\mathcal{M}$ be the set of all irreducible cells in $\mathbb{P}(Z)$. Then $\mathbb{P}(Z)_{\#\mathcal{M}}$ is nonpositively curved if and only if the irreducible decomposition of Z has fewer than 3 factors.*

In terms of hyperplane arrangements, we have the following.

Corollary 4.3.2. *If $\mathcal{H}$ is a simplicial hyperplane arrangement in $\mathbb{P}(V)$, then the natural cubical metric on the minimal blow-up $\mathbb{P}(V)_{\#}$ is nonpositively curved if and only if the irreducible decomposition of V^* has fewer than 3 factors.*

We will first reduce the proof of Theorem 4.3.1 to comparing the number of connected components in certain graphs associated to Z and to each vertex of Z. The first graph Γ is defined as follows. The vertex set of Γ is the set of parallel classes of edges in Z. Two such classes e_1 and e_2 determine an edge of Γ if and only if $\{e_1, e_2\}$ is not a decomposition of a face (i.e., if and only if any representative face for $e_1 + e_2$ is not a square). There is a natural correspondence between the factors in the irreducible decomposition of Z and the connected components of Γ.

For any vertex v of Z, there is a similar graph Γ_v defined as follows. The vertex set of Γ_v is the set of edges of Z which contain v. Two such edges of Z determine an edge of Γ_v if and only if they are both contained in an irreducible 2-dimensional face of Z (i.e., a face which is not a isomorphic to a square). Since any parallel class of edges in Z meets a given vertex v in at most one edge, Γ_v is an embedded subgraph of Γ. Let $\overline{v}$ denote the image of v in $\mathbb{P}(Z)$ and let $L = \mathrm{Lk}(\overline{v}, \mathbb{P}(Z))$. ($L$ is just the boundary complex of a simplex.) It follows from Lemma 3.3.11 that the link of $\overline{v}$ in $\mathbb{P}(Z)_{\#\mathcal{M}}$ is the subdivision $L_{\oslash\mathcal{S}_v}$ where $\mathcal{S}_v$ is the set of all simplices in L spanned by the vertex sets of connected subgraphs of Γ_v.

Lemma 4.3.3. *$L_{\oslash\mathcal{S}_v}$ is a flag complex if and only if Γ_v has fewer than 3 components.*

Proof. For a given simplex σ in L, let $\Gamma_v(\sigma)$ denote the full subgraph of Γ_v contained in σ. Then $S = \{\sigma_1, \dots, \sigma_k\}$ is a nested subset of $\mathcal{S}_v$ if and only if each $\Gamma_v(\sigma_i)$ is in a different connected component of Γ_v and the union the $\Gamma_v(\sigma_i)$ is not all of Γ_v (since L itself is an empty simplex). The lemma then follows from Proposition 3.3.9. □

We shall prove Theorem 4.3.1 by showing that Γ_v and Γ have the same number of components for any vertex v of Z.

For any vertex v in Z, we let $\mathcal{E}(v)$ denote the set of edges of Z which contain v; thus, $\mathcal{E}(v)$ is the vertex set of Γ_v. For any collection of edges $E_1, \dots, E_k$ in $\mathcal{E}(v)$, we let $E_1 + \cdots + E_k$ denote the unique k-dimensional face which they span. Suppose E is an edge of Z with vertices v and v'. Then there is a natural bijection $\theta : \mathcal{E}(v) \to \mathcal{E}(v')$ given by $\theta(E) = E$ and for any other $F \in \mathcal{E}(v)$, $\theta(F)$ is the unique edge in $\mathcal{E}(v') - \{E\}$ with the property that $E + F = E + \theta(F)$. More generally, if γ is any sequence $(v_1, \dots, v_m)$ of adjacent vertices (or "path") in Z, then we get a bijection $\theta_\gamma : \mathcal{E}(v_1) \to \mathcal{E}(v_m)$ by composing the bijections θ along each joining edge. Since the path around any 2-dimensional face induces the identity map on $\mathcal{E}(v)$, there is a well-defined bijection $\theta : \mathcal{E}(v) \to \mathcal{E}(w)$ for any two vertices v and w, independent of the path joining them (this is just the dual version of statements in the proof of Lemma 4.2.6).

