ON ℓ^2-HOMOLOGY OF LOW DIMENSIONAL BUILDINGS

DISSERTATION

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By

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ABSTRACT

We study topological invariants related to the ℓ^2-homology of low dimensional regular right-angled buildings. By definition, such buildings admit a chamber transitive automorphism group G. In this setting, we provide several formulas for the ℓ^2-Euler characteristic with respect to G and compute ℓ^2-Betti numbers for a variety of 2-dimensional right-angled buildings. One of these formulas relates the ℓ^2-Euler characteristic to the h-polynomial of the nerve of the associated right-angled Coxeter group. Particularly interesting is the case where this nerve is a triangulation of a n-sphere. We prove that the h-polynomial associated with a flag triangulation of a n-sphere has real roots for $n \leq 3$.

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CHAPTER 1

INTRODUCTION

In geometric group theory there is a strong connection between the study of certain infinite groups and certain spaces on which such groups act. Coxeter groups and buildings give examples of this relationship. The buildings we are considering are not the classical Bruhat-Tits buildings associated to algebraic groups. The Coxeter groups Γ associated to the buildings of interest to us are arbitrary right-angled Coxeter groups - and these are generally not Euclidian or spherical reflection groups. Several constructions by Davis and others provide examples of complexes on which such groups act. In Chapter 2 we recall definitions and basic constructions related to Coxeter groups and buildings. Chapter 3 introduces the related concept of growth series of a Coxeter system in several variables.

In Chapter 4 we discuss Hilbert Γ-modules, their Γ-dimension and ℓ^2-homology theory. The linear algebra on which ℓ^2-homology theory is based is the category of Hilbert Γ-modules and the theory of their Γ-dimension (also called "von Neumann dimension"). For such a Hilbert Γ-module a Γ-dimension (which is a real number, not an integer!) is defined (the usual dimension as a vector space is infinite in this case). This leads to nice applications in topology. For a special type of non-compact spaces (cellular complexes on which Γ acts cellularly, properly and cocompactly) a ℓ^2-homology theory may be introduced by a minor modification of the usual definition of the homology of a chain complex. This, coupled with the existence of Γ-dimension, leads to the introduction of numerical invariants such as ℓ^2-Betti numbers. In Chapter 5 we introduce definitions, notations and basic ℓ^2-topological properties specifically written for right-angled buildings.

Chapter 6 contains the most important contributions of this thesis. Associated to any simplicial complex there is a certain polynomial called its h-polynomial, the coefficients of which are certain linear combinations of the number of simplices in each dimension. It is a conjecture of Januszkiewicz that if the simplicial complex is a flag complex and if it is homeomorphic to the sphere, then the roots of the h-polynomial should all be real. We show this conjecture to be true in dimensions less then 4, relate it to the Flag Complex Conjecture and reduce it to some combinatorial inequalities in dimensions less then 6. This is also related to the Concentration Conjecture of Dymara-Januskiewicz which states that the ℓ^2-Betti numbers of the Davis complex associated to a triangulation of a sphere as a flag complex are concentrated in one dimension and increase monotonically in t. The Concentration Conjecture was proved to be true in dimensions 2 and 3 but fails in dimension 4 (by recent work of Davis, Dymara, Januskiewicz and Okun).

An important part of Chapter 6 is devoted to computing ℓ^2-Betti numbers for two dimensional right-angled buildings of arbitrary thickness. We discuss several classes of examples as well as some concrete examples of interest. The alternating sum of the ℓ^2-Betti numbers coincides with the orbihedral Euler characteristic of X/G. This is proved to coincide with the reciprocal of the growth function of the associated Coxeter group. We discuss a more general approach using the growth function in several variables. We introduce a several variables version of the h-polynomial and prove its connection with the ℓ^2-Euler characteristic.

