GROUP ACTIONS ON EXOTIC STIEFEL MANIFOLDS

by

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1. INTRODUCTION

During the past five or six years, a considerable amount of interest in differential topology has been shown in the so-called "Brieskorn varieties." Associated with such a variety is a smooth closed manifold. Many of the algebraic and geometric invariants of these "Brieskorn manifolds" have been determined by Brieskorn, Milnor, Hirzebruch and others. The most interesting applications of these results have been in providing examples of odd dimensional homotopy spheres and examples of differentiable actions of certain compact Lie groups on such homotopy spheres. In this paper we obtain analogous results about manifolds homeomorphic to $V_{n+1,2}$, the Stiefel manifold of 2-frames in $\mathbb{R}^{n+1}$, alternately described either as $SO(n+1)/SO(n-1)$ or as the unit tangent bundle to S^n.

Throughout, unless otherwise specified, we will use "manifold" to mean "compact, orientable, smooth manifold" (with or without boundary). All group actions will be differentiable, and "diffeomorphism" (denoted by "$\cong$") will mean "orientation-preserving diffeomorphism." Also, we will adopt the standard notation, θ_n, for the group of homotopy n-spheres, and $\theta_n(\partial\pi)$ for the subgroup of homotopy spheres which bound parallelizable manifolds. (See Kervaire and Milnor [32]).

John Milnor's <u>Singular Points of Complex Hypersurfaces</u> [39] contains several results basic to this study. He considers the following situation: let $f\colon \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be a non-constant polynomial. In order to study the topology of the complex variety $f^{-1}(0)$, at a point z_0, one considers the (real) (2n-1)-dimensional manifold,

$$V = f^{-1}(0) \cap S_\varepsilon^{2n+1},$$

where S_ε^{2n+1} is a sufficiently small sphere centered at z_0. If z_0 is a regular point, then $f^{-1}(0)$ is a smooth manifold near z_0, and hence V is the standard sphere. We will be interested in cases where the origin is an isolated critical point and where we will be able to assume that the intersection is with the standard $S^{2n+1} \subset \mathbb{C}^{n+1}$. The Brieskorn manifolds are an interesting special case of the above situation:

(1.1) Definition: Let $f(z) = (z_1)^{a_1}+(z_2)^{a_2}+ \ldots + (z_{n+1})^{a_{n+1}}$, where $a_1,a_2,\ldots,a_{n+1}$ are positive integers. Then the variety $f^{-1}(0)$ is called a Brieskorn variety and

$$V^{2n-1}(a_1,a_2,\ldots,a_{n+1}) = f^{-1}(0) \cap S^{2n+1}$$

is a Brieskorn manifold.

Because of Proposition 2.4, in order to get interesting examples of manifolds homeomorphic to $V_{n+1,2}$, we must consider manifolds defined by a slightly larger class of polynomials.

(1.2) Definition (Milnor): The polynomial $f(z_1,\ldots,z_{n+1})$ is weighted homogeneous of type $(a_1,\ldots,a_{n+1})$ iff it can be written as a linear combination of monomials $z_1^{i_1}\ldots z_{n+1}^{i_{n+1}}$ for which

$$i_1/a_1 + i_2/a_2 + \ldots + i_{n+1}/a_{n+1} = 1$$

This is equivalent to the requirement that:

$$f(e^{c/a_1}z_1,\ldots, e^{c/a_{n+1}}z_{n+1}) = e^c f(z_1,\ldots z_{n+1})$$

for every $c \in \mathbb{C}$. In this case, we will use either the notation K^f_{2n-1} or V^f_{2n-1} to denote the manifold $f^{-1}(0) \cap S^{2n+1}$.

Before specializing to the cases of Brieskorn manifolds and manifolds defined by weighted homogeneous polynomials, we summarize the major results of Milnor's book [39] for the general situation:

(1.3) <u>Fibration Theorem</u>: If z_0 is any point of the complex hypersurface $f^{-1}(0)$ and if S_ε is a sufficiently small sphere centered at z_0, then the mapping

$$\phi(z) = f(z)/\|f(z)\|$$

from $S_\varepsilon - V$ to S^1, is the projection map of a smooth fibre bundle. The fibre

$$F_\theta = \phi^{-1}(e^{i\theta}) \subset S_\varepsilon - V$$

is a parallelizable 2n-dimensional manifold. (non-compact).

Implicit use of this theorem will be made throughout the the remainder of this paper. It plays a particularly important role in chapter 7.

(1.4) <u>Theorem</u>: If z_0 is an isolated critical point, then each fibre F_θ has the homotopy type of a wedge $S^n \vee \ldots\ldots \vee S^n$ of n-spheres. Furthermore, the closure

$$\bar{F}_\theta = F_\theta \cup V$$

is a smooth compact manifold with boundary.

In fact it follows from [47] that $\overline{F}_\theta$ is diffeomorphic to a handlebody.

For any fibre bundle over a circle, the natural action of a generator of $\pi_1(S^1)$ on the homology of the fibre is described by an automorphism

$$h_*: H_*(F_0) \longrightarrow H_*(F_0)$$

where h denotes the "characteristic homeomorphism" (See [39;pp. 67 and 76] for a definition of this). Let

$$\Delta(t) = \det(tI_* - h_*)$$

denote the characteristic polynomial of the linear transformation

$$h_*: H_n(F_0^{2n};\ \mathbb{C}) \longrightarrow H_n(F_0^{2n};\ \mathbb{C}).$$

Then,

(1.6) <u>Theorem</u>: For $n \neq 2$, V^{2n-1} is a topological sphere iff

$$\Delta(1) = \pm 1$$

In Lemma 2.2 and Theorem 3.1 we will prove the corresponding result for the case when V is homeomorphic to $V_{n+1,2}$, that is:

<u>Theorem</u>: For $n \neq 2$,

i) If n is even, V^{2n-1} is homeomorphic to $V_{n+1,2}$ iff $\Delta(1) = \pm 2$.

ii) If n is odd, V^{2n-1} is homeomorphic to either $V_{n+1,2}$ or $S^n \times S^{n-1}$ iff $\Delta(1) = 0$ and $(t-1)^2$ does not divide $\Delta(t)$.

Actually, as we shall see in chapter 3, a somewhat stronger result is possible.

When V is a Brieskorn manifold, Milnor notes that $\Delta(t)$ can be explicitly computed: [39; p.71]

(1.7) <u>Theorem</u> (Brieskorn - Pham): Let $V^{2n-1}(a_1,\dots,a_{n+1})$ be a Brieskorn manifold bounding the fibre $\overline{F}_\theta^{2n}(a_1,\dots,a_{n+1})$. Then $H_n(\overline{F}_\theta)$ is free abelian of rank

$$\mu = (a_1 - 1)(a_2 - 1)\cdot\dots\cdot(a_{n+1}-1)$$

The eigenvalues of

$$h_*: H_n(\overline{F}_0;\ \mathbb{C}) \longrightarrow H_n(\overline{F}_0;\ \mathbb{C})$$

are products $w_1\cdot\dots\cdot w_{n+1}$, where each w_j ranges over all a_j-th roots of unity other than 1. Hence

$$\Delta(t) = \prod (t - w_1 w_2 \cdots w_{n+1}).$$

The following convenient description, due to Brieskorn [6] and Milnor, enables us to decide when $V^{2n-1}(a_1,\dots,a_{n+1})$ is a homotopy sphere. Write $a = (a_1,\dots,a_{n+1})$, and define the graph $\Gamma(a)$: $\Gamma(a)$ has n+1 vertices $a_1,\dots,a_{n+1}$. Two of them (say a_i and a_j) are joined by an edge iff $\gcd(a_i, a_j) > 1$.

(1.8) <u>Lemma</u> ([21]): $\Delta(1) = \pm 1$ iff $\Gamma(a)$ satisfies:

i) $\Gamma(a)$ has at least two isolated points, or

ii) $\Gamma(a)$ has one isolated point and at least one connected component with an odd number of vertices such that $(a_i, a_j) = 2$ for any a_i and a_j $(i \neq j)$ in that component.

Using this lemma, it is immediately clear that the following Brieskorn manifolds are homotopy spheres:

1.) $V^{2n-1}(p,q,2,\ldots,2)$ where p and q are odd, and $(p,q)=1$,
2.) $V^{2n-1}(p,q,3,\ldots,3)$ where $(p,q) = (p,3) = (q,3) = 1$.
3.) $V^{2n-1}(d,2,2,\ldots,2)$ where d and n are odd.

On the other hand, according to Proposition 2.4, for even n, no Brieskorn manifold other than $V^{2n-1}(2,2,\ldots,2)$ is homeomorphic to $V_{n+1,2}$. As is shown in chapters 2 and 3, the weighted homogeneous polynomials do provide examples of manifolds homeomorphic to $V_{n+1,2}$. $\Delta(t)$ can be computed in this case using the following theorem of Milnor [39; p.77], which is even easier to use than Theorem 1.7:

(1.9) <u>Theorem</u> (Milnor): Let f be a weighted homogeneous polynomial of type $(a_1, a_2, \ldots, a_{n+1})$. The characteristic homeomorphism, $h: F^f \longrightarrow F^f$ is defined by

$$h(z_1,\ldots,z_{n+1}) = (e^{2\pi i/a_1} z_1, \ \ldots\ldots, \ e^{2\pi i/a_{n+1}} z_{n+1})$$

The euler numbers χ_d of the fixed point manifolds of the various iterates $h^d: F^f \longrightarrow F^f$ are related to the characteristic polynomial $\Delta(t)$ by the formula

$$\Delta(t) = (t-1)^{-1} \prod_{d|p} (t^d-1)^{r_d}$$

if n is even, or

$$\Delta(t) = (t-1) \prod_{d|p} (t^d-1)^{-r_d}$$

if n is odd; where p is the period of h, and where

$$\chi_j = \sum_{d|j} dr_d \quad .$$

Using this theorem to compute $\Delta(t)$, in chapter 2 we are able to provide examples of simply connected π-manifolds with the same homology as $V_{n+1,2}$. Then, in chapter 3, we use surgery to prove the major result of Part I, namely:

Theorem 3.1: Let K^{2n-1} be a stably parallelizable, simply connected, homology $V_{n+1,2}$ (n>3). If n is even, then

i) $K \equiv (\pm V_{n+1,2}) \# \Sigma$

While if n is odd, then

ii) Either $K \equiv (\pm V_{n+1,2}) \# \Sigma$ or

$$K \equiv (S^n \times S^{n-1}) \# \Sigma$$

Where Σ is a homotopy (2n-1) sphere. Moreover, if K bounds a parallelizable manifold F, then $\Sigma \in \theta_{2n-1}(\partial\pi)$.

One can view this theorem as a partial generalization of the well-known fact that simply connected homology spheres are homotopy spheres.

When n = 2k, the Brieskorn manifold $V^{4k-1}(a_1,\ldots,a_{n+1})$ bounds the 4k-dimensional manifold $\overline{F}^{4k}(a_1,\ldots,a_{n+1})$.

Hirzebruch has computed the signature of these manifolds [21; 314-06]:

(1.10) Theorem (Hirzebruch): If n is even, then the signature, $I(\overline{F})$, of $\overline{F}^{2n}(a_1,\dots,a_{n+1})$ is equal to

$$I(\overline{F}) = I^+(\overline{F}) - I^-(\overline{F}), \text{ where}$$

$I^+(\overline{F})$ = the number of (n+1)-tuples of integers $(x_1,\dots x_{n+1})$ $0<x_j<a_j$, with

$$0 < \sum_{j=1}^{n+1} x_j/a_j < 1 \pmod 2$$

and,

$I^-(\overline{F})$ = the number of (n+1)-tuples of integers $(x_1,\dots,x_{n+1})$ $0<x_j<a_j$, with

$$1 < \sum_{j=1}^{n+1} x_j/a_j < 2 \pmod 2$$

Note that if n is even and if $V^{2n-1}(a_1,\dots,a_{n+1})$ is a homotopy sphere, then using Theorem 1.10 we can completely determine the diffeomorphism type of V^{2n-1}. In fact:

(1.11) Theorem (Hirzebruch): Every element of $\theta_{2n-1}(\partial\pi)$ can be represented as a Brieskorn manifold. In particular, if n is even and if Σ is the generator of $\theta_{2n-1}(\partial\pi)$, then

$$V^{2n-1}(3,6k-1,2,\dots,2) \equiv (-1)^{n/2}\, k\Sigma$$

In trying to classify our examples of exotic Stiefel manifolds, the major stumbling block that we encounter is that for the case of manifolds defined by weighted homogeneous polynomials, we do not know a corresponding result to

Theorem 1.10. For example, let $K^{p,q}_{2n-1}$ be the manifold defined by the polynomial

$$z_1^p + z_1 z_2^q + z_3^2 + \ldots + z_{n+1}^2$$

where n is even, p and q are odd, and $(p-1,q) = 1$. It turns out that

$$K^{p,q}_{2n-1} \cong (\pm V_{n+1,2}) \# \Sigma.$$

If $K^{p,q}_{2n-1}$ bounds $\overline{F}^{p,q}_{2n}$, then the signature $I(\overline{F}^{p,q}_{2n})$ would determine Σ and the sign of $\pm V_{n+1,2}$. That is, we would be able to completely determine the diffeomorphism type of $K^{p,q}_{2n-1}$.

Originally, it was to answer this question about the signature of $\overline{F}^{p,q}_{2n}$, that we became interested in actions of the orthogonal group. $O(n-1)$ acts on the Brieskorn sphere $V^{2n-1}(p,q,2\ldots,2)$ by operating only on the last n-1 coordinates. The fixed point set is a torus knot of type (p,q). Hirzebruch and Erle [19], [21], [22] have studied such actions, and have shown that:

<u>Theorem</u> (Hirzebruch - Erle): If n is even, p, q are odd, and $(p,q) = 1$, then

$$I(\overline{F}^{2n}(p,q,2,\ldots,2)) = \pm \operatorname{sgn} t(p,q)$$

where $\operatorname{sgn} t(p,q)$ is the signature of a torus knot of type (p,q). (The signature of a knot is defined, for example, in Murasugi [43]).