Lemma 4.3.4. *Let v and v' be two adjacent vertices and E the joining edge. Let $\theta : \mathcal{E}(v) \to \mathcal{E}(v')$ be the bijection defined above. Suppose E_1 and E_2 are two edges in $\mathcal{E}(v) - \{E\}$, and F is the 3-dimensional zonotope $E_1 + E_2 + E$. Then the following are equivalent.*
(i) $E + E_1$ and $E + E_2$ are reducible.
(ii) E_1 is parallel to $\theta(E_1)$ and E_2 is parallel to $\theta(E_2)$.
(iii) $a(E_1 + E_2) = \theta(E_1) + \theta(E_2)$ where a is the antipodal map in F.

Proof. (i) and (ii) are equivalent since two faces F_1 and F_2 of a face F_3 in a zonotope Z are parallel in Z if and only if they are parallel in F_3. For the remaining implications, suppose P_1 and P_2 are the hyperplane pieces in F which correspond to the parallel classes of E_1 and E_2, respectively. Then (ii)$\Rightarrow$(iii) follows from the fact that Z is simple. That is, P_1 and P_2 must intersect in the barycenters of $E_1 + E_2$ and $\theta(E_1) + \theta(E_2)$, and the antipodal map exchanges these barycenters. To see that (iii)$\Rightarrow$(ii), suppose $a(E_1 + E_2) = \theta(E_1) + \theta(E_2)$. Since the antipodal map is combinatorially defined, $a(E_1)$ and $a(E_2)$ are adjacent edges. It follows that the edge parallel to $a(E_1)$ (resp., $a(E_2)$) in the 2-dimensional face $a(E_1 + E_2)$ must be $\theta(E_1)$ (resp., $\theta(E_2)$) (see Figure 8). Since antipodal edges in F are parallel, (ii) must hold. $\square$

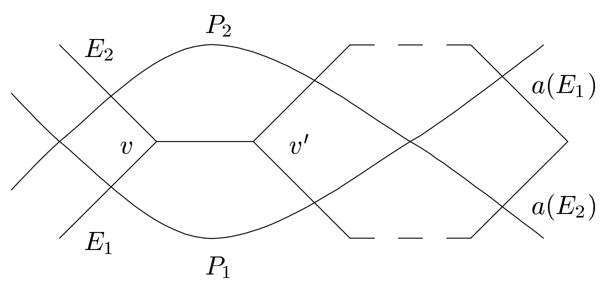


FIGURE 8.

Next suppose that $\mathcal{E}(v)$ can be partitioned into two nonempty subsets $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$ such that $E_1 + E_2$ is reducible for every $E_1 \in \mathcal{E}_1(v)$ and $E_2 \in \mathcal{E}_2(v)$. For any other vertex w, let θ be the bijection $\mathcal{E}(v) \to \mathcal{E}(w)$ defined above. Then $\mathcal{E}_1(w) = \theta(\mathcal{E}_1(v))$ and $\mathcal{E}_2(w) = \theta(\mathcal{E}_2(v))$ give a partition of $\mathcal{E}(w)$ into two disjoint subsets.

Lemma 4.3.5. *If $E_1 \in \mathcal{E}_1(v)$ and $E_2 \in \mathcal{E}_2(v)$, then $\theta(E_1) + \theta(E_2)$ is reducible.*

Proof. We might as well assume that w is adjacent to v and that the edge E which joins them is contained in $\mathcal{E}_1(v)$. Let v, v', w, w', E', E_1', and $\theta(E_1)'$ be as in Figure 9. By the previous lemma (applied to v and v'), $a(E + E_1) = E' + E_1'$.

On the other hand, $E + E_1 = E + \theta(E_1)$ and $E' + E_1' = E' + \theta(E_1)'$. This implies $a(E + \theta(E_1)) = E' + \theta(E_1)'$, so by the previous lemma (applied to w and w'), we have that $\theta(E_1) + \theta(E_2)$ is reducible. □

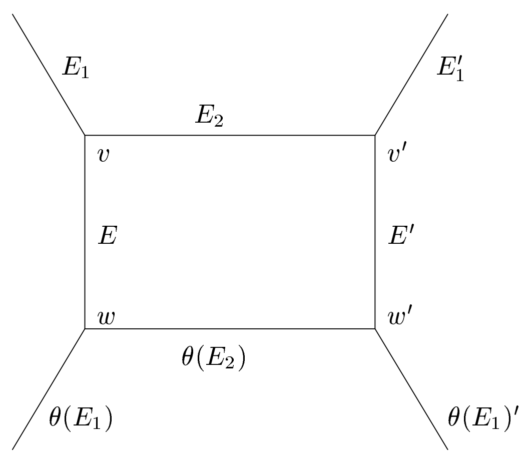


FIGURE 9.