CHAPTER 2

REGULAR RIGHT-ANGLED BUILDINGS

In this chapter we recall some basic definitions and important facts about buildings, Coxeter groups and their associated complexes. Besides the usual notion of Coxeter complex Michael Davis introduced a different (inequivalent) notion of Coxeter complex (referred to as a Davis complex) in [9]. The Davis realization of a building was introduced in [10]. Excellent references for Coxeter groups and buildings are [5], [4] and [26].

Coxeter Groups

The theory of abstract reflection reflection groups was developed by Tits who introduced the terminology Coxeter groups for these objects.

Let I be a set (usually a finite set). A *Coxeter matrix* $M = (m_{ij})$ over I is a symmetric I by I matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ and $m_{ij} \geq 2$ whenever $i \neq j$. Introduce symbols $s_i, i \in I$ and put $S = \{s_i\}_{i \in I}$. The *Coxeter group* *(over M)* is the group defined by the presentation:

$$W = < S \mid (s_i s_j)^{m_{ij}} = 1, (i,j) \in I \times I >$$

The pair (W, S) is called a *Coxeter system*.

Definition 2.1. A Coxeter system (W, S) is called a *right angled Coxeter system* if $m_{ij} = 2$ or ∞ for all $i \neq j$.

If J is any subset of I, then M_J denotes the J by J matrix formed by restricting M to J. Let $s : I \longrightarrow S$ be the bijection $i \mapsto s_i$ and denote by W_J the subgroup generated by $s(J)$. It turns out that W_J is also a Coxeter group (see [5]). If (W, S) is a right angled Coxeter system again it turns out that the situation is as nice as possible: each $(W_J, s(J))$ is a right angled Coxeter system corresponding to the sub-Coxeter matrix $(m_{ij})_{(i,j) \in J \times J}$ (see page 20 of [5]). The subset J is called *spherical* if W_J is a finite group.

Associated to a Coxeter system (W, S), there are three simplicial complexes, L, K and the Davis complex Σ. Let $\mathcal{S}$ denote the set of spherical subsets of S, partially ordered by inclusion and let

$$\mathcal{S}^{(i)} = \{T \in \mathcal{S} \mid \mathrm{Card}(T) = i\}$$

$\mathcal{S}$ has a minimum element, namely, $\emptyset$. $\mathcal{S}_{>\emptyset}$ is the poset of simplices of a simplicial complex denoted by $L(W, S)$ (or L for short) and called the *nerve* of (W, S). In other words, the vertex set of L is S and a nonempty subset $T \subseteq S$ spans a simplex if and only if it is spherical. Moreover, $\mathcal{S}^{(i)}$ is the set of $(i-1)$-simplices in L.

We are also interested in $W\mathcal{S}$, the *poset of spherical cosets*. It is defined as the disjoint union of the sets W/W_T, $T \in \mathcal{S}$. Thus, a typical element of $W\mathcal{S}$ is a coset wW_T for some $T \in \mathcal{S}$. The partial order is inclusion.

The geometric realization of $\mathcal{S}$ is denoted K and the geometric realization of $W\mathcal{S}$ by Σ. The group W acts properly on the simplicial complex Σ; the orbit space is the finite complex K. The main features of Σ are:

- W acts on Σ with finite isotropy subgroups and with K as a strict fundamental domain,
- Σ is contractible.

L and K are finite simplicial complexes. The complex Σ is infinite whenever W is infinite.

The complex Σ also has a description in terms of a different construction . For each $s \in S$ define the mirror K_s to be the following subcomplex of K:

$$K_s := |\mathcal{S}_{\geq \{s\}}|$$

We can then form the space $\mathcal{U}(W, K)$ in the following way: it will be another