In a similar fashion $K^{p,q}_{2n-1}$ also admits an $O(n-1)$ action with fixed point submanifold, a link of two components. Unfortunately, as Remark 5.7 shows, the signature of this link is, in general, not equal to the signature $I(\overline{F}^{p,q}_{2n})$.

Actually, this counterexample is incidental to the main arguments of Part II. Although we also consider actions of S^1 and of Z_2, the main sections of Part II essentially deal with the natural orthogonal actions on $K^{p,q}_{2n-1}$. Our work is centered about two questions:

1) K. Janich and the Hsiang brothers have shown that the O(n-1)-action on a Brieskorn sphere is completely determined by the knot type of the fixed point set. Do the classification results of Janich and the Hsiangs generalize when the fixed point set is not connected?

2) How "interesting" are our examples of actions of O(n-1) and its subgroups on $K^{p,q}_{2n-1}$?

We obtain a partial answer to question 1) in Theorems 5.8 and 5.9.

It is not clear exactly what we mean by question 2). When the local representations at fixed points are equivalent, we can construct the connected sum of two actions on the (equivariant) connected sum of two manifolds. Similarly, since O(n-1) operates on $V_{n+1,2}$ with fixed points, one can construct an action on the equivariant plumbing of various copies of the unit tangent bundle to S^n. What we mean by question 2) is then:

a) Is the O(n-1) action on $K^{p,q}_{2n-1}$ equivalent to the O(n-1) on the boundary of some equivariant plumbing?

b) Is the O(n-1) action on $K^{p,q}_{2n-1}$ equivalent to the connected sum of the natural action on $V_{n+1,2}$ with an action of O(n-1) on some $\Sigma \in \theta_{2n-1}(\partial\pi)$?

In Conjecture 5.4 we claim that, at least for $K^{3,3}_{2n-1}$, the answer to question a) is yes. It seems plausible that, in fact, we can always find a plumbing O(n-1) - diffeomorphic to $K^{p,q}_{2n-1}$.

However, as we show in Proposition 5.12, our examples are "interesting" in the sense of question b). If the answer to b) is yes, we call the action"decomposable." In chapter 7 we examine this question of when group actions on exotic Stiefel manifolds are decomposable, at least in the case where the ambient manifold is $K^{p,q}_{2n-1}$ and Σ is a Brieskorn sphere. Besides proving that the actions of O(n-1) and its subgroups $O(n-k) \subset O(n-1)$ are indecomposable in this case, we develop techniques for deciding the more general problem of when actions of the orthogonal group (with exactly three orbit types) can be split as the connected sum of two actions.

Chapter 4 is largely extraneous to the rest of the paper; it contains some entertaining examples of free actions of free actions of finite cyclic groups on homotopy spheres and on exotic $V_{n+1,2}$'s.

In chapter 6 we are concerned with S^1-actions operating as subgroups of the O(n-1) action, i.e., $S^1 \subset O(n-1)$.

Browder and Petrie [13] have obtained results about such actions on homotopy spheres; we make some trivial extensions of their results to $V_{n+1,2} \# \Sigma$.

There are two possible points of view to take in reading this paper. The first is to read it as a paper on Brieskorn manifolds and manifolds defined by weighted homogeneous polynomials. From this viewpoint, the paper obtains information about particular examples of these manifolds. We then must look at Part II as giving us information about something resembling a subgroup of the group of isometries of these particular manifolds.

The second point of view is to read this as a paper on transformation groups. From this outlook, the paper provides several interesting examples of group actions on $V_{n+1,2} \# \Sigma$. Reading the paper in this way, the justification for Part I is merely that it gives us convenient descriptions of certain exotic Stiefel manifolds.

PART I:

TOPOLOGICAL STIEFEL MANIFOLDS

2. EXAMPLES OF HOMOLOGY STIEFEL MANIFOLDS

In this chapter we will introduce several examples of manifolds which have the same homology as $V_{n+1,2}$. These examples will satisfy the additional hypotheses of being $(n-2)$-connected π-manifolds which bound parallelizable manifolds. Furthermore, the manifolds described in this chapter will be naturally equipped with smooth G-actions, for $G = O(n-1)$, S^1 or Z_2. In later chapters one of our primary goals will be to determine when the following examples are equivalent up to homeomorphism, diffeomorphism or equivariant diffeomorphism. The notion of homology $V_{n+1,2}$ is clarified by the following lemma:

(2.1) Lemma: If n is even,

$$H_i(V_{n+1,2}) = \begin{cases} \mathbb{Z} \text{ ; if } i = 0 \text{ or } 2n-1 \\ \mathbb{Z}_2 \text{ ; if } i = n-1 \\ 0 \text{ ; otherwise} \end{cases}$$

If n is odd,

$$H_i(V_{n+1,2}) = \begin{cases} \mathbb{Z} \text{ ; if } i = 0 \text{ or } 2n-1 \\ \mathbb{Z} \text{ ; if } i = n-1 \text{ or } n \\ 0 \text{ ; otherwise} \end{cases}$$

Proof: Consider the Gysin sequence of $V_{n+1,2}$, i.e., of the unit tangent sphere bundle to S^n:

$$H^{i-n}(S^n) \xrightarrow{\smile \chi} H^i(S^n) \longrightarrow H^i(V_{n+1,2}) \longrightarrow H^{i+1-n}(S^n) \longrightarrow \cdots$$

where χ is the Euler class of the tangent bundle. The only non-trivial part of the sequence is:

$$0 \longrightarrow H^0(S^n) \xrightarrow{\cup \chi} H^n(S^n) \longrightarrow H^n(V_{n+1,2}) \longrightarrow 0$$

From which it follows that

$$H^n(V_{n+1,2}) = \begin{cases} \mathbb{Z} \ ; \text{ if n is odd} \\ \mathbb{Z}_2 \ ; \text{ if n is even} \end{cases}$$

The lemma follows from Poincare duality and the Universal Coefficient Theorem.

Let $f: \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be a non-constant polynomial, and let $V^f_{2n-1} = f^{-1}(0) \cap S^{2n+1}$. The next lemma relates the homology of V^f to the characteristic polynomial $\Delta(t)$. (Compare Milnor [39;p.68]).

(2.2) <u>Lemma</u>: V^f_{2n-1} is an integral homology $V_{n+1,2}$ iff

i) $\Delta(1) = \pm 2$; if n is even

ii) $\Delta(1) = 0$, and $(t-1)^2$ does not divide $\Delta(t)$; if n is odd.

<u>Proof</u>: For n > 1, it follows immediately from:

$$\begin{array}{ccccccccc} 0 \longrightarrow & H_{n+1}(S - V^f) & \longrightarrow & H_n(F_0) & \xrightarrow{h_*-I_*} & H_n(F_0) & \longrightarrow & H_n(S - V^f) & \longrightarrow 0 \\ & \| \approx & & & & & & \| \approx & \\ & H^{n-1}(V^f) & & & & & & H^n(V^f) & \\ & \| \approx & & & & & & \| \approx & \\ & H_n(V^f) & & & & & & H_{n-1}(V^f) & \end{array}$$

Where the top row is the Wang sequence, and the vertical isomorphisms are respectively, the Alexander and Poincare duality isomorphisms.

i) Since $\Delta(1) = \pm 2$ iff $\operatorname{coker}(h_*-I_*) = \mathbb{Z}_2$ and $\ker(h_*- I_*) = 0$, the lemma is true if n is even. (Compare the proof of Lemma 2.7).

ii) $\Delta(1) = 0$ iff $\ker(h_*-I_*) \neq \emptyset$, while $(t-1)^2$ does not divide $\Delta(t)$ iff $\ker(h_*-I_*) \cong \mathbb{Z}$ iff $H_n(V^f) = \mathbb{Z}$. The lemma follows by the Universal Coefficient Theorem. The proof for $n = 1$ is similar.

Our first example is well-known:

<u>Example 1</u>: Consider the Brieskorn manifold $V^{2n-1}(2,2,\ldots,2)$ defined by the polynomial

$$f(z) = (z_1)^2 + (z_2)^2 + \ldots + (z_{n+1})^2$$

If $z_j = x_j + iy_j$, then the requirement that $f(z) = 0$ means

$$\sum_{j=1}^{n+1} (x_j^2 - y_j^2) = 0 = \sum_{j=1}^{n+1} 2x_j y_j$$

or in other words, the vectors $(x_1,\ldots x_{n+1})$ and $(y_1,\ldots,y_{n+1})$ are orthonormal. Thus

$$V^{2n-1}(2,2,\ldots,2) \equiv V_{n+1,2}$$

In fact, it is easy to see that the map

$$(z_1,z_2,\ldots,z_{n+1}) \xrightarrow{p} (x_1,x_2,\ldots,x_{n+1})$$

is just the projection map of the unit tangent bundle

$$V_{n+1,2} \xrightarrow{p} S^n$$

On the other hand, using Theorem 1.7 (Briekorn - Pham) a trivial computation shows

$$\Delta(t) \doteq t + (-1)^n$$

which agrees with Lemma 2.2.

The question naturally arises, which Brieskorn manifolds have the homology of $V_{n+1,2}$. Unfortunately, if n is even, we have the following negative result:

(2.3) Proposition: If n is even and if the characteristic polynomial $\Delta(t)$ of the Brieskorn manifold $V^{2n-1}(a_1, a_2 \ldots, a_{n+1})$ satisfies

$$\Delta(1) = \pm 2$$

then each $a_i = 2$, for all $1 \leq i \leq n+1$.

So that for $n \equiv 0$ (2) there are no Brieskorn examples of homology $V_{n+1,2}$'s other than Example 1.

Proof: Recall the formula

$$(*) \quad \Delta(t) = \prod (t - w_1 \cdot \ldots \cdot w_{n+1})$$

where w_j ranges over all non-trivial a_j-th roots of unity. According to Grothendieck, for Brieskorn manifolds $\Delta(t)$ is the product of cyclotomic polynomials,

$$\Delta(t) = \prod F_m(t)$$

where the roots of F_m are precisely the primitive m-th roots of unity. It is not hard to show that:

$$F_m(1) = \begin{cases} p \ ; & \text{if } m = p^\alpha, \text{ for a prime } p \\ 1 \ ; & \text{otherwise} \end{cases}$$

Since $\Delta(1) = \pm 2$, we must have that

$$\Delta(t) = F_{2\alpha}(t) \prod F_m(t)$$

where m is not a power of a prime.

<u>Assertion</u>: $\alpha = 1$

<u>Proof</u>: Let δ be some primitive 2^α-th root of unity. Since δ is a root of $\Delta(t)$, there must be a particular term in (*)

$$(t - w_1 \cdots w_k w_{k+1} \cdots w_{n+1})$$

such that (by reordering if necessary)

$$w_1 \cdots w_k = \delta \qquad \text{and} \qquad w_{k+1} \cdots w_{n+1} = 1.$$

This can only happen if for each $1 \leq i \leq k$, w_i is a primitive 2^{β_i}-th root of unity, for some $\beta_i \leq 1$. Thus the following term must also occur in (*)

$$(t - w_1^{2^{(\beta_1 - 1)}} \cdots\cdots w_k^{2^{(\beta_k - 1)}} \cdot w_{k+1} \cdots w_{n+1}) = (t - (-1)^k)$$

Since 1 cannot be a root of $\Delta(t)$, k cannot be even and hence -1 is a root. Therefore, $\alpha = 1$.

The following assertion completes the proof:

Assertion: Each w_j is equal to -1.

Proof: Again we consider the decomposition of a particular term in (*)

$$t - (w_1 \cdot \ldots \cdot w_k)(w_{k+1} \cdot \ldots w_{n+1})$$

By the previous assertion

$$w_i = -1 \quad \text{for } 1 \leq i \leq k, \text{ and}$$
$$w_i \neq -1 \quad \text{for } k+1 \leq i \leq n+1$$

Also, since -1 is a root of $\Delta(t)$, we can choose the term in (*) with: $w_{k+1} \cdot \ldots \cdot w_{n+1} = 1$. Suppose that $k \neq n+1$. If $k+1 \leq j \leq n+1$, then $w_j \neq w_j^{-1}$, so we have another distinct term in $\Delta(t)$:

$$t - (w_1 \ldots w_k)(w_{k+1}^{-1} \ldots w_{n+1}^{-1}) = t - (-1)^k \cdot 1$$
$$= (t+1)$$

This would imply that

$$\Delta(t) = (t+1)^2 \prod (\text{other terms})$$

But in that case

$$|\Delta(1)| \geqq 4 > 2$$

which is impossible. Hence $k = n+1$. This completes the proof of Proposition 2.3.

(2.4) Remark: The same proof shows that if n is even, then the only Brieskorn manifold (up to permutation of the coordinates) with $\Delta(1) = \pm$ p, a prime, is $V^{2n-1}(p,2,\ldots,2)$.

(2.5) Remark: However, if n is odd, then there are Brieskorn manifolds with $\Delta(1) = 0$ and with $(t-1)^2$ not dividing $\Delta(t)$. For example, the characteristic polynomial of $V^{2n-1}(d,2,\ldots 2)$ is

$$\begin{aligned}\Delta(t) &= \frac{(t^d + 1)}{(t + 1)} \\ &= t^{d-1} - t^d + \ldots \pm 1\end{aligned}$$

If d is even, then $\Delta(t)$ satisfies the hypothesis of Lemma 2.2. (See Example 5).

As the next two examples show, however, the class of manifolds defined by weighted homogeneous polynomials is much richer.