Proof of Theorem 4.3.1. Let v be a vertex of Z. Suppose $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$ are a partition of $\mathcal{E}(v)$ such that no element of $\mathcal{E}_1(v)$ is connected by an edge in Γ_v to an element of $\mathcal{E}_2(v)$. Let $\mathcal{E}$ be the set of parallel classes of edges in Z (i.e., the vertex set of the graph Γ). For $i = 1, 2$ let $\mathcal{E}_i$ be the set of parallel classes generated by edges in $\mathcal{E}_i(v)$ as v ranges over all vertices. It follows from the previous lemma that $\mathcal{E}_1$ and $\mathcal{E}_2$ partition $\mathcal{E}$ into disjoint subsets, and $e_1 + e_2$ is reducible for every $e_1 \in \mathcal{E}_1$ and $e_2 \in \mathcal{E}_2$. Hence, two elements of Γ_v are in the same connected component of Γ_v if and only if they (their parallel classes) are in the same connected component of Γ. Thus, Γ_v and Γ have the same number of connected components. The theorem then follows from Gromov's Lemma (Lemma 1.6.5) and Lemma 4.3.3. □

4.4. Maximal partial blow-ups and a conjecture of Khovanov. In this section we consider the nonpositive curvature question for maximal *partial* blow-ups of zonotopal cell complexes. Let K be a zonotopal cell complex with the property that the link of every vertex is a complex. Let $\mathcal{A}$ be a subset of parallel classes of dimension greater than one (blowing up 0 and 1-dimensional cells does nothing), and let $\mathcal{M}$ be the partial minefield corresponding to the maximal partial building set $\mathcal{C}_{\geq \mathcal{A}}$ (defined in Lemma 3.1.3). We will call such a minefield the *maximal partial minefield associated to* $\mathcal{A}$. For any vertex v of K, the induced partial subdivision set $\mathcal{S}_v$ for $L = \mathrm{Lk}(v, K)$ is of the form $\mathcal{P}(L)_{\geq \mathcal{T}}$ where $\mathcal{T}$ consists of all $\mathrm{Lk}(v, Z)$ where $Z \in \mathcal{M}$. It follows from Lemma 3.3.10 that the link of v in $K_{\#\mathcal{M}}$ will be a flag complex if and only if it is simplicial and every empty simplex in L contains an edge in $\mathcal{S}_v$. Hence, we have the following generalization of Theorem 4.1.1.

Theorem 4.4.1. *Let K be a zonotopal cell complex all of whose links are complexes. Let $\mathcal{M}$ be the maximal partial minefield corresponding to a subset $\mathcal{A}$ of $\mathcal{C}_K$. Suppose that $\mathcal{M}$ contains every irreducible, nonsimple cell of K. Then the natural cubical metric on $K_{\#\mathcal{M}}$ is nonpositively curved if and only if for every vertex v, every empty simplex in* $\mathrm{Lk}(v, K)$ *has an edge in $\mathcal{S}_v$.*

Proof. By Corollary 3.3.12, $K_{\#}$ is a simple zonotopal cell complex (so the natural cubical decomposition is into small cubes). By Corollary 1.6.7, the metric $d_{\square}$ is nonpositively curved if and only if the link of every vertex in $K_{\#}$ is a flag complex. $\square$

Corollary 4.4.2. *If Z is a simple zonotope and $\mathcal{M}$ is as above, then $(\partial Z)_{\#\mathcal{M}}$ is nonpositively curved if and only if every vertex v is contained in some* 2*-dimensional cell in $\mathcal{M}$.*

Proof. If Z is simple, then the 1-skeleton of $\mathrm{Lk}(v, \partial Z)$ is an empty simplex and the only empty simplices which appear in links of vertices in ∂Z are of this form. Hence, the condition of Theorem 4.4.1 that every empty simplex in $\mathrm{Lk}(v, \partial Z)$ has an edge in $\mathcal{S}_v$ is precisely the condition that v is a vertex of a 2-dimensional cell in $\mathcal{M}$. $\square$

In [Kh], Khovanov conjectures that for any real arrangement $\mathcal{H}$ associated to an orthogonal reflection group W, the complement of any W-invariant union of codimension-2 subspaces in $\mathcal{H}$ is a $K(\pi, 1)$-space. This follows immediately as a special case of the following corollary.