CW complex with a W-action for which K is a strict fundamental domain. For each subset T of S, set

$$K_T = K \cap \bigcap_{s \in T} K_s,$$

$$K^T = \bigcup_{s \in T} K_S$$

For each cell c of K and each point $z \in K$, set

$$S(z) = \{s \in S \mid z \in K_s\}$$

$$S(c) = \{s \in S \mid c \subseteq K_s\}$$

Define $\mathcal{U}(W, K) := (W \times K)/\sim$ where the equivalence relation $\sim$ is defined by $(w, z) \sim (w', z')$ if and only if $z = z'$ and the cosets $wW_{S(z)}$ and $w'W_{S(z)}$ are equal. Thus, $\mathcal{U}(W, K)$ is the space formed by gluing together copies of K one for each element of W, the copies $w \times K$ and $ws \times K$ being glued together along the subspaces $w \times K_s$ and $ws \times K_s$. Write $[w, z]$ for the image of (w, z) in $\mathcal{U}$. The group W acts on $\mathcal{U}$ and K is a strict fundamental domain. Identify K with the image of $1 \times K$ in $\mathcal{U}$. Then wK is identified with the image of $w \times K$. The CW structure on $\mathcal{U}$ is defined by declaring the family (wc), where $w \in W$ and c is a cell of K, to be the set of all cells in $\mathcal{U}$. (Note that wc is the image of $w \times c$ in $\mathcal{U}$.) The setwise stabilizer of a cell c of K is the special subgroup $W_{S(c)}$. Moreover, $W_{S(c)}$ fixes each point of c.

The natural map $W \times \mathcal{S} \to W\mathcal{S}$, defined by $(w, T) \to wW_T$, induces a map of geometric realizations $W \times K \to \Sigma$ and this descends to W-equivariant map $\mathcal{U}(W, K) \to \Sigma$. As in [9], it is easily seen that this map is a simplicial isomorphism, i.e.,

$$\Sigma \cong \mathcal{U}(W, K)$$

Remark 2.2. Although we are interested in the above constructions in the case of right-angled Coxeter systems we preferred to reproduce them here for arbitrary Coxeter systems.

Remark 2.3. Suppose L is a flag complex. We refer the reader to Appendix A for definitions regarding flag complexes. The right-angled Coxeter group associated to L, denoted W_L, is defined as follows. The set of generators is the vertex set of L and the edges of L give relations: $s^2 = 1$ and $(st)^2 = 1$, whenever $\{s, t\}$ spans an edge in L. Then L is exactly the nerve of $(W_L, \mathcal{S}_0(L))$. The corresponding complexes K and Σ are denoted K_L and Σ_L.

Buildings

Buildings were created by Tits as a tool for understanding real semisimple groups and their p-adic analogs. While classical (spherical or euclidian) buildings correspond to spherical or euclidian reflection groups, we are mainly interested in buildings related to other (infinite) reflection groups.

An abstract definition of buildings can be given as follows. Let M be a Coxeter matrix over a set I. Let I^* denote the free monoid on I. An element of I^* is a word $\mathbf{i} = i_1 ... i_k$, where each $i_j \in I$. Denote by W the Coxeter group determined by M. Suppose $\mathbf{i} = i_1 ... i_k$ is an element of I^*. Its value $s(\mathbf{i})$ is the element of W defined by $s(\mathbf{i}) = s_{i_1} ... s_{i_k}$. Two words $\mathbf{i}$ and **i'** are equivalent (with respect to M) if $s(\mathbf{i}) = s(\mathbf{i}')$. The word $\mathbf{i}$ is reduced if the word length of $s(\mathbf{i})$ is k. We refer the reader to [26] for the definition of a chamber system.

Definition 2.4. Let I, I^*, M and W be as above. A *building of type* M is a chamber system C over I such that

(B'1) for each $i \in I$, each subset of the partition corresponding to i contains at least two chambers

(B'2) there exists a W-valued distance function $\delta : C \times C \mapsto W$ such that if $\mathbf{i}$ is a reduced word in I^* (with respect to M), then chambers c and c' can be joined by a gallery of type $\mathbf{i}$ if and only if $\delta(c, c') = s(\mathbf{i})$.