Example 2: Let $f\colon \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be a weighted homogeneous polynomial and let

$$K^f_{2n-1} = f^{-1}(0) \cap S^{2n+1}$$

bound $\overline{F}^f_{2n}$. Denote by

$$H_f = \left(\frac{\partial f}{\partial z_i \partial z_j}\right)$$

the (n+1) x (n+1) Hessian matrix of partial derivatives. If $H_f(0)$ is nonsingular, then K^f is a homology $V_{n+1,2}$ and $H_n(F^f) = Z$.

(2.6) Remark: In the next chapter, we will show that if n is even, then in fact F^f_{2n} is diffeomorphic to $T(S^n)$, the tangent disc bundle to S^n (and consequently $K^f \equiv V_{n+1,2}$).

(2.7) Remark: Note, for example, that if

$$f(z) = (z_1)^m + (z_1)(z_2) + (z_3)^2 + \dots + (z_{n+1})^2$$

then $H_f(0)$ is nonsingular. More will be said about this particular K^f_{2n-1} when we study actions of $O(n-1)$. (See Conjecture 5.7).

Proof of Example 2: We will use Theorem 1.9 (Milnor) to compute $\Delta(t)$. Let h^j: $F^f \longrightarrow F^f$ be the j-fold iteration of the characteristic homeomorphism, and let F^j be the fixed point submanifold of h^j. F^j is defined by the equations: (Compare [39;p78])

$$f(z) = 1, \quad z_{i_1} = z_{i_2} = \dots = z_{i_k} = 0$$

for some integer $k \leqq n+1$. Let μ_j = the rank of $H_{n-k}(F^j)$. It is clear that the matrix

$$H_f^j = \left(\frac{\partial f}{\partial z_i \partial z_j}\right) \quad ; \quad z_i, z_j \neq z_{i_q}$$

for $1 \leqq q \leq k$, represents a (n-k-1) x (n-k-1) block in the matrix H_f. That is, by renumbering the z_i's if necessary, we have that:

$$H_f = \begin{pmatrix} \boxed{H_f^j} & 0 \\ 0 & \boxed{\times} \end{pmatrix}$$

Since $H_f(0)$ is nonsingular so is $H_f^j(0)$. It follows from a lemma of Milnor [39; Lemma B.1] that $\mu_j = 1$, if $F^j \neq \emptyset$. Consequently $\chi_j = 0$ or 2 depending on whether (n-k) is odd or even. Using the formula

$$\chi_j = \sum_{d|j} d r_d$$

an easy argument shows that if n is even,

$$r_d = \begin{cases} 1 \ ; \text{ if } d = 2 \\ 0 \ ; \text{ otherwise} \end{cases}$$

and so,

$$\Delta(t) = (t-1)^{-1} \prod_{d|p} (t^d-1)^{r_d}$$

$$= (t+1)$$

If n is odd, it follows that $r_d = 0$ for all d and so

$$\Delta(t) = (t-1) \prod_{d|p} (t^d-1)^{-r_d}$$

$$= (t-1)$$

The example follows from Lemma 2.2.

According to Remark 2.6, Example 2 gives us no exotic $V_{n+1,2}$. In the next example, however, we have a large number of candidates.

Example 3: Let

$$f(z) = (z_1)^p + (z_1)(z_2)^q + (z_3)^2 + \ldots + (z_{n+1})^2$$

where p and q are odd and $(p-1,q) = 1$. We will use the notation

$$K^{p,q}_{2n-1} = K^{f}_{2n-1} = f^{-1}(0) \cap S^{2n+1} \text{ , and}$$

$$F^{p,q}_{2n} = F^{f}_{2n}.$$

We claim that $K^{p,q}_{2n-1}$ is a homology $V_{n+1,2}$, and that

$$\mu = \text{rank } H_n(F^{p,q}_{2n}) = p(q-1)+1.$$

This follows immediately from the fact that,

$$\Delta(t) = \frac{(t+1)(t^{pq}+1)}{(t^{p}+1)}$$

if n is even, so that $\Delta(t) = 2$. If n is odd, then

$$\Delta(t) = \frac{(t-1)(t^{pq}-1)}{(t^{p}-1)}$$

so that if n is odd $\Delta(t)$ also satisfies the hypothesis of Lemma 2.2.

<u>Proof</u>: Again, we can compute $\Delta(t)$ using Theorem 1.9. Using a result of Milnor [39; Lemma B.2], we can compute the Euler characteristics of the fixed point manifold of each h^j. In the case where n is even, the pertinent Euler numbers are

$$\chi_2 = 2$$
$$\chi_p = p$$
$$\chi_{2p} = 2 - p$$
$$\chi_{pq} = 1 - \mu = -p(q-1)$$
$$\chi_{2pq} = 1 + \mu = 2 + p(q-1)$$

Using the equation $\mathcal{X}_j = \Sigma\, dr_d$, the formula of Theorem 1.9 shows

$$\Delta(t) = \frac{(t^2-1)(t^p-1)(t^{2pq}-1)}{(t-1)(t^{2p}-1)(t^{pq}-1)}$$

$$= \frac{(t+1)(t^{pq}+1)}{(t^p+1)}$$

The argument is similar for the case where n is odd.

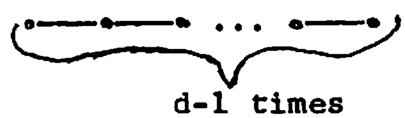
The last three examples are not defined by polynomials.

Example 4: If M^n is an n-manifold and Σ^n is a homotopy sphere ($n \neq 3$ or 4), then the connected sum $M \# \Sigma$ is PL-homeomorphic to M but not necessarily diffeomorphic if $n \geqq 7$. If $\Sigma \in \theta_{2n-1}(\partial\pi)$ then $V_{n+1,2}$ is a smooth π-manifold which bounds a parallelizable manifold. If n is odd, then it follows from [14] or [33] that $V_{n+1,2}$ is diffeomorphic to $V_{n+1,2} \# \Sigma$ iff $\Sigma \in \theta_{2n-1}(\partial\pi)$. On the other hand, suppose that n is even and that $\Sigma \in \theta_{2n-1}(\partial\pi)$. Since the Adams conjecture has been solved, it follows from a theorem of W. Browder [9; Theorem 2.13] that if $V_{n+1,2} \# \Sigma$ is diffeomorphic to $V_{n+1,2}$, then $\Sigma \cong S^{2n-1}$. Thus for even n ($n \neq 2$) and for distinct elements $\Sigma \in \theta_{2n-1}(\partial\pi)$, the manifolds $V_{n+1,2} \# \Sigma$ are in distinct diffeomorphism classes.

It is, of course, possible to construct examples of homology $V_{n+1,2}$'s in a variety of other ways. Some of the more interesting of these occur as the boundary of plumbings of an odd number of copies of $T(S^n)$.

Example 5: The well-known examples $W^{2n-1}(d)$ are defined in

[3], [6] or [21; 314-10] as the (2n-1)-dimensional O(n)-manifold corresppnding to an integer $d \geqq 0$. According to Hirzebruch [21; 314-11], $W^{2n-1}(d)$ can alternately be defined either as $V^{2n-1}(d,2,2,\ldots,2)$ or as the boundary of the equivariant plumbing of d-1 copies of $T(S^n)$, the tangent disc-bundle of S^n, along the graph:

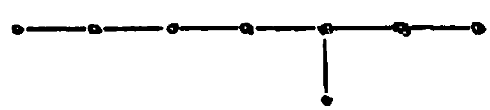


If n is even, then

$$H_{n-1}(W^{2n-1}(d)) \doteq \mathbb{Z}_d$$

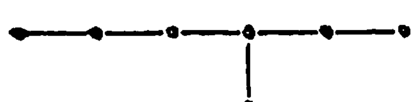
On the other hand, if n is odd, there are two cases. If d is odd, then $W^{2n-1}(d)$ is a homotopy sphere; while if d is even, it follows from Remark 2.5 that $W^{2n-1}(d)$ is a homology $V_{n+1,2}$.

We recall the famous example of Milnor, $M^{4k}(E_8)$, a 4k-dimensional manifold whose boundary $M_0(E_8)$ is the generator of $\theta_{4k-1}(\partial\pi)$. $M_0(E_8)$ is defined as the boundary of the plumbing of eight copies of $T(S^n)$ according to the graph:



Analogously, we have our last example.

<u>Example 6</u>: Let $M^{2n-1}(E_7)$ be the plumbing of seven copies of $T(S^n)$ by the graph



(Notice that this is the Dynkin diagram of the exceptional Lie group E_7. Also notice that if we plumb one more copy to the left hand side we get $M^{2n-1}(E_8)$). We claim that $M_0^{2n-1}(E_7) = \partial M^{2n-1}(E_7)$ is a homology $V_{n+1,2}$.

Proof: We prove the claim only for the case where n = 2k; the proof when n is odd being analogous. It follows from Browder's surgery notes [8; Chapter 5] that the intersection form

$$\Phi: H_{2k}(M(E_7)) \otimes H_{2k}(M(E_7) \longrightarrow \mathbb{Z}$$

is given by the matrix

$$\Phi = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

The same proof as Lemma (V2.8) in [8] shows that det $\Phi = 2$ and sgn $\Phi = 7$. The example now follows from this obvious generalization of a lemma of Milnor [39; Lemma 8.3].

(2.7) Lemma: Let F be a 2n-dimensional (n-1)-connected manifold with boundary F = K. $H_n(F)$ is free abelian (by Poincare duality). Then $H_{n-1}(K) = \mathbb{Z}_p$ iff the intersection pairing

$$\Phi: H_n(F) \otimes H_n(F) \longrightarrow \mathbb{Z}$$

has determinant $\pm p$, a prime.

<u>Proof of Lemma</u>: Consider the homology exact sequence:

$$H_n(\overline{F}) \xrightarrow{j_*} H_n(\overline{F},K) \longrightarrow H_{n-1}(K) \longrightarrow 0$$

It follows from Poincare duality that $H_n(\overline{F},K)$ is free abelian (assuming that $H_n(F)$ is free abelian) and that the intersection pairing

$$\Phi' : H_n(\overline{F},K) \otimes H_n(\overline{F}) \longrightarrow Z$$

has determinant ± 1. Using the identity

$$\Phi(\alpha,\beta) = \Phi'(j_*\alpha,\beta)$$

it follows that j_* has determinant $\pm$ p iff Φ has determinant $\pm$p.

Since we can diagonalize a square integer matrix by row and column operations without changing its determinant or its image, coker $j_* = Z_p = H_{n-1}(K)$; and the lemma follows.

This completes the proof of example 6 when n is even. When n is odd the intersection form is a 7x7 matrix with determinant 0 and rank 6. A similar argument proves this case.

3. WHEN ARE THESE EXAMPLES HOMEOMORPHIC TO $V_{n+1,2}$?

In the previous chapter we presented several examples of manifolds with the same homology as the Stiefel manifold $V_{n+1,2}$. So far, our geometric information about these manifolds is rather limited. We know that they are (n-2)-connected, are stably parallelizable, have the same homology as $V_{n+1,2}$, and bound parallelizable manifolds, but that is about all. Hopefully, we would like to be able to classify these manifolds up to homeomorphism and possibly up to diffeomorphism.

It turns out that the case of simply connected homology $V_{n+1,2}$'s is almost as nice as the case of simply connected homology spheres. We will show that when n is even and greater than 3, all of our examples are diffeomorphic to $V_{n+1,2} \# \Sigma$ where $\Sigma \in \theta_{2n-1}(\partial\pi)$. When n is odd, the worst that can happen is that one of our examples is diffeomorphic to

$$(S^n \rtimes S^{n-1}) \# \Sigma$$

where Σ is either the standard (2n-1)-sphere or the Kervaire sphere. We can state this as a theorem:

(3.1) <u>Theorem</u>: Let K be a simply connected, stably parallelizable, homology $V_{n+1,2}$ $(n > 3)$. If n is even, then

i) $K \equiv (\pm V_{n+1,2}) \# \Sigma$

While if n is odd, then

ii) Either $K \equiv V_{n+1,2} \# \Sigma$, or

$$K \equiv (S^n \rtimes S^{n-1}) \# \Sigma$$

Where Σ is a homotopy (2n-1)-sphere. Moreover, if K bounds a parallelizable manifold $\overline{F}$, then $\Sigma \in \theta_{2n-1}(\partial\pi)$.

Remark: K is (n-2)-connected by the Hurewicz isomorphism theorem. We may assume that F is (n-1)-connected. (See [32; Theorem 5.5]).

Let $E(\gamma)$ denote the total space of the closed n-disc bundle over S^n classified by $\gamma \in \pi_{n-1}(SO(n))$. Let $E_0(\gamma) = \partial E(\gamma)$ be the associated unit S^{n-1}-bundle over S^n. Let $\tau \in \pi_{n-1}(SO(n))$ classify the tangent disc bundle of S^n, so that

$$E_0(\tau) = V_{n+1,2}.$$

We will denote by

$$J: \pi_{n-1}(SO(n)) \longrightarrow \pi_{2n-1}(S^n) \quad \text{and}$$

$$H: \pi_{2n-1}(S^n) \longrightarrow \mathbb{Z}$$

the J-homomorphism and the Hopf invariant, respectively. As a special case of [49; Lemma 2], we have:

(3.2) Lemma (Wall): If $x \in H^n(E(\gamma))$ is a generator, then

$$x^2 = HJ(\gamma)$$

It follows immediately from Lemma 2.7 that

(3.3) Corollary: $E_0(\gamma)$ is a homology $V_{n+1,2}$ iff

$$HJ(\gamma) = \begin{cases} \pm 2 \text{ ; if n is even} \\ 0 \text{ ; if n is odd} \end{cases}$$

Let σ generate the stable part of $\pi_{n-1}(SO(n)$. According to Kosinski [33; pp. 241, 242]:

(3.4) <u>Lemma</u> (Kosinski): $E_0(\gamma)$ is a π-manifold iff

1) $\gamma = \tau$ or 0 , if n is odd

2) $\gamma = k\tau$, for some $k \in Z$, if $n \equiv 2 \ (4)$

3) $\gamma = k\tau + 2km\sigma$, for $k, m \in \mathbb{Z}$, if $n \equiv 0 \ (4)$.