Corollary 4.4.3. *Let $\mathcal{H}$ be a simplicial real hyperplane arrangement in V. Let E be any union of codimension-*2 *subspaces in $\mathcal{H}$ which intersects every chamber in a codimension-*2 *subcomplex. Then $\widehat{V} = V - E$ is aspherical.*

Proof. Suppose Z is the (simple) zonotope corresponding to $\mathcal{H}$, and suppose X is its dual polytope. Let $\mathcal{A}$ be the set of all parallel classes of faces in ∂Z which are dual to the codimension-2 subspaces in E, and let $\mathcal{M}$ be the maximal partial minefield associated to $\mathcal{A}$. Since E intersects each chamber (i.e., the dual of a vertex of Z) in a codimension-2 subcomplex, every vertex of Z is a vertex of some 2-cell in $\mathcal{M}$. By Corollary 4.4.2, $(\partial Z)_{\#\mathcal{M}}$ is, therefore, nonpositively curved. Hence, by Lemma 3.5.3, the oriented blow-up $(\partial Z)_{\odot\mathcal{M}}$ is nonpositively curved and therefore aspherical. Let $\widehat{\partial Z}$ be the complement of all subspaces dual to faces in M (or equivalently, the complement of all subspaces dual to elements of $\mathcal{A}$). Then by

Lemma 3.5.2, $\widehat{\partial Z}$ is aspherical. But the latter is homeomorphic to $\widehat{V} \cap \partial X$ which is a deformation retract of $\widehat{V}$. Hence, $\widehat{V}$ is aspherical. $\square$

5. Asphericity of blow-ups

Nonpositive curvature implies asphericity; however, *a priori*, a cubical complex might be aspherical even if its natural piecewise Euclidean metric is not nonpositively curved. (For example, the natural metric on the cubical subdivision of any triangulation of an aspherical manifold will never be nonpositively curved.) In this section we give some necessary conditions for the blow-up of a projective space to be aspherical.

5.1. The number of factors. Suppose Z is a zonotope, $\mathcal{G} \subset \mathcal{C}_Z$ is a building set, and that $\mathcal{M}$ is the corresponding minefield. Also suppose that Z admits a *maximal proper decomposition*. By this we mean that $\mathcal{G}_{<Z}$ is a building set. Hence, the set $\{Z_1, \dots, Z_k\}$ of maximal elements in $\mathcal{G}_{<Z}$ is a decomposition of Z (and $k \geq 2$). (If $Z \notin \mathcal{G}$, then its $\mathcal{G}$-decomposition is a maximal proper decomposition.) Let $\mathcal{M}_i$ be the minefield for Z_i corresponding to $\mathcal{G}_{\leq [Z_i]}$ where $[Z_i]$ denotes the parallel class of Z_i. Let $a_i : \partial Z_i \to \partial Z_i$ and $a : \partial Z \to \partial Z$ be the antipodal maps. We let $\mathbb{P}(Z_i)_\# = (\partial Z_i)_\# / (a_i)_\#$ and $\mathbb{P}(Z)_\# = (\partial Z)_\# / a_\#$. (Note that $\mathbb{P}(Z)_\#$ can also be obtained by blowing up $\mathbb{P}(Z)$ along the image of the minefield $\mathcal{M}$.)