The Davis realization of a building

We proceed now to introduce right-angled buildings and their Davis realizations. Let L be a flag complex with vertex set I and let W_L be the associated right-angled Coxeter group. Suppose we are given a family of groups $(P_i)_{i\in I}$ such that each P_i is a cyclic group of order $q_i + 1$. Associated with such data we have a group G_L defined as the quotient of the free product of the $(P_i)_{i\in I}$ by the normal subgroup generated by all commutators of the form $[g_i, g_j]$, where $g_i \in P_i$, $g_j \in P_j$ and $m_{ij} = 2$. Alternatively, G_L could be defined as the graph product of the $(P_i)_{i\in I}$ with respect to L.

Let $\mathbf{q} = (q_i)_{i\in I}$. We define a building of type W_L, denoted $C = C(\mathbf{q}, L)$, in the following way. Its set of chambers is G_L(defined above) and g and g' are i-adjacent if and only if they have the same image in G_L/P_i. We now produce the W_L-valued distance function $\delta : G_L \times G_L \mapsto W_L$ following [10]. If $g = g_{i_1}...g_{i_k}$ and $\mathbf{i} = i_1...i_k$, then define $\delta(1, g) = s(\mathbf{i})$ and then extend this by $\delta(g, g') = \delta(1, g^{-1}g')$. For more details see [10], page 5.

We now describe the Davis realization of a building $C(\mathbf{q}, L)$ which is denoted by $\Sigma(\mathbf{q}, L)$. We have defined in the previous section the complex K_L and its mirrors $(K_L)_s$. On $C \times K$ define an equivalence relation by $(c, x) \sim (c', x')$ if and only if $x = x'$ and $\delta(c, c') \in (W_L)_{S(x)}$. The Davis realization of $C(\mathbf{q}, L)$ is defined to be the quotient space:

$$\Sigma(\mathbf{q}, L) = (C \times K)/\sim$$

CHAPTER 3

THE GROWTH SERIES OF A GROUP

In this chapter we recall definitions and basic results about the growth series of a group (the one variable version as well as the several variables version). Most of the material can be found in [5], [27] and [25]. For a survey on growth series, discussing finitely generated groups in general as well as Coxeter groups we refer the reader to [29].

Let W be a finitely generated group and S be a finite symmetric generating set of W (i.e S is finite, generates S and, if $s \in S$ then $s^{-1} \in S$). The set S determines a length on W, called *word length.* It is defined by:

$$l(w) = min\{r | w = s_1 \dots s_r, s_i \in S\}$$

for $w \in W$. The *growth series of W with respect to S* is the formal power series:

$$W(t) = \sum_{w \in W} t^{l(w)}$$

For Coxeter systems, a several variables version of the growth series is defined by Serre in [27]. Let (W, S) be a Coxeter system with S a finite set. Let $\mathbf{t} = (t_i)_{i \in I}$

be a family of indeterminates and $s \mapsto i(s)$ a mapping of S into I that satisfies the condition: $i(s) = i(s')$ if s and s' are conjugate in W

For $s \in S$ we write t_s instead of $t_{i(s)}$.

Let $w \in W$. Choose a reduced decomposition $(s_1, \ldots, s_q)$ of w; the monomial $\mathbf{t}_w = t_{s_1} \ldots t_{s_q}$ is independent of the choice of $(s_1, \ldots, s_q)$. The total degree of $\mathbf{t_w}$ is equal to the length $l(w)$ of w. The *growth series of W with respect to S* is the formal series:

$$W(\mathbf{t}) = W((t_i)_{i \in I}) = \sum_{w \in W} \mathbf{t}_w$$

If X is any subset of W, put :

$$X(\mathbf{t}) = X((t_i)_{i \in I}) = \sum_{w \in X} \mathbf{t}_w$$

When I consists of just one element we obtain the growth series in one variable. Another case of interest to us, since we use only right-angled Coxeter groups, is when $I = S$ and the mapping $s \mapsto i(s)$ is the identity map.