Taking the intersection of the results of the last two lemmas, gives us:

(3.5) <u>Corollary</u>: $E_0(\gamma)$ is a stably parallelizable homology $V_{n+1,2}$ iff

$$\gamma = \begin{cases} 0 \text{ or } \tau; & \text{if } n \text{ is odd} \\ \pm\tau; & \text{if } n \equiv 2 \ (4) \\ \pm\tau + 2m\sigma; & \text{if } n \equiv 0 \ (4) \end{cases}$$

<u>Proof</u>: Noting that $\sigma \in \ker(HJ)$ and that

$$HJ(\tau) = \begin{cases} 0; & \text{if } n \text{ is odd} \\ 2; & \text{if } n \text{ is even} \end{cases}$$

the result follows.

As a final preliminary to the proof of Theorem 3.1, we need the following lemma:

(3.6) <u>Lemma</u>: There exists a framed surgery on K (where K satisfies the hypothesis of Theorem 3.1) so that the resulting manifold is a homotopy sphere, Σ. Moreover, if K bounds a parallelizable manifold F, then $\Sigma \in \theta_{2n-1}(\partial\pi)$.

Proof: The proof is based on several lemmas of Kervaire and Milnor in "Groups of Homotopy Spheres I" [32]. By framed surgery we mean that K is a framed manifold and that there exists a framing on the trace of the surgery W which restricts to the original framing on K.

When n is odd, since $H_n(K) \approx \mathbb{Z}$, it follows immediately from [32; Lemma 5.7] that such a surgery exists.

The case when n is even is slightly more complicated. Let $L(\lambda, \mu) \in \mathbb{Q}/\mathbb{Z}$ be the linking number of two homology classes. (See [32; p. 524] for a definition of this). If λ is the non-trivial element of $H_n(K) = \mathbb{Z}_2$ then it follows from Poincare duality for torsion groups that $L(\lambda,\lambda) \neq 0$. (In fact, $L(\lambda,\lambda)$ must equal $\frac{1}{2}$). From [32; Lemmas 6.3 and 6.4] it follows that a framed surgery can be chosen so that H_n of the new manifold is definitely smaller than $\mathbb{Z}_2$, hence it must be zero. That is, the new manifold is a homotopy sphere, Σ.

Suppose that K bounds the framed manifold F. Since the framings of W and of F agree on K, it follows that the manifold

$$W \bigcup_K F$$

can be framed. Since $W \bigcup_K F$ has a non-empty boundary, it is parallelizable. [32; Lemma 3.4] But

$$\Sigma = \partial(W \bigcup_K F)$$

so $\Sigma \in \theta_{2n-1}(\partial\pi)$. This completes the proof.

(3.7) Remark: Of course, the lemma also shows that K can be obtained as the result of a framed surgery on some element in θ_{2n-1}.

Proof of Theorem 3.1: Since K is obtained by one surgery on Σ^{2n-1} we may assume (subtracting and adding Σ) that we get it by a single surgery on S^{2n-1}. The result of the surgery is completely determined by the isotopy class of the embedding

$$\lambda: S^{n-1} \times D^n \subset S^{2n-1}$$

Applying the theorems of Haefliger, the isotopy class is unique on $S^{n-1} \times 0$, $(n>3)$, so by the tubular neighborhood theorem the isotopy class of λ is the class of a bundle map

$$\hat{\gamma}: S^{n-1} \times D^n \longrightarrow S^{n-1} \times D^n$$

It follows that $K \equiv E_0(\gamma)$, where the bundle map $\hat{\gamma}$ is the characteristic map for $\gamma \in \pi_{n-1}(SO(n))$. Corollary 3.5 completes the proof unless $n \equiv 0$ (4).

If $n \equiv 0$ (4), then we must have:

$$\gamma = \pm \tau + 2m\sigma$$

But according to Kosinski [33; 5.7.1], if n is even,

$$E_0(\pm \tau + 2m\sigma) \equiv E_0(\pm \tau) \# m^2 \Sigma(\sigma,\sigma)$$

where $\Sigma(\sigma,\sigma) \in \theta_{2n-1}$. (If $n > 8$ and $n \equiv 0$ (4), then $\Sigma(\sigma,\sigma) \notin \theta_{2n-1}(\partial\pi)$). Thus, if n is even

$$K \equiv (\pm V_{n+1,2}) \# \Sigma$$

while if n is odd, either

$$K \equiv V_{n+1,2} \# \Sigma \text{ , or}$$

$$K \equiv (S^n \times S^{n-1}) \# \Sigma$$

where $\Sigma \in \theta_{2n-1}(\partial\pi)$. An argument similar to the proof of the last part of Lemma 3.6 shows that if K bounds a parallelizable manifold F, then $\Sigma \in \theta_{2n-1}(\partial\pi)$. This completes the proof.

(3.8) Remark: If n is odd, our theorem is a special case of a result of De Sapio's [16; Proposition 1].

(3.9) Remark: As noted in chapter 2, example 4, if n is odd and $\Sigma \in \theta_{2n-1}(\partial\pi)$, then

$$V_{n+1,2} \# \Sigma \equiv V_{n+1,2}$$

On the other hand, [16]

$$(S^n \times S^{n-1}) \# \Sigma \not\equiv S^n \times S^{n-1}$$

unless $\Sigma \equiv S^{2n-1}$.

(3.10) Remark: Suppose that K^{4k-1} bounds a parallelizable manifold $\overline{F}^{4k}$ and that Σ^{4k-1} bounds the parallelizable manifold M^{4k}. Let $f: K^{4k-1} \longrightarrow \pm V_{2k+1,2} \# \Sigma$ be an orientation reversing diffeomorphism and note that

$$\overline{F}^{4k} \bigcup_f -(M^{4k} \# E(\pm\tau))$$

is a closed π-manifold, i.e., it is almost parallelizable. (Where "#" means connected sum along the boundary). According to Kervaire and Milnor in "Bernoulli Numbers, Homotopy Groups, and a Theorem of Rohlin" [31]:

Theorem (Kervaire - Milnor): The signature of any almost parallelizable 4k-manifold is a multiple of

$$2^{2k-1}(2^{2k-1}-1)B_k j_k a_k/k = 8\cdot(\text{order of } \theta_{4k-1}(\partial\pi))$$

Thus $I(\overline{F}^{4k}) \equiv I(M \# E(\pm\tau))$ (mod 8· order of $\theta_{4k-1}(\partial\pi)$). Since $I(M) \equiv 0$ (mod 8) and $I(E(\pm\tau)) = \pm 1$, we have, in particular, the following corollary of our theorem:

(3.11) Corollary: $I(\overline{F}^{4k}) \equiv \pm 1$ (mod 8).

Theorem 3.1 and the above corollary have the following implications for the examples of chapter 2:

1) Let K^f_{4k-1} be the manifold defined in example 2. (Recall that $f: \mathbb{C}^{2k+1} \longrightarrow \mathbb{C}$ is a weighted homogeneous polynomial such that $H_f(0)$ is nonsingular). Since the intersection matrix of $\overline{F}^f$ is just the integer 2, it follows that

$$(F^f_{4k}, K^f_{4k-1}) \equiv (E(\tau), V_{n+1,2})$$

2) Recall the manifolds $K^{p,q}_{4k-1}$ of example 3. Since

$$K^{p,q}_{4k-1} \equiv (\pm V_{n+1,2}) \# \Sigma$$

it follows that the diffeomorphism type of $K^{p,q}_{4k-1}$ is determined by $I(\overline{F}^{p,q}_{4k})$. Unfortunately, we are not able to compute this number. (Compare Remark 5.11).

3) If $M^{4k}(E_7)$ denotes the plumbing of 7 copies of the

tangent disc-bundle of S^{2k}, defined in example 6, then since $I(M^{4k}(E_7)) = 7$, it follows from Corollary 3.11, that:

$$(M^{4k}(E_7), M_0(E_7)) \equiv (M^{4k}(E_8) \amalg E(-\tau), M_0(E_8) \# -V_{2k+1,2})$$

where $M_0(E_8)$ generates $\theta_{4k-1}(\partial\pi)$. (Compare Hirzebruch [21] and Browder [8; chapter 5]). Also note that by comparing the intersection forms of the plumbings, one can see how to do the specific surgery on $M_0(E_7)$ to get $M_0(E_8)$.

(3.12) Remark: Consider the hypothesis in Theorem 3.1, that K^{2n-1} is stably parallelizable. Since the only obstructions to the triviality of the stable tangent bundle of K lie in

$$H^*(K; \pi_{*-1}(SO)),$$

it follows from Bott periodicity that if $n \equiv 6$ or 7 (8), then all these cohomology groups vanish, so that the hypothesis is unnecessary.

On the other hand, if K is not a π-manifold, it clearly cannot satisfy the conclusion of Theorem 3.1. Suppose that $n \equiv 0, 1, 2,$ or 4 (8). In these dimensions $\pi_{n-1}(SO) \neq 0$, so $\sigma \neq 0$. $E_0(\tau + \sigma)$ is a homology $V_{n+1,2}$ but by Lemma 3.4, it is not a π-manifold.

We shall close this chapter with three conjectures.

(3.13) Conjecture: In the examples of chapter 2, the case where

$$K \equiv (S^n \rtimes S^{n-1}) \# \Sigma$$

never occurs, i.e., if n is odd all of our examples of homology

$V_{n+1,2}$'s are actually diffeomorphic to $V_{n+1,2}$. (Compare Conjecture 5.4).

(3.14) <u>Conjecture</u>:

$$K^{3,4j-1}_{2n-1} \equiv (-V_{n+1,2}) \# V^{2n-1}(3,6j-1,2,2,\dots,2)$$

If n is even, then we know that the homotopy sphere V^{2n-1} bounds a manifold of index 8j. Hence, we are actually asserting in this conjecture that $I(\overline{F}^{3,4j-1}_{2n}) = 8j-1$. In chapter 5, we examine this conjecture in detail for the case when j = 1. (Compare Conjecture 5.3).

Suppose that n is even and that L^{2n-1} is a simply connected, stably parallelizable, homology $E_0(k\tau)$. If we could show that the middle homology $H_{n-1}(L) = Z_{2k}$ can be killed with a single framed surgery, then the proof of Theorem 3.1 would also show:

(3.15) <u>Conjecture</u>: $L \equiv E_0(\pm k\tau) \# \Sigma$, where $\Sigma \in \theta_{2n-1}$.

PART II:

GROUP ACTIONS

4. S^1-ACTIONS WITH FINITE ISOTROPY SUBGROUPS

In this chapter, we discuss some natural S^1-actions on manifolds which are defined by weighted homogeneous polynomials. We will also consider some related free actions of finite cyclic groups. First we shall examine the canonical free S^1-action on $V_{n+1,2}$.

Recall the description of $V_{n+1,2}$ as the Brieskorn manifold

$$V^{2n-1}\underbrace{(2,2,\ldots,2)}_{n+1 \text{ times}} \subset S^{2n+1}.$$

Let $S^1 = \{\alpha \in \mathbb{C};\ |\alpha| = 1\}$ and consider the representation of S^1 in $U(n+1)$ given by:

$$\alpha \longmapsto \begin{bmatrix} \alpha & & & \\ & \alpha & & O \\ & & \ddots & \\ O & & & \alpha \end{bmatrix}$$

This representation of S^1 defines the standard action

$$\lambda:\ S^1 \times S^{2n+1} \longrightarrow S^{2n+1}$$

Since

$$f(\lambda(\alpha, z_1,\ldots,z_{n+1})) = \alpha[(z_1)^2 + \ldots + (z_{n+1})^2]$$
$$= \alpha \cdot f(z_1,\ldots,z_{n+1}) ,$$

it follows that λ restricts to $V_{n+1,2} = f^{-1}(0) \cap S^{2n+1}$. It is clear from our remarks in chapter 2, example 1, that

$$\lambda:\ S^1 \times V_{n+1,2} \longrightarrow V_{n+1,2}$$

is just the natural action induced by the inclusion

$$SO(2) \subset SO(n+1)$$

acting on $SO(n+1)/SO(n-1)$. The orbit space is

$$\tilde{G}_{n+1,2} = \frac{SO(n+1)}{SO(2) \times SO(n-1)}$$

the Grassman manifold of oriented 2-planes in $\mathbb{R}^{n+1}$. The following commutative diagram of principal S^1-bundles

$$\begin{array}{ccc} S^1 & \overset{id}{\longleftrightarrow} & S^1 \\ \downarrow & & \downarrow \\ V_{n+1,2} & \subset & S^{2n+1} \\ \downarrow & & \downarrow \\ \tilde{G}_{n+1,2} & \subset & \mathbb{C}P^n \end{array}$$

proves:

(4.1) <u>Proposition</u>: $\tilde{G}_{n+1,2} \subset \mathbb{C}P^n$ is a projective variety defined by the homogeneous polynomial equation

$$[z_1]^2 + \cdots + [z_{n+1}]^2 = 0$$

This is a special case of a more general situation. If $f\colon \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ is any weighted homogeneous polynomial of type $(a_1, a_2, \ldots, a_{n+1})$, recall that

$$V^f_{2n-1} = f^{-1}(0) \cap S^{2n+1}$$

while H^f_{2n} is the fibre of

$$\begin{array}{c} S^{2n+1} - V^f \\ \downarrow \\ S^1 \end{array}$$

so that $\partial \overline{H}^f = V^f$. Let $b = \operatorname{lcm}(a_1, \ldots, a_{n+1})$. Consider the representation of S^1 given by

$$\alpha \longmapsto \begin{bmatrix} \alpha^{b/a_1} & & & \\ & \alpha^{b/a_2} & & \bigcirc \\ & & \ddots & \\ \bigcirc & & & \alpha^{b/a_{n+1}} \end{bmatrix}$$

This defines an action

$$\lambda_f : S^1 \times S^{2n+1} \longrightarrow S^{2n+1}$$

It follows immediately from the definition of weighted homogeneous polynomial that

$$f(\lambda_f(\alpha, z) = \alpha^b \cdot f(z)$$

Thus $S^{2n+1} - V^f \longrightarrow S^1$ is a S^1-bundle, in the sense of Atiyah [45], that is, the following diagram commutes.

$$\begin{array}{ccc} S^1 \times (S^{2n+1} - V^f) & \xrightarrow{\lambda_f} & S^{2n+1} - V^f \\ \Big\downarrow \text{id x proj} & & \Big\downarrow \text{proj} \\ S^1 \times S^1 & \xrightarrow{g} & S^1 \end{array}$$

Where $g(\alpha, \beta) = \alpha^b \beta$. In particular, λ_f restricts to V^f_{2n-1}.