Lemma 5.1.1. *$\mathbb{P}(Z)_\#$ is an $\mathbb{RP}^{k-1}$-bundle over $\mathbb{P}(Z_1)_\# \times \cdots \times \mathbb{P}(Z_k)_\#$.*

Proof. $(Z_i)_\#$ is a $[-1,1]$-bundle over $\mathbb{P}(Z_i)_\#$ and $(a_i)_\#$ acts on $(Z_i)_\#$ by multiplication by -1 on each fiber. Hence, $(Z_1)_\# \times \cdots \times (Z_k)_\#$ is a $[-1,1]^k$-bundle over $\mathbb{P}(Z_1)_\# \times \cdots \times \mathbb{P}(Z_k)_\#$ and $a_\# = (a_1)_\# \times \cdots \times (a_k)_\#$. It follows that $\partial Z_\#$ is a $\partial([-1,1]^k)$-bundle (i.e., an S^{k-1}-bundle) over $\mathbb{P}(Z_1)_\# \times \cdots \times \mathbb{P}(Z_k)_\#$. Taking the quotient by $a_\#$, we obtain the result. $\square$

Corollary 5.1.2. *Suppose Z has a maximal proper decomposition with respect to $\mathcal{G}$: $Z = Z_1 \times \cdots \times Z_k$. Then $\pi_{k-1}(\mathbb{P}(Z)_\#)$ contains an infinite cyclic subgroup. In particular, if $k \geq 3$, then $\mathbb{P}(Z)_\#$ is not aspherical.*

Proof. In general, suppose $E \to B$ is a bundle with fiber F. If $E \to B$ admits a section, then in the homotopy sequence of the fibration, the map $\pi_i(E) \to \pi_i(B)$ is surjective and, hence, $\pi_{i-1}(F) \to \pi_{i-1}(E)$ is injective. In the case at hand, $\mathbb{P}(Z)_\#$ is an $\mathbb{RP}^{k-1}$ bundle over $B = \mathbb{P}(Z_1)_\# \times \cdots \times \mathbb{P}(Z_k)_\#$. Moreover, since $\mathbb{P}(Z)_\#$ is the projective space bundle associated to a sum of line bundles, there is a section $b \mapsto [L_b]$ where L is the line bundle determined by one of the summands, L_b is its fiber at b, and $[L_b]$ is the corresponding point in projective space. Hence, the infinite cyclic group $\pi_{k-1}(\mathbb{RP}^{k-1})$ injects into $\pi_{k-1}(\mathbb{P}(Z)_\#)$. $\square$

Corollary 5.1.3. *Let Z be a simple zonotope, $\mathbb{P}(Z)$ the associated zonotopal cell structure on projective space, and $\mathbb{P}(Z)_\#$ the minimal blow-up (with respect to the minefield corresponding to $\mathcal{I}_Z$). Then the following are equivalent:*
(i) The cubical metric on $\mathbb{P}(Z)_\#$ is nonpositively curved.
(ii) $\mathbb{P}(Z)_\#$ is aspherical.
(iii) Z has fewer than 3-irreducible factors.

Proof. The universal cover of any complete, nonpositively curved space is contractible so (i)⇒(ii). By the previous corollary, (ii)⇒(iii). By Theorem 4.3.1, (iii)⇒(i). □

5.2. Retractions onto faces. In this subsection $\mathcal{G}$ is a partial building set for Z, $\mathcal{M}$ is the corresponding partial minefield, and $G \in \mathcal{M}$ is a face of Z to be blown up. Let $\mathcal{M}_G$ be the restriction of the partial minefield to G (corresponding to $\mathcal{G}_{\leq [G]}$) and let $Z_\#$ and $G_\#$ be the blow-ups with respect to $\mathcal{M}$ and $\mathcal{M}_G$, respectively. Thus, $G_\#$ is a subcomplex of $Z_\#$. Likewise, if $\mathbb{P}(G)_\#$ and $\mathbb{P}(Z)_\#$ are the corresponding blow-ups of the associated projective spaces, then $\mathbb{P}(G)_\#$ is a subcomplex of $\mathbb{P}(Z)_\#$.

Theorem 5.2.1. *$G_\#$ is a retract of $Z_\#$, and $\mathbb{P}(G)_\#$ is a retract of $\mathbb{P}(Z)_\#$.*

What makes the proof difficult to write down is that the retraction is most naturally defined in terms of the dual complexes $X_\#$ and $(X_G)_\#$ (the dual of $G_\#$). We take notation from Section 3.4: $\mathcal{H}$ is the hyperplane arrangement in V, $\mathcal{C}$ is the dual arrangement. We identify $\mathcal{C}$ with $\mathcal{C}_Z$ and $\mathcal{G}$ with the corresponding subset of $\mathcal{C}$. Also, let $A \in \mathcal{C}$ be the subspace corresponding to G. Put $V_G = V/A^\perp$. Then $\mathcal{C}_{\leq A}$ is the dual of a hyperplane arrangement in V_G. In addition, we have the convex polytope X $(= X_\mathcal{H})$ in V and similarly X_G (cut out by the induced hyperplane arrangement) in V_G.