As an example we compute the growth series for the infinite dihedral group $W = \mathbb{Z}_2 * \mathbb{Z}_2 =< s_1, s_2 | s_1^2 = s_2^2 = 1 >$. The growth series in one variable is :

$$\begin{aligned} W(t) &= 1 + 2t + 2t^2 + 2t^3 + \ldots \\ &= \frac{1+t}{1-t} \end{aligned}$$

while the growth series in several variables is :

$$\begin{aligned}
W(\mathbf{t}) = W(t_1,t_2) \quad &= 1 + t_1 + t_2 + t_1t_2 + t_2t_1 + t_1t_2t_1 + t_2t_1t_2 + \dots \\
&= 1 + (t_1+t_2) + 2t_1t_2 + t_1t_2(t_1+t_2) + 2(t_1t_2)^2 + \\
&\quad +(t_1t_2)^2(t_1+t_2) + 2(t_1t_2)^3 + (t_1t_2)^3(t_1+t_2) + \dots \\
&= 1 + 2t_1t_2(1 + t_1t_2 + (t_1t_2^2) + \dots) \\
&\quad +(t_1+t_2)(1 + t_1t_2 + (t_1t_2^2) + \dots) \\
&= 1 + \frac{2t_1t_2}{1-t_1t_2} + \frac{t_1+t_2}{1-t_1t_2} \\
&= \frac{(1+t_1)(1+t_2)}{1-t_1t_2}
\end{aligned}$$

Theorem 3.1. *(Serre [27]) Let (W,S) be a Coxeter system with S finite.Then $W(\mathbf{t})$ is a rational function in t_i, that is*

$$W(\mathbf{t}) = \frac{f(\mathbf{t})}{g(\mathbf{t})}$$

where $f,g \in \mathbb{Z}[\mathbf{t}]$ are polynomials with integer coefficients.

Proof. For a complete proof we refer the reader to Proposition 26, page 145 of [27](compare [5], Exercise 26, page 45). The proof goes by induction on $\mathrm{Card}(S)$ and uses the following formula:

$$0 = \sum_{T \subseteq S} \frac{(-1)^{\mathrm{Card}(T)}}{W_T(\mathbf{t})}$$

□

Let ρ_W denote the radius of convergence of $W(t)$ and denote by $\mathcal{R}_W$ the region of convergence of $W(\mathbf{t})$. Since $W(t)$ is a rational function $\rho_W = \min\{t | g(t) = 0\}$. For example, if $W = \mathbb{Z}_2 * \mathbb{Z}_2$ the computation above yields $\rho_W = 1$.

Remark 3.2. If $W = \mathbb{Z}_2$ it is easy to see that $W(t) = t+1$. If $G = \mathbb{Z}_{t+1}$ then obviously $W(t) = |G|$ (where $|G|$ denotes the order of G). More general, if $W = (\mathbb{Z}_2)^k$ then

$$W(t_1, t_2, ..., t_k) = (t_1 + 1)(t_2 + 1)...(t_k + 1)$$

This can be verified directly but we also mention that the growth series is multiplicative under finite products of groups. Let $G = \mathbb{Z}_{t_1+1} \times ... \times \mathbb{Z}_{t_k+1}$. Then obviously $W(t_1, t_2, ..., t_k) = |G|$. With $\mathbf{t} = (t_1, t_2, ..., t_k)$ we have

$$W(\mathbf{t}) = |G|$$

The following lemmas are proved in [6] (see Lemma 2 and Remark v), page 376). The first lemma is originally due to Steinberg (see [28]).

Lemma 3.3. *Let (W, S) be a right-angled Coxeter system and denote by $\mathcal{S}$ the set of spherical subsets of S (including $T = \emptyset$). Then:*

$$\frac{1}{W(\mathbf{t})} = \sum_{T \in \mathcal{S}} \frac{(-1)^{Card(T)}}{W_T(\mathbf{t}^{-1})}$$

Lemma 3.4. *Let (W, S) be a right-angled Coxeter system and denote by $\mathcal{S}$ the set of spherical subsets of S (including $T = \emptyset$). Then:*

$$\frac{1}{W(\mathbf{t})} = \sum_{T \in \mathcal{S}} \frac{1 - \chi(\partial K_T)}{W_T(\mathbf{t})}$$

where $\chi(K_T)$ denotes the ordinary Euler characteristic of K_T.