Note that every isotropy subgroup of λ_f is contained in $\mathbb{Z}_b \subset S^1$, where $\mathbb{Z}_b$ denotes the subgroup of b-th roots of unity. λ_f resembles both the quasi-free actions of Browder and Petrie [13] and the pseudo-free actions of Montgomery and Yang [50]; yet, it does not fall into either of these categories.

λ_f and the standard inclusion $\mathbb{Z}_m \subset S^1$ induce an action of $\mathbb{Z}_m$ on V^f_{2n-1}. If $(m,b) = 1$, then it follows that the action is free. This simple observation has several interesting corollaries, among them:

(4.2) Proposition: If $(m,2pq) = 1$, then there exists a free action, $\mathbb{Z}_m \times K^{p,q}_{2n-1} \longrightarrow K^{p,q}_{2n-1}$.

Digressing for a moment, we have two other corollaries about homotopy spheres.

(4.3) Proposition: Let L^{2n-1} denote the lens space S^{2n-1}/Z_7. Let $\Sigma \in \theta_7$ be a homotopy sphere not divisible by 7. There exists a free action $\mathbb{Z}_7 \times \Sigma \longrightarrow \Sigma$ so that the orbit space $\Sigma/\mathbb{Z}_7$ is not PL-equivalent to L^7. Since Σ can be written as the Brieskorn manifold $V(p,q,2,2,2)$ for some p and q prime to 7, it follows that $\Sigma/\mathbb{Z}_7$ can be embedded in L^9.

(4.4) Proposition: If $\Sigma \in \theta_7$ is an odd multiple of the generator, then there exists a free involution T on Σ so that the orbit space Σ/T is not PL-equivalent to $\mathbb{R}P^7$. Moreover, since Σ can be written as a Brieskorn manifold $V(a_1,\ldots,a_5)$ where all the a_i's are odd, Σ/T can be embedded in $\mathbb{R}P^9$.

We will prove only Proposition 4.4, which essentially is a result of Hirzebruch [20]. The proof of Proposition 4.3 is virtually identical. Before proceeding with the proof, we note the following example of Hirzebruch:

Example: Consider the Brieskorn manifold

$$V^7(6j-1,18j-1,3,3,3).$$

Since the graph has two isolated points, it follows from

Lemma 1.8 that $V^7 \in \theta_7$. Using his formulae for index of Brieskorn varieties, Hirzebruch shows that $V^7(6j-1,18j-1,3,3,3)$ bounds a manifold of index $8\cdot 9j(4j-1)$. Computing residue classes of $9j(4j-1)$ mod 28, we see that we can write the following odd multiples of $M_0^7(E_8)$ in the above form: 1, 3, 7, 13, 15, 17, 21, 27. Clearly the multiples 5, 9, 11, 19, 23, 25, as well as the other even multiples, can be constructed from other polynomials with odd exponents.

Proof of Proposition 4: Since we can write Σ as a Brieskorn manifold $V^7(a_1, \ldots, a_5) \subset S^9$, where $a_1,\ldots,a_5$ are odd, the circle action $\lambda_{a_1,\ldots,a_5}$ induces a free involution T on $V^7(a_1,\ldots,a_5)$ given by

$$T(z_1,\ldots,z_5) = -1(z_1,\ldots,z_5)$$

which is just the antipodal map on S^9. Thus

$$\begin{array}{ccccc} \Sigma & = & V(a_1,\ldots,a_5) & \subset & S^9 \\ & & \downarrow & & \downarrow \\ \Sigma/T & = & V/T & \subset & \mathbb{R}P^9 \end{array}$$

Let Σ be an odd multiple of the generator $M_0^7(E_8) \in \theta_7$. Suppose that Σ/T were PL-homeomorphic to $\mathbb{R}P^7$. Since $\Gamma_n = 0$ for $n \geq 7$, the only obstruction to smoothing this equivalence lies in $[\mathbb{R}P^7, PL/O]$. The Puppe sequence of $S^6 \longrightarrow \mathbb{R}P^6$ gives

$$\begin{array}{ccccc} [\Sigma S^6, PL/O] & \longrightarrow & [\mathbb{R}P^7, PL/O] & \longrightarrow & [\mathbb{R}P^6, PL/O] \\ \parallel & & & & \parallel \\ \theta_7 & & & & 0 \end{array}$$

So θ_7 maps onto $[\mathbb{R}P^7, PL/O]$. It follows that

$$\Sigma/T \cong \mathbb{R}P^7 \# \Sigma'$$

for some $\Sigma' \in \theta_7$. Pulling back we must have that,

$$\Sigma \cong S^7 \# 2\Sigma'.$$

Since Σ is not divisible by 2, this is a contradiction. Thus Σ/T is not PL-homeomorphic to $\mathbb{R}P^7$. This completes the proof.

Remark: The PL-homeomorphism classes of homotopy $\mathbb{R}P^n$'s can be given a group structure [36]. It turns out that for homotopy $\mathbb{R}P^7$'s this group is $\mathbb{Z}_4 \oplus \mathbb{Z}$. It would be interesting to know which elements we can get from the natural involutions on Brieskorn 7-spheres. The first step in answering this would be to calculate the Browder-Livesay invariant of the involution T.

5. LINK MANIFOLDS

During the past few years smooth actions of the orthogonal group have been studied extensively. Quite strong results have been obtained by F. Hirzebruch, K. Janich, and W.C. and W.Y. Hsiang, as well as G. Bredon and D. Erle, especially in the case where the action has only two or three orbit types. The study of actions with three orbit types has led to the definition of "knot manifolds." Since our examples of exotic $V_{n+1,2}$'s admit O(n-1)-actions in natural ways, we would like to be able to obtain information about them by using these techniques. Of the various definitions of knot manifold, we shall mean the following:

Definition: A knot manifold (respectively, link manifold) is a manifold M^{2n-1} together with an action

$$O(n-1) \times M \longrightarrow M$$

satisfying the following three conditions:

i) Each isotropy subgroup is conjugate to either O(n-3), O(n-2) or O(n-1), and all of these occur.

ii) The representations of the isotropy subgroups O(n-3), O(n-2) and O(n-1) normal to the orbit are equivalent to respectively, the 4-dimensional trivial representation, the 3-dimensional trivial plus the standard representation of O(n-2), the 1-dimensional trivial plus the sum of two copies of the standard representation of O(n-1).

iii) The fixed point set is a circle.(respectively,2 circles).

Although this is the standard definition, it turns out that condition ii) is actually superfluous. According to W.C. and W.Y. Hsiang, it follows from their Differentiable Slice Theorem [24] that i) and iii) imply ii).

It is known [24; Theorem 4.2] that the orbit space, M', of the knot manifold (link manifold) is a smooth manifold, whose boundary, M', is the image of the singular orbits, and that the fixed point set L is a closed 1-dimensional submanifold of M'. We recall the Hsiangs' terminology of the "orbit triple" $(M', \partial M', L)$. A simple argument using condition ii) in the definition (Compare [24; Proposition 4.4]) shows that L is of codimension 3 in M', and hence, M' is a 4-manifold. Bredon shows that if the ambient manifold M^{2n-1} is (n-3)-connected, then M' is contractible. Since we will only be interested in the case where $M' \cong D^4$, we will use the term "knot manifold" (respectively, "link manifold") only when the orbit triple is actually (D^4, S^3, L). The main result on knot manifolds, due independently to Janich [27; p. 317] and W.C. and W.Y. Hsiang [24; Cor. 4.3] is the following:

(5.1) <u>Theorem</u> (Janich -Hsiangs): For $n \geqq 3$, to each differentiable knot class (S^3, F) there corresponds a unique (up to equivariant diffeomorphism) (2n-1)-dimensional knot manifold $M^{2n-1}(F)$.

Since part of the proof of this uses the fact that F is connected, the case of link manifolds is more complicated.

Another major result, due to Hirzebruch, is proved by

Dieter Erle in [19; p. 214].

(5.2) <u>Theorem</u> (Hirzebruch [21]): Let F be a knot, then $M^{2n-1}(F)$, $n \geqq 2$, is the boundary of a parallelizable manifold. For n odd, $M^{2n-1}(F)$ is homeomorphic to S^{2n-1} and thus represents an element of $\theta_{2n-1}(\partial\pi)$, it is the standard sphere if $c(F) = 0$, the Kervaire sphere if $c(F) = 1$. ($c(F)$ is the Robertello - Arf invariant of the knot F). If $n = 2m$, then $M^{4m-1}(F)$ is $(2m-2)$-connected and

$$H_{2m-1}(M^{4m-1}(F)) = H_1\ (M^3(F)).$$

For $m \geqq 2$, it is homeomorphic to S^{4m-1} iff $\det(F) = \pm 1$. Then $M^{4m-1}(F)$ represents an element of $\theta_{4m-1}(\partial\pi)$, which is

$$\pm \frac{I(F)}{8} \cdot g_m$$

where $I(F)$ is the signature of the knot and g_m generates $\theta_{4m-1}(\partial\pi)$. (In fact, if $\det(F) = \pm p$, then

$$H_{2m-1}(M^{4m-1}(F)) = Z_p\ ).$$

In order to classify the manifolds $K^{p,q}_{4m-1}$, we would like to know: What is the index of $F^{p,q}_{4m}$? Therefore, one would hope that a corresponding result relating the signature of the fixed point link (compare Murasugi [43]) to the signature of the manifold bounded by the link manifold would hold. Unfortunately, as Remark 5.7 shows, the manifolds $K^{p,q}_{4m-1}$ provide counterexamples to this conjecture.

Our primary motivation for studying link manifolds is to answer the following four questions:

A) The manifolds $K^{p,q}_{2n-1}$ and equivariant plumbings of copies of the unit tangent S^{n-1}-bundle of S^n both admit natural $O(n-1)$ actions. When are these two types of manifolds equivariantly diffeomorphic? In particular the conjecture arises:

(5.3) Conjecture: $K^{3,3}_{2n-1}$ is $O(n-1)$-diffeomorphic to the equivariant plumbing $M_0^{2n-1}(E_7)$. (More will be said about this later).

B) In the case where n is odd, we had trouble (see Theorem 3.1) distinguishing between $(S^n \rtimes S^{n-1}) \# \Sigma$ and $V_{n+1,2}$. Let V_1 and V_2 be link manifolds diffeomorphic respectively to $S^n \rtimes S^{n-1} \# \Sigma$ and $V_{n+1,2}$, and let ℓ_i be the linking number of the link corresponding to V_i $(i=1,2)$. The following conjecture would distinguish many of our examples from $(S^n \rtimes S^{n-1}) \# \Sigma$, if it were true. Hence, we could prove Conjecture 3.13 for the case of link manifolds, if we knew:

(5.4) Conjecture: $\ell_1 \equiv 0 \pmod 2$,
$\ell_2 \equiv 1 \pmod 2$.

Evidence for this conjecture is provided by Proposition 5.6. Conjecture 5.4 would imply, for example, that if

$$f(z) = (z_1)^4 + (z_1)(z_2)^2 + (z_3)^2 + \dots + (z_{n+1})^2$$

(n is odd) then the manifold K^f_{2n-1} is diffeomorphic to $(S^n \rtimes S^{n-1}) \# \Sigma$ rather than to $V_{n+1,2}$. Note that K^f_{2n-1} is a homology $V_{n+1,2}$ since $\Delta(t) = (t-1)(t^4+1)$ and that the fixed point link has linking number 2. (Compare Milnor [39; p.72]).

C) Let

$$f(z) = (z_1)^m + (z_1)(z_2) + (z_3)^2 + \cdots + (z_{n+1})^2$$

where $m \geq 2$. In chapter 3, we showed that $K^f_{2n-1} \equiv V_{n+1,2}$ (at least when n is even). Essentially for the same reasons as Conjecture 5.3, (compare Proposition 5.6, part 4) we have:

(5.5) Conjecture: K^f_{2n-1} is $O(n-1)$-diffeomorphic to $V_{n+1,2}$.

D) Finally, we mention the question of "decomposing" $O(n-1)$ actions. If n is even we know that $K^{p,q}_{2n-1} \equiv V_{n+1,2} \# \Sigma$. Since $V_{n+1,2}$ admits a natural $O(n-1)$-action and Σ can be thought of as a knot manifold, (compare [21]) the question arises: when can we choose an $O(n-1)$-action on Σ so that the equivariant connected sum $V_{n+1,2} \# \Sigma$ is $O(n-1)$-diffeomorphic to $K^{p,q}_{2n-1}$? A simple argument in Proposition 5.12 shows that the answer to this question is never.

In some of the above remarks we have assumed the following proposition:

(5.6) Proposition: The following are $(2n-1)$- dimensional link manifolds:

1) $V_{n+1,2} = V^{2n-1}(2,2,\ldots,2)$; where the link consists of two unknotted circles linked once.

2) $S^n \times S^{n-1}$; where the link consists of two unlinked circles.

3) $W^{2n-1}(d) = V^{2n-1}(d,2,\ldots,2)$, $d \equiv 0\ (2)$; where the link consists of two unknotted circles linked d/2 times.