Proof of Theorem 5.2.1. The natural projection $V \to V_G$ induces a projection

$$\pi : V \times \prod_{B \in \mathcal{G}} \mathbb{P}(V/B^\perp) \to V_G \times \prod_{B \in \mathcal{G}_{\leq A}} \mathbb{P}(V/B^\perp).$$

Recalling that $X_\#$ is the closure of the image of $X \cap \widehat{V}$ in the domain, π restricts to a map $X_\# \to (V_G)_\#$. Now choose a disk around the origin in V_G of small enough radius to be contained in $\pi(X_\#)$ and scale X_G so that it is contained in the disk. Define retractions $s_G : V_G \to X_G$ and $(s_G)_\# : (V_G)_\# \to (X_G)_\#$ as in Lemma 3.4.1. The retraction $r : X_\# \to (X_G)_\#$ is defined to be the composition $(s_G)_\# \circ \pi|_{X_\#}$. Noting that $X_\#$ and $(X_G)_\#$ can be identified with the barycentric subdivisions of $Z_\#$ and $G_\#$, respectively, we see that r is indeed a retraction.

To see that $\mathbb{P}(G)_\#$ is a retract of $\mathbb{P}(Z)_\#$, we can suppose without loss of generality that $Z \in \mathcal{M}$. Then $Z_\#$ is the canonical $[-1,1]$-bundle over $\mathbb{P}(Z)_\#$ and $G_\#$ is the canonical $[-1,1]$-bundle over $\mathbb{P}(G)_\#$. The result then follows from the previous paragraph. □

Remark 5.2.2. The map π in the previous proof factors as $p \circ q$ where the range of q (= the domain of p) is the partial product

$$V \times \prod_{B \in \mathcal{G}_{\leq A}} \mathbb{P}(V/B^{\perp}),$$

and where p is the product of the projection $V \to V_G$ with the identity map. Moreover, $q(V_{\#}) = A^{\perp} \times (V_G)_{\#}$ and the inverse image of each point by the map $p : q(X_{\#}) \to p \circ q(X_{\#})$ is a convex subset of $A^{\perp}$. It follows that $q(X_{\#}) \to (X_G)_{\#}$ is a homotopy equivalence.

Corollary 5.2.3. *For any $G \in \mathcal{M}$, the inclusion $\mathbb{P}(G)_{\#} \to \mathbb{P}(Z)_{\#}$ induces a monomorphism of homotopy groups. In particular, if $\mathbb{P}(Z)_{\#}$ is aspherical, then so is $\mathbb{P}(G)_{\#}$.*

Combining this with the results of the previous subsection we get some necessary conditions for the asphericity of $\mathbb{P}(Z)_{\#}$.

Corollary 5.2.4. *If $\mathbb{P}(Z)_{\#}$ is aspherical, then (i) the minimal elements of $\mathcal{G}$ are at most 2-dimensional and (ii) no $G \in \mathcal{M}$ admits a maximal proper decomposition $G = G_1 \times \cdots \times G_k$ with $k \geq 3$.*

Proof. Suppose $A \in \mathcal{G}$ is minimal and G is a representative face for the corresponding parallel class. Then $G_{\#}$ is an interval bundle over $\mathbb{P}(G)_{\#} = \mathbb{P}(G)$ which is aspherical if and only if G has dimension 1 or 2. For (ii), let A be the subspace corresponding to G. Applying Corollary 5.1.2 to the building set $\mathcal{G}_{\leq A}$ for the induced hyperplane arrangement in $V/A^{\perp}$, we have that $G_{\#}$ is not aspherical if $k \geq 3$. □

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M. DAVIS, DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210

T. Januszkiewicz, Mathematics Department, Wrocław University, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.

R. Scott, Department of Mathematics, Santa Clara University, 500 El Camino Real, Santa Clara, California 95053.