CHAPTER 4

HILBERT Γ-MODULES AND ℓ^2-HOMOLOGY

In this chapter we recall definitions, notation and basic results concerning the category of Hilbert Γ-modules and ℓ^2-homology theory. ℓ^2-invariants were introduced in topology and geometry by Atyiah in [1]. All this material (and much more) can be found in [21] or [17].

Hilbert Γ-modules and their Γ-dimension

In this section we introduce the category of Hilbert Γ-modules. Athough it is not a typical category for homological algebra purposes (it is not an abelian category; it is, however, an additive category) it has enough features, some quite exotic, that lead to nice applications to topology. The richness of this category comes from the following three facts:

• the existence of the Γ-dimension (sometimes called von Neumann dimension) which is a real number (and conjectured to be a rational number for Hilbert Γ-modules of interest in topology) instead of a positive integer. It applies to Hilbert spaces for which the usual dimension as vector spaces is infinite.

• the existence of weak isomorphisms. Weak isomorphisms would not exist if the category were abelian.

Let Γ be a countable group. We define $\ell^2(\Gamma)$ as the the Hilbert space with Hilbert basis $\{g|g \in \Gamma\}$. Thus $\ell^2(\Gamma)$ consists of formal sums $\sum_{g\in\Gamma} \lambda_g g$ where $\lambda \in \mathbb{C}$ and $\sum_{g\in\Gamma} |\lambda_g|^2 < \infty$. The inner product is given by:

$$< \sum_{g\in\Gamma} \lambda_g g, \sum_{g\in\Gamma} \mu_g g >= \sum_{g\in\Gamma} \overline{\lambda_g} \mu_g$$

The *group algebra* $\mathbb{C}[\Gamma]$ consists of elements $\alpha = \sum_{g\in\Gamma} \alpha_g g \in \ell^2(\Gamma)$ such that $\alpha_g = 0$ for all but finitely many g. For $\alpha \in \mathbb{C}[\Gamma]$ and $\beta = \sum_{g\in\Gamma} \beta_g g \in \mathbb{C}[\Gamma]$ the multiplication is given by:

$$\alpha\beta = \sum_{g\in\Gamma} \gamma_g g, \text{ where } \gamma_g = \sum_{h\in\Gamma} \alpha_{gh^{-1}} \beta_h$$

The group algebra $\mathbb{C}[\Gamma]$ is a dense subspace of $\ell^2(\Gamma)$; however, its multiplication does not extend to $\ell^2(\Gamma)$.

The *group von Neumann algebra* $\mathcal{N}(\Gamma)$ is defined as the algebra of Γ-invariant bounded operators from $l^2(\Gamma)$ to itself:

$$\mathcal{N}(\Gamma) = \mathcal{B}\big(\ell^2(\Gamma), \ell^2(\Gamma)\big)^\Gamma$$

where $l^2(\Gamma)$ is equiped with the obvious Γ-action. If H is a Hilbert space we denote by $\mathcal{B}(H,H)$ the space of bounded linear operators from H to itself. We also consider the von Neumann algebra of $n\times n$ matrices with entries in $\mathcal{N}(\Gamma)$:

$$\mathfrak{M}_{n\times n}\big(\mathcal{N}(\Gamma)\big)$$

The involution $*:\ell^2(\Gamma)\longrightarrow\ell^2(\Gamma)$ is defined by

$$\big(\sum_{g\in\Gamma}\alpha_g g\big)^* = \sum_{g\in\Gamma}\overline{\lambda_g}g^{-1}$$

This extends to matrices $A=(a_{ij})\in\mathcal{M}_{n\times n}\big(\mathcal{N}(\Gamma)\big)$ by $A^*=(a^*_{ji})$.