4) K^f_{2n-1}, where

$$f(z) = (z_1)^m + (z_1)(z_2) + (z_3)^2 + \cdots + (z_{n+1})^2 ;$$

where the link consists of two unknotted circles linked once.

5) K^f_{2n-1}, where

$$f(z) = (z_1)^4 + (z_1)(z_2)^2 + (z_3)^2 + \ldots + (z_{n+1})^2;$$

where the link consists of a torus knot of type (2,3), t(2,3), linked with a circle with linking number 2.

6) $(S^n \times S^{n-1}) \# M^{2n-1}(F)$, where the knot manifold $M^{2n-1}(F)$ is a homotopy sphere; and where the link consists of the knot F and an unlinked circle.

7) $V_{n+1,2} \# M^{2n-1}(F)$; where the link consists of the knot F linked with a circle with linking number 1.

8) $K^{p,q}_{2n-1}$; where the link is a torus link consisting of the torus knot t(p-1,q) linked with a circle with linking number q.

9) $M^{2n-1}_0(E_7)$; where the link consists of a torus knot t(2,3) linked with a circle with linking number 3.

In particular the orbit triple in each case is (D^4, S^3, L), where L is the specified link.

Proof: We shall only describe the O(n-1)-action in examples 6), 8) and 9); the other cases being similar.

Example 6): The O(n-1)-action on $S^n \times S^{n-1}$ is the obvious one, that is, O(n-1) operates on S^n and on S^{n-1}

by the inclusions $O(n-1) \subset O(n+1)$ and $O(n-1) \subset O(n)$, and hence on the cross product via the representation

$$A \longrightarrow \begin{bmatrix} 1 & & & & & \\ & 1 & & & & 0 \\ & & A & & & \\ & & & 1 & & \\ & 0 & & & A & \end{bmatrix}$$

The fixed point set of the action on S^n is a great circle, while on S^{n-1} it consists of two points; so that the fixed point link in $S^n \times S^{n-1}$ consists of two unlinked circles. Thus condition iii) of the definition of link manifold is satisfied.

If $(x,y) \in S^n \times S^{n-1}$, denote by $O(n-1)_{(x,y)}$ the isotropy subgroup at (x,y). Let $[O(n-k)]$ denote the conjugacy class of $O(n-k) \subset O(n-1)$. We regard S^{n-1} as the equatorial sphere in S^n; and we consider the isotropy subgroups at (x,y) in four cases:

a) If x is a fixed point and $y = \pm x$, then $O(n-1)_{(x,y)}$ must be the whole group $O(n-1)$.

b) If x is a fixed point and y is linearly independent of x, then $O(n-1)_{(x,y)} \in [O(n-2)]$.

c) If x is not a fixed point and $y = \pm x$, then $O(n-1)_{(x,y)} \in [O(n-2)]$.

d) If x is not a fixed point and y is independent of x, then $O(n-1)_{(x,y)} \in [O(n-3)]$.

So, the action satisfies condition i) of the definition.

Thus $S^n \times S^{n-1}$ is a link manifold. Similarly, it is clear that that the representations of the isotropy subgroups (condition ii)) are as they should be.

Now consider the knot manifold $M^{2n-1}(F)$. Since the local representations at fixed points are equivalent, the connected sum can be performed equivariantly. (See W.Y. Hsiang [25; p. 92]). Since

$$F \# (\text{trivial knot}) = F,$$

the link consists of F and an unknotted circle. The two components are unlinked.

Example 8): Recall that $K^{p,q}_{2n-1} = f^{-1}(0) \cap S^{2n+1}$, where

$$f(z) = (z_1)^p + (z_1)(z_2)^q + (z_3)^2 + \dots + (z_{n+1})^2.$$

O(n-1), being the intersection of the complex orthogonal group O(n-1,C) with U(n-1), is precisely the subgroup of GL(n+1,C) which leaves the forms,

$$(z_3)^2 + \cdots + (z_{n+1})^2 \quad \text{and}$$
$$z_3\bar{z}_3 + \cdots + z_{n+1}\bar{z}_{n+1}$$

invariant. Therefore, there exists a natural O(n-1)-action on $K^{p,q}_{2n-1}$ (also on $\bar{F}^{p,q}_{2n}$).

The fixed point set is equal to $\{z;\ z_3 = z_4 = \dots = z_{n+1} = 0\}$ Similarly, $\{z;\ z_1 \neq 0\}$ is the set of points with isotropy subgroups conjugate to O(n-3); while the other points all have isotropy subgroups conjugate to O(n-2).

If $f_1(z_1,z_2) = (z_1)^p + (z_1)(z_2)^q$, then the fixed point link $K_1^{p,q}$ is equal to $f^{-1}(0) \cap S^3 \subset S^{2n+1}$. $f^{-1}(0)$ consists of two non-singular branches corresponding to the plane $z_1 = 0$, and to the surface $(z_1)^{p-1} + (z_2)^q = 0$. According to Brauner, the intersection of S^3 with the surface is the torus knot $t(p-1,q)$; while the intersection with the plane $z_1 = 0$, clearly is an unknotted circle.

The characteristic polynomial of $K_1^{p,q}$ is

$$\triangle(t) = \frac{(t-1)(t^{pq}-1)}{(t^p-1)}$$

According to Milnor [39; Lemma 10.1],

$$\pm t^i \triangle(t) = (t-1)\cdot \triangle(t_1,t_2)$$

where $\triangle(t_1,t_2)$ is the Alexander polynomial of the link. Since [44; p. 95], $\triangle(1,1)$ = the linking number, it follows that the two components are linked with linking number q. Example 8 follows.

<u>Example 9)</u>: $O(n-1)$ acts on $M_0^{2n-1}(E_7)$ as follows: $O(n-1)$ operates on S^n and hence on the unit tangent bundle $V_{n+1,2}$. The action on S^n leaves a great circle fixed; so that the action on $V_{n+1,2}$ makes it into a link manifold with fixed point set consisting of two circles linked once. By putting the center of the plumbing operation on the great circle in S^n, the plumbing can be performed equivariantly. The fixed point link is pictured in diagram 1. Later we shall show that this link is just $t(2,3)$ linked three times with a circle. Since $V_{n+1,2}$ is a link manifold, so is the boundary of the plumbing $M_0^{2n-1}(E_7)$.

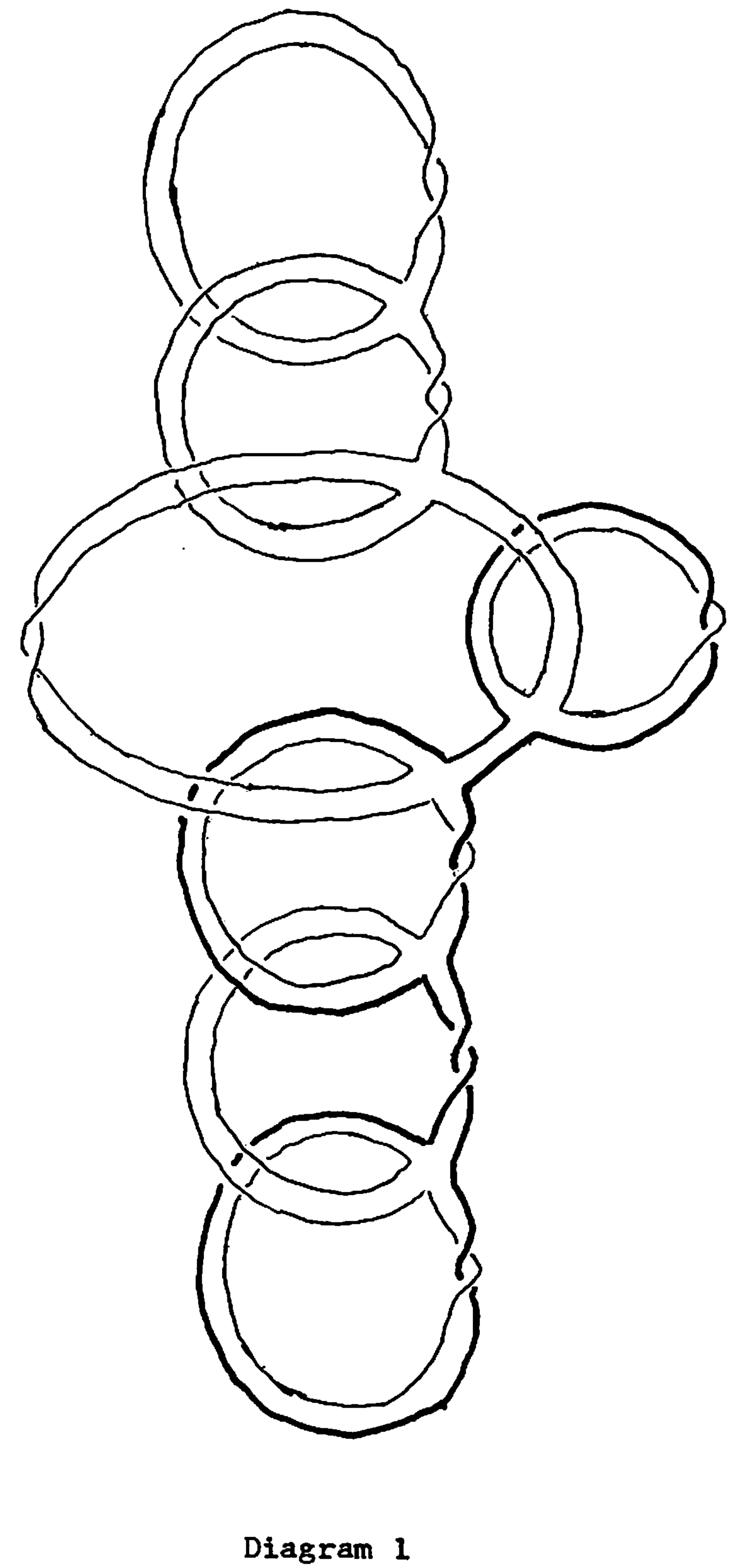


Diagram 1

To complete the proof of Proposition 5.6 it remains to be shown that the orbit space is indeed D^4. But this follows from the work of D. Erle [19; proof of Satz 14.4] essentially because an equivariant surgery on the link manifold does not alter the orbit space and the fact that the orbit space of the diagonal action $O(n-1) \times S^{2n-1} \longrightarrow S^{2n-1}$ is D^4.

(5.7) Remark: Consider the link manifold $K^{3,q}_{2n-1}$. The fixed point link is the torus knot t(2,q) linked q times with a circle. Since this is an alternating link, it is easy to compute its signature using the techniques of Murasugi [43]. It turns out that

$$\text{sgn (link)} = \text{sgn}(t(2,q) + \text{sgn}(2 \text{ circles linked } q \text{ times})$$

It is easy to see (from the formulae of Hirzebruch, for example) that $\text{sgn}(t(2,q)) = q$. It is also clear that

$$\text{sgn (2 circles linked } q \text{ times}) = 2q-1$$

Thus, we find that the signature of the fixed point link is equal to $\pm(3q-1)$. According to Corollary 3.11

$$I(\overline{F}^{p,q}_{4k}) \equiv \pm 1 \ (8).$$

Since in general $3q-1 \not\equiv \pm 1 \ (8)$, Hirzebruch's result (Theorem 5.2) does not generalize to link manifolds. This counter-example is somewhat surprising - it is not clear why something should go wrong in passing from homotopy spheres which are knot manifolds to exotic $V_{n+1,2}$'s which are link manifolds. One might conjecture that the signature of a link with more than one component has been defined incorrectly.

One would hope that the remainder of Theorem 5.2 would generalize in one form or another to link manifolds; however, our information is very incomplete in this area.

We would also like to be able to extend the classification theorem for knot manifolds (Theorem 5.1) to link manifolds. In one direction this is a trivial consequence of Janich's classification theorems [28]:

(5.8) <u>Theorem</u>: Let $M^{2n-1}(L_1)$ and $M^{2n-1}(L_2)$ be equivariantly diffeomorphic link manifolds, then the links (S^3,L_1) and (S^3,L_2) are equivalent.

We know of no counterexample to the converse of this theorem; but since the Hsiangs' proof of Theorem 5.1 relies, in an essential way, on the connectedness of the fixed point set, we can prove only the following partial converse:

(5.9) <u>Theorem</u>: For $n \geqq 3$, for any smooth link class (S^3,L), then (up to equivariant diffeomorphism) there exists at least one and at most two link manifolds $M_1^{2n-1}(L)$ and $M_2^{2n-1}(L)$ with (D^4,S^3,L) as their orbit triple. (Of course, we are assuming that the link has exactly two components).

<u>Outline of Proof</u>: Our proof is based on the Hsiangs' proof of Theorem 4.3 in [24]. The machinery we must develop to prove it in detail is much too complicted to profitably be included. The Hsiangs show that if the fixed point manifold

F is connected and if (M', M',F) is a triple of manifolds satisfying suitable hypotheses, then there exists a unique SO(n-1)-manifold, $n > 6$, (with exactly 3 orbit types) whose orbit triple is (M', M',F). Examining their proof, it is not hard to see that if we change the hypothesis to be that the fixed point manifold has n components, then essentially the same proof shows that there are at most n different SO(n-1)- manifolds with a given orbit triple.

Since Janich [28; Prop. 6] has shown that

<u>Proposition</u> (Janich): If an SO(n-1) action has only 3 orbit types, then it can be extended uniquely to an O(n-1)-action.

we see that the Hsiangs' use of SO(n-1) rather than O(n-1) is immaterial.

In the case of link manifolds the orbit triple is (D^4, S^3, L). It follows either from Lemma 7.1 or the fact, due to Cerf, that $\Gamma_4 = 0$, that the diffeomorphism class of (D^4, S^3, L) is determined by the diffeomorphism class of (S^3, L). Theorem 5.9 follows.

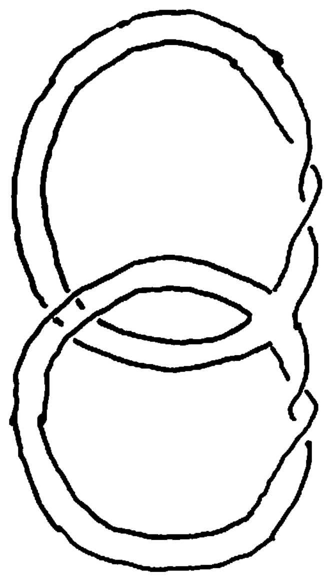
(5.10) <u>Remark</u>: Recall Conjecture 5.3. Note that

$$\text{rank } H_n(F^{3,3}_{2n}) = 7 = \text{rank } H_n(M^{2n}(E_7))$$

where $K^{3,3}_{2n-1}$ bounds $\overline{F}^{3,3}_{2n}$. If we also knew that $I(\overline{F}^{3,3}_{4m}) = \pm 7$, then, at least in dimensions 4m-1, it would be true that

$$K^{3,3}_{4m-1} \cong M_0^{4m-1}(E_7) \cong M_0^{4m-1}(E_8) \# (-V_{n+1,2})$$

By Proposition 5.6, the fixed point link in $K^{3,3}_{2n-1}$ is t(2,3) linked with a circle with linking number three. We claim that the fixed point link of $M_0(E_7)$, pictured in diagram 1, is the same link. Examination shows that it is the following knot linked with a circle three times.



But this is the fixed point knot of the plumbing,

that is, of $W^{2n-1}(3) = V^{2n-1}(3,2,2,\ldots,2)$. Hence, it is $t(2,3)$. Thus, for $n \geqq 2$, either $K^{3,3}_{2n-1}$ is equivariantly diffeomorphic to $M_0(E_7)$ or else these two manifolds are precisely the two possible link manifolds mentioned in Theorem 5.9.

It is interesting to note that [39; p. 83], $K^{3,3}_3$ can alternately be described as either S^3/G where G is the binary octahedral group with 48 elements ($G \subset U(2)$), or as the double branched covering of S^3 along the link. A similar result for $M^3_0(E_7)$ would be useful in deciding Conjecture 5.3 and in describing $O(n-2)$-actions on $K^{3,3}_{2n-1}$.

We conclude this chapter with some remarks about decomposable actions.

(5.11) Definition: An action

$$\lambda: G \times (V_{n+1,2} \# \Sigma) \longrightarrow (V_{n+1,2} \# \Sigma),$$

where G is a compact Lie group, is a decomposable G-action of type k iff the following two conditions hold:

1) The fixed point set of λ is diffeomorphic to $(V_{n-k+1,2} \# \Sigma')$, where $\Sigma' \in \theta_{2(n-k)+1}(\partial\pi)$.

2) There exists a canonical action

$$\phi: G \times V_{n+1,2} \longrightarrow V_{n+1,2}$$

with fixed point set $V_{n-k+1,2}$ and some other action

$$\lambda': G \times \Sigma \longrightarrow \Sigma$$

with fixed point set Σ' so that the action $\phi \# \lambda'$ can be defined on the equivariant connected sum, and so that λ is equivalent to $\phi \# \lambda'$.

(5.12) Proposition: The $O(n-1)$-action on $K^{p,q}_{2n-1}$ is indecomposable.

Proof: For the action to be decomposable means that $K^{p,q}_{2n-1}$ is equivalent to a link manifold $V_{n+1,2} \# M^{2n-1}(F)$, where the knot manifold $M^{2n-1}(F)$ is a homotopy sphere. Since the fixed point link of $K^{p,q}_{2n-1}$ has linking number $q > 1$, while the fixed point link of $V_{n+1,2}$ # $M^{2n-1}(F)$ has linking number 1, it follows from Theorem 5.8 that the two manifolds are not equivariantly diffeomorphic.

6. SEMIFREE S^1-ACTIONS

If $k \leq \left[\frac{n-1}{2}\right]$, let p_k be the representation of S^1 in $O(n-1)$ which takes the (2x2)-matrix $g \in SO(2)$ to a matrix of the form:

$$p_k(g) = \begin{bmatrix} I & & & & \\ & g & & & \\ & & g & & \\ & & & \ddots & \\ & & & & g \end{bmatrix}$$

where I is the $(n-2k-1)x(n-2k-1)$ identity matrix and where there are k copies of g. If $O(n-1)$ operates on a link manifold M^{2n-1}, as in chapter 5, then S^1 operates as a subgroup via the representation p_k. The action of S^1 on M^{2n-1} is called <u>semifree</u> since there are exactly two isotropy subgroups, 0 and S^1. We shall restrict our attention to the case where $n = 2m$.

Browder and Petrie [13] have results about such S^1-actions on homotopy spheres. They consider the Brieskorn sphere

$$V^{p,q}_{4m+1} = V^{4m-1}\underbrace{(p,q,2,\ldots,2)}_{2m+1 \text{ terms}}$$

where $(p,q) = 1$, and the corresponding manifold which it bounds,

$$W^{p,q}_{4m} = \overline{F}^{4m}(p,q,2,\ldots,2).$$

The representation p_k defines a semifree S^1-action on $V^{p,q}_{4m-1}$. The fixed point manifold $V^{p,q}_{4(m-k)-1}$ is a homotopy sphere, for $k < m-1$; while for $k = m-1$, $V^{p,q}_3$ is a homology S^3. Their main result is:

(6.1) <u>Theorem</u> (Browder - Petrie): If $k \neq m/2$ and if $k \leq m-1$ and if the semifree S^1-actions defined by p_k on $V^{p,q}_{4m-1}$ and on $V^{p',q'}_{4m-1}$ are equivalent, then

$$I(W^{p,q}_{4m}) = I(W^{p',q'}_{4m-1})$$

<u>Remark</u>: Of course, for $V^{p,q}_{4m-1}$ and $V^{p',q'}_{4m-1}$ to be diffeomorphic, $I(W^{p,q}_{4m})$ must equal $I(W^{p',q'}_{4m})$ mod the order of $\theta_{4m-1}(\partial\pi)$.

<u>Remark</u>: This theorem is probably not the strongest possible result. For example, if $k = m-1$, the fixed point manifolds $V^{p,q}_3$ and $V^{p',q'}_3$ are homology 3-spheres obtained as the double branched coverings of S^3 along $t(p,q)$ and $t(p',q')$, respectively. Since these manifolds are not even homotopy equivalent for distinct pairs (p,q) and (p',q'), we see that in this case the actions are equivalent iff $(p,q) = (p',q')$.

It turns out that Browder's and Petrie's proof of Theorem 6.1 also proves:

(6.2) <u>Theorem</u>: If $k \neq m/2$, $k \leq m-1$, and if the semifree S^1-actions defined by p_k on $K^{p,q}_{4m-1}$ and on $K^{p',q'}_{4m-1}$ are equivalent, then

$$I(\overline{F}^{p,q}_{4m}) = I(\overline{F}^{p',q'}_{4m})$$

<u>Outline of the proof of Theorems 6.1 and 6.2</u>: See [13; pp.138,139] for the details.

Suppose that X is a $(4m-1)$-dimensional manifold which admits an orientation preserving Z_2-action. Let g be the generator of Z_2 and let X^g denote the fixed point set. Suppose further that:

i) X^g is a rational homology $S^{4(m-k)-1}$, and that

ii) There is a 4m-dimensional $\mathbb{Z}_2$-manifold Y, with $Y = X$ (as $\mathbb{Z}_2$-manifolds).

Browder and Petrie recall the invariants L(g,Y) and Sgn(g,Y) first defined by Atiyah and Singer in [2]. L(g,Y) is defined using Hirzebruch's power series of rational Pontrijagin classes, $\mathcal{L}(p_1,\ldots,p_m)$. Sgn(g,Y) is the g-signature. It turns out that

$$I(X,Z_2) = L(g,Y) - Sgn(g,Y)$$

is a well-defined invariant of the $\mathbb{Z}_2$-action.

The standard inclusion, $\mathbb{Z}_2 \subset S^1$, defines $\mathbb{Z}_2$-actions on the manifolds $K^{p,q}_{4m-1}$ and $V^{p,q}_{4m-1}$. Since the homotopy Stiefel manifold $K^{p,q}_{4(m-k)-1}$ is a rational homology sphere, both $K^{p,q}_{4m-1}$ and $V^{p,q}_{4m-1}$ satisfy conditions i) and ii).

It is easy to show that if the normal bundle to the fixed point set $Y^g \subset Y$ is trivial and if the $\mathbb{Z}_2$-action extends to a circle action on X, then

$$I(X,\mathbb{Z}_2) = -Sgn(1,Y) = I(Y).$$

Since $W^{p,q}_{2n}$ (respectively, $\bar{F}^{p,q}_{2n}$) is equal to $f^{-1}_{n+1}(0)$ under the map

$$f_{n+1} = z_{n+1}\colon W^{p,q}_{2n+2} \longrightarrow \mathbb{C},$$

$W^{p,q}_{2n}$ has trivial normal 2-plane bundle in $W^{p,q}_{2n+2}$, and so the normal bundle of $W^{p,q}_{4(m-k)} \subset W^{p,q}_{4m}$ is trivial (respectively the normal bundle of $F^{p,q}_{4(m-k)} \subset F^{p,q}_{4m}$ is trivial). Thus,

$$I(V^{p,q}_{4m-1}, Z_2) = -Sgn(1, W^{p,q}_{4m})$$
$$= -I(W^{p,q}_{4m})$$

(respectively,

$$I(K^{p,q}_{4m-1}, Z_2) = -I(\overline{F}^{p,q}_{4m}) \quad).$$

Since $I(\ ,\mathbb{Z}_2)$ is a $\mathbb{Z}_2$-diffeomorphism invariant the theorems follow.

Using Hirzebruch's formulae (Theorem 1.10) for the index of the Brieskorn example $W^{p,q}_{4m}$, it is an immediate corollary of Theorem 6.1 that:

Corollary (Browder - Petrie): For any $m \geqq 3$, there are an infinite number of distinct semifree S^1-actions on Brieskorn (4m-1) spheres with fixed point set of codimension 4k for any $k \neq m/2$, $k < m-1$.

If we could answer the following two questions:

1) For any $\Sigma \in \theta_{4m-1}(\partial\pi)$, is $V_{2m+1,2} \# \Sigma$ diffeomorphic to $K^{p,q}_{4m-1}$ for some pair (p,q)?

2) What is the index, $I(F^{p,q}_{4m})$?

then Theorem 6.2 would prove a similar corollary for semifree S^1-actions on $V_{n+1,2} \# \Sigma$. On the other hand, we can prove such a result directly from the above corollary. There is a canonical semifree S^1-action on $V_{2m+1,2}$ via the representation ρ_k, with fixed point submanifold $V_{2(m-k)+1,2}$. Since we can write Σ^{4m-1} as the Brieskorn manifold $V^{r,s}_{4m-1}$ (for some pair (r,s)), and since the connected sum can be performed equivariantly at a fixed point as a corollary of the above corollary and Theorem 6.1, we have:

(6.3) <u>Corollary</u>: For any $m \geqq 3$, there are an infinite number of distinct semifree S^1-actions on $V_{2m+1,2} \# \Sigma$ $(\Sigma \in \theta_{4m-1}(\partial\pi))$ with fixed point set of codimension 4k, for any $k \neq m/2$, $k \leqq m-1$.

The question arises: are we talking about the same set of actions in Corollary 6.3 as in Theorem 6.2? We can rephrase this question as: are the semifree S^1-actions on $K^{p,q}_{2n-1}$, induced by p_k, decomposable S^1-actions of type 2k? In particular, can we decompose them so that the action is given by connected sum with one of the above actions on a Brieskorn sphere. The results of the last chapter suggest that the answer to this is no.

7. DECOMPOSABLE ACTIONS

O(n-k-1) operates on the link manifold $K^{p,q}_{2n-1}$ as a subgroup, $O(n-k-1) \subset O(n-1)$, $0 \leq k < n-1$. Throughout this chapter we will assume that n is even. If nis odd, all of our results and proofs are still valid as long as we assume that $K^{p,q}_{2n-1}$ is diffeomorphic to $V_{n+1,2}$. (Note that this is implied, for example, by Conjecture 5.4 or 3.13).

Recall that $K^{p,q}_{2n-1} \equiv \pm(V_{n+1,2}) \# \Sigma$, for some $\Sigma \in \theta_{2n-1}(\partial\pi)$. Of course, O(n-k-1) acts on $V_{n+1,2}$ via the standard inclusion $O(n-k-1) \subset O(n-1)$. If O(n-k-1) acts on Σ in such a way that the connected sum can be performed equivariantly, then it makes sense to ask whether or not $K^{p,q}_{2n-1}$ is O(n-k-1)-diffeomorphic to the equivariant connected sum, that is, is the action decomposable? When k = 0, we have shown (Proposition 5.12) that the action is indecomposable.

In particular, note that if the action on Σ is induced from a knot manifold, then the connected sum can always be taken equivariantly. Since actions of the orthogonal group with three orbit types are relatively well understood when the ambient manifold is a homotopy sphere, it would make little sense to further study the natural actions on $K^{p,q}_{2n-1}$ if they were already decomposable.

The fixed point submanifold of the O(n-k-1)-action on $K^{p,q}_{2n-1}$ is just $K^{p,q}_{2n-1}$, which for k odd and k > 1, is diffeomorphic to $V_{k+2,2} \# \Sigma^{2k+1}_1$. (As before, if k is even, then we must assume that Conjecture 3.13 is true). The action on

$V_{n+1,2} \# \Sigma$ has as its fixed point submanifold $V_{k+2,2} \# \Sigma_2^{2k+1}$, where for $k > 1$, $\Sigma_i^{2k+1} \in \theta_{2k+1}(\partial\pi)$ (for $i = 1$ or 2).

If the action on $K_{2n-1}^{p,q}$ is decomposable, we note that at least the following two conditions must be satisfied:

1) The fixed point manifolds must be diffeomorphic, i.e.,

$$\Sigma_1^{2k+1} \cong \Sigma_2^{2k+1}.$$

Also, it follows from the results of chapter 6, that if the action on Σ is defined by some Brieskorn sphere, say, $V_{2n-1}^{r,s}$ then

2) The signatures must be equal, that is,

$$I(\overline{F}_{2n}^{p,q}) = I(W_{2n}^{r,s}) \pm 1$$

Conditions 1) and 2) are obviously not very strong. For example, if $I(\overline{F}_{2n}^{p,q}) = 8j \pm 1$, then setting $(r,s) = (3, 6j-1)$, we would satisfy condition 2). Condition 1) is even weaker. In fact, if the action on Σ is defined by the action on a Brieskorn sphere, then it is clear that 2) implies 1).

In this chapter we shall develop somewhat stronger techniques for deciding when an action is decomposable. More generally, we would like to be able to answer the following question: If the orthogonal group acts on $V_1 \equiv V_2 \# V_3$ (with precisely 3 orbit types), then is the action on V_1 equivalent to the connected sum of an action on V_2 with one on V_3? When V_1, V_2, and V_3 are defined by weighted homogeneous polynomials and the actions are the obvious ones, then we can answer this question.

We would also like to be able to decide when orthogonal involutions and semifree S^1-actions on $K^{p,q}_{2n-1}$ are decomposable. Although it is possible to obtain some weak results in this direction as corollaries of the results on O(n-k-1)-actions, our techniques do not work in these cases. First we need two rather obvious technical lemmas:

(7.1) Lemma: Suppose that there exists a diffeomorphism of pairs,

$$f: (S^n, K_1) \cong (S^n, K_2)$$

where K_i (i = 1 or 2) is a closed submanifold of codimension ≥ 1. Then there exists a diffeomorphism of triples,

$$g: (D^{n+1}, S^n, K_1) \cong (D^{n+1}, S^n, K_2)$$

Proof: Since K_i is contained in some n-cell in S^n, we may as well assume that each K_i is contained in the northern hemisphere, $D^n_+ \subset S^n$. We can choose f within its smooth isotopy class so that it is the is the identity on the southern hemisphere, D^n_-. Let h: $S^n \longrightarrow S^n$ be a diffeomorphism such that

$$[h] = -[f] \in \Gamma_n = \mathrm{Diff}(S^n)/\mathrm{Ext}(D^{n+1})$$

(where [ ] means isotopy class) and such that h is the identity on D^n_+. Then the composition

$$g = f \circ h: (S^n, K_1) \cong (S^n, K_2)$$

satisfies [g] = 0, and hence g extends to D^{n+1}.

(7.2) <u>Lemma</u>: Let V_i^{2n-1} $(i = 1,2)$ be a $(n-2)$-connected closed submanifold of S^{2n+1} such that $S^{2n+1}-V_i$ fibres over S^1, with typical fibre F_i. Let h_i be the characteristic homeomorphism of F_i. F_i is $(n-1)$-connected. Let $\Delta_i(t)$ be the characteristic polynomial of

$$h_{i*}: H_n(F_i;C) \longrightarrow H_n(F_i;C)$$

We can take the connected sum $S^{2n+1} \# S^{2n+1}$ at a point in each V_i so that

$$V_1 \# V_2 \subset S^{2n+1} \# S^{2n+1} \cong S^{2n+1}.$$

Then $S^{2n+1}-(V_1 \# V_2)$ fibres over S^1 with characteristic polynomial

$$\Delta(t) = \Delta_1(t)\Delta_2(t)$$

<u>Proof</u>: $S^{2n+1} \# S^{2n+1} - (V_1 \# V_2)$ fibres over S^1 in the obvious way. The closure of the fibre $\overline{F}$ is just the connected sum along the boundary $\overline{F}_1 \# \overline{F}_2$. Thus

$$H_*(F) = H_*(F_1) \oplus H_*(F_2)$$

and the linear transformation, $h_*: H_n(F;\mathbb{C}) \longrightarrow H_n(F;\mathbb{C})$, is equivalent to

$$h_{1*} \oplus h_{2*}: H_n(F_1;\mathbb{C}) \oplus H_n(F_2;\mathbb{C}) \longrightarrow H_n(F_1;\mathbb{C}) \oplus H_n(F_2;\mathbb{C}).$$

It follows that

$$\Delta(t) = \Delta_1(t)\Delta_2(t).$$

(7.3) Proposition: Suppose that the following four conditions are satisfied:

1) $V_1 \equiv V_2 \# V_3$, where each V_i $(i = 1,2,3)$ is an $(n-3)$-connected, $(2n-1)$-dimensional manifold.

2) For some k, $1 \leq k \leq n-2$, there exists actions

$$\phi_i: O(n-k-1) \rtimes V_i \longrightarrow V_i$$

with exactly 3 orbit types.

3) The connected sum, $V_2 \# V_3$, can be performed equivariantly, so that there is a well defined action

$$\phi_2 \# \phi_3: O(n-k-1) \rtimes (V_2 \# V_3) \longrightarrow (V_2 \# V_3)$$

4) The fixed point submanifolds, $F_i \subset V_i$, are connected.

Then the orbit space in each case is D^{2k+4}. ϕ_1 is equivalent to $\phi_2 \# \phi_3$ iff

$$(S^{2k+3}, F_1) \equiv (S^{2k+3} \# S^{2k+3}, F_2 \# F_3)$$

Furthermore, if $V_i \equiv K^{p,q}_{2n-1}$ (respectively, if $V_i = V^{r,s}_{2n-1}$) then $F_i \equiv K^{p,q}_{2k+1}$ (respectively, $V^{r,s}_{2k+1}$) and the embedding

$$K^{p,q}_{2k+1} \subset S^{2k+3} \quad \text{(respectively, } V^{r,s}_{2k+1} \subset S^{2k+3})$$

is determined by the defining polynomial, i.e., by the equations

$$z_{k+1} = \cdots = z_{n+1} = 0 \ .$$

Proof: Since V_i is (n-3)-connected it follows from Bredon [4; Theorem 4.1] that the orbit space is a contractible (2k+4)-manifold. Since $2k+4 \geqq 6$, the orbit space $V_i/\phi_i \equiv D^{2k+4}$. It follows that the orbit triples are

$$(D^{2k+4}, S^{2k+3}, F_i)$$

(where i = 1,2,3) and that the orbit triple for $\phi_2 \# \phi_3$ is just the connected sum along the boundary of the orbit triples for ϕ_2 and ϕ_3, that is,

$$(D^{2k+4} \coprod D^{2k+4}, S^{2k+3} \# S^{2k+3}, F_2 \# F_3) \equiv (D^{2k+4}, S^{2k+3}, F_2 \# F_3)$$

According to Lemma 7.1, we can erase the D^{2k+4}'s. The rest of the theorem is just a restatement of the classification theorem of Janich and the Hsiangs [24; Theorem 4.3].

As a corollary of this proposition we can show:

(7.4) Theorem: For no pairs (p,q) and (r,s) (where p,q,r,s are odd integers greater than 1 and gcd(p-1,q) = 1 = gcd(r,s)) is $K^{p,q}_{2n-1}$ O(n-k-1)-diffeomorphic to $V_{n+1,2} \# V^{r,s}_{2n-1}$ ($0 \leqq k \leqq n-2$, $n > 2$). That is, the O(n-k-1)-action on $K^{p,q}_{2n-1}$ is not decomposable into the connected sum of the standard action on $V_{n+1,2}$ and an action on a Brieskorn sphere $V^{r,s}_{2n-1}$.

Proof: When k = 0, this is a special case of Proposition 5.12. Now suppose that k > 0 and that for some (r,s) $K^{p,q}_{2n-1}$ is O(n-k-1)-diffeomorphic to $V_{n+1,2} \# V^{r,s}_{2n-1}$. According to Proposition 7.3 this implies that

$$S^{2k+3} - K^{p,q}_{2k+1} \equiv S^{2k+3} - (V_{k+2,2} \# V^{r,s}_{2k+1}).$$

But both sides of this equation fibre over S^1. Since the total spaces are diffeomorphic it follows that the bundles are fibre homotopy equivalent. On the homology level, a fibre homotopy equivalence must commute with the characteristic homeomorphism. Hence, if $\Delta'(t)$ is the characteristic polynomial of $S^{2k+3} - K^{p,q}_{2k+1}$ and $\Delta''(t)$ is the characteristic polynomial of $S^{2k+3} - (V_{k+2,2} \# V^{r,s}_{2k+1})$, then we must have that

$$\Delta'(t) = \Delta''(t)$$

where

$$\Delta'(t) = \frac{(t-1)(t^{pq}-1)}{(t^p-1)}, \text{ if k is even, or}$$

$$\Delta'(t) = \frac{(t+1)(t^{pq}+1)}{(t^p+1)} \quad , \text{ if k is odd.}$$

While using Lemma 7.1

$$\Delta''(t) = \frac{(t-1)^2(t^{rs}-1)}{(t^r-1)(t^s-1)} \quad , \text{ if k is even, and}$$

$$\Delta''(t) = \frac{(t+1)^2(t^{rs}+1)}{(t^r+1)(t^s+1)} , \text{ if k is odd.}$$

If k is even, define the polynomials $P'(t) = \Delta'(t)/(t-1)$ and $P''(t) = \Delta''(t)/(t-1)$. Note that

$$P'(1) = q \quad \text{while}$$

$$P''(1) = 1.$$

Hence, if k is even, then for all pairs (p,q) and (r,s) $\Delta'(t) \neq \Delta''(t)$.

Now suppose that k is odd and that for some pairs (p,q) and (r,s), satisfying the hypothesis, $\Delta'(t) = \Delta''(t)$. A simple combinatorial argument shows that this is impossible. Note that the primitive 2pq-th roots of unity are roots of $\Delta'(t)$, while the primitive rs-th roots of unity are roots of $\Delta''(t)$. Hence pq = rs. Comparing the denominators, a simple argument now shows that $\Delta'(t)$ can equal $\Delta''(t)$ only if either r=1 or s=1. This is a contradiction, since $V^{r,s}_{2n-1}$ is a homotopy sphere iff r and s are odd, relatively prime, and greater than 1. Thus, $\Delta'(t)$ can never equal $\Delta''(t)$ and the theorem follows.

It is almost certainly true, that by a slightly finer argument, in the hypothesis of Theorem 7.4, we could replace the Brieskorn sphere $V^{r,s}_{2n-1}$ by any knot manifold $M^{2n-1}(F)$ which is a homotopy sphere. Thus,

(7.5) Conjecture: The action of O(n-k-1) on $K^{p,q}_{2n-1}$ cannot be decomposed into the standard action on $V_{n+1,2}$ connected sum with an action on a knot manifold $M^{2n-1}(F)$, where the O(n-k-1)-action on $M^{2n-1}(F)$ is induced by inclusion.

In a similar fashion, since the characteristic polynomials are different, it also follows from Proposition 7.3 that the O(n-k-1)-actions on $K^{p,q}_{2n-1}$ and $K^{p',q'}_{2n-1}$ are equivalent iff (p,q) = (p',q'). In fact, we can prove a somewhat stronger result:

(7.6) <u>Theorem</u>: For $0 \leqq k \leqq n-2$, $n > 2$, the $O(n-k-1)$- actions on $K^{p,q}_{2n-1}$ and $K^{p',q'}_{2n-1}$ do not differ by an $O(n-k-1)$-action on a Brieskorn sphere $V^{r,s}_{2n-1}$.

We have the corresponding conjecture:

(7.7) <u>Conjecture</u>: The $O(n-k-1)$- actions on $K^{p,q}_{2n-1}$ and $K^{p',q'}_{2n-1}$ do not differ by an $O(n-k-1)$-action on a knot manifold $M^{2n-1}(F)$ (where the action on $M^{2n-1}(F)$ is induced by $O(n-k-1) \subset O(n-1)$).

<u>Proof of Theorem 7.6</u>: The proof is completely analogous to the proof of Theorem 7.4. We suppose that $K^{p,q}_{2n-1}$ is $O(n-k-1)$-diffeomorphic to $K^{p',q'}_{2n-1} \# V^{r,s}_{2n-1}$. An easy argument shows that the corresponding characteristic polynomials

$$\Delta'(t) = \frac{(t\pm1)(t^{pq}\pm1)}{(t^{p}\pm1)}$$

and

$$\Delta''(t) = \frac{(t\pm1)^{2}(t^{p'q'}\pm1)(t^{rs}\pm1)}{(t^{p'}\pm1)(t^{r}\pm1)(t^{s}\pm1)}$$

can never be equal.

(7.8) <u>Remark</u>: Suppose that n is even and k is odd. Recall that we have semifree S^1-actions defined on $K^{p,q}_{2n-1}$ induced by the inclusion of $(n-k-1)/2$ copies of $SO(2)$ along the diagonal in $O(n-k-1)$. Since for distinct pairs (p,q) and (p',q') the $O(n-k-1)$-actions on $K^{p,q}_{2n-1}$ and $K^{p',q'}_{2n-1}$ are inequivalent, one might suspect that the corresponding S^1-actions are also inequivalent. Recent results of Browder and Petrie [14] indicate that this is false. Their result also shows that although the O(n-k-1)-actions on $V_{2n-1}^{r,s}$ and $V_{2n-1}^{r',s'}$ are inequivalent, the associated semifree S^1-actions are not necessarily inequivalent.

Theorems 7.4 and 7.7 indicate the direction that we think further study of group actions on exotic Stiefel manifolds should take. If we mod out by the relation, "two G-actions are equivalent iff they differ by a G-action on a homotopy sphere," then in some sense we are studying actions whose "exoticness" depends only on the action on the topological structure rather than on the differentiable structure of an exotic $V_{n+1,2}$.

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