The definition we use for $\mathcal{N}(\Gamma)$ is not the original one but is equivalent to it. The original definition for $\mathcal{N}(\Gamma)$ is the weak closure of $\mathbb{C}[\Gamma]$ in $\mathcal{B}(\ell^2(\Gamma),\ell^2(\Gamma))$; equivalently $\mathcal{N}(\Gamma)$ is the von Neumann algebra generated by $\mathbb{C}[\Gamma]$ in $\mathcal{B}(\ell^2(\Gamma),\ell^2(\Gamma))$. Another description of $\mathcal{N}(\Gamma)$ is as the double commutant $\mathbb{C}[\Gamma]''$ of $\mathbb{C}[\Gamma]$ in $\mathcal{B}(\ell^2(\Gamma),\ell^2(\Gamma))$, where for any subset S of $\mathcal{B}(\ell^2(\Gamma),\ell^2(\Gamma))$ its commutant is:

$$S' = \{u\in\mathcal{B}(\ell^2(\Gamma),\ell^2(\Gamma))|us=su \text{ for all } s\in S\}$$

We have a $*$-homomorphism $\mathcal{N}(\Gamma)\longrightarrow\ell^2(\Gamma)$ defined by $u\mapsto u1$. Thus $\mathcal{N}(\Gamma)$ can be identified with a subspace of $\ell^2(\Gamma)$ where the action of $\mathcal{N}(\Gamma)$ on $\ell^2(\Gamma)$ is left multiplication. Similarly, $\mathfrak{M}_{n\times n}\big(\mathcal{N}(\Gamma)\big)$ can be identified with $\mathcal{B}\big(\oplus_{i=1}^n\ell^2(\Gamma),\oplus_{i=1}^n\ell^2(\Gamma)\big)^\Gamma$. For the proof of this fact we refer the reader to Lemma 5 of [20]. In view of these identifications, $*$ coresponds to the operation of taking the adjoint of a linear operator. We recall that if $A \in \mathcal{B}(H, H')$, then $A^* \in \mathcal{B}(H', H)$ is uniquely defined by

$$< Au, v >=< u, A^* v > \text{ for } u \in H \text{ and } v \in H'$$

An element $P \in \mathcal{B}(H, H)$ is an *orthogonal projection* if $P^* = P$ (i.e. P is self-adjoint) and $P^2 = P$.

We now proceed to introduce the category $\mathcal{C}_\Gamma$ of Hilbert Γ-modules.

Definition 4.1. A *Hilbert* Γ-*module* is a Hilbert space V together with a left action of Γ by linear isometries such that there is a Hilbert space H and a Γ-equivariant isometric embedding of V into the tensor product of Hilbert spaces $\ell^2(\Gamma)\hat{\otimes}H$. A Hilbert Γ-module is of finite type if there is a surjective bounded Γ-equivariant operator from $\oplus_{i=1}^n \ell^2(\Gamma)$ onto V for an appropriate positive integer n. This is equivalent to the existence of an isometric Γ-equivariant embedding of V into $\oplus_{i=1}^n \ell^2(\Gamma)$ for an appropriate positive integer n or to the existence of an orthogonal Γ-equivariant projection $P : \oplus_{i=1}^n \ell^2(\Gamma) \longrightarrow \oplus_{i=1}^n \ell^2(\Gamma)$ whose image is isometrically Γ-isomorphic to V for an appropriate positive integer n.

Definition 4.2. A *morphism* of Hilbert Γ-modules $f : U \longrightarrow V$ is a bounded Γ-equivariant operator. $f : U \longrightarrow V$ is called a *weak isomorphism* if its kernel is trivial and its image is dense.

The category $\mathcal{C}_\Gamma$ is not abelian. Indeed, any morphism $f : U \longrightarrow V$ which is injective and has dense image is both monomorphic and epimorphic (a morphism u is called *epimorphism* if $uu_1 = uu_2$ implies $u_1 = u_2$; similarly, a morphism u is called *monomorphism* if $u_1 u = u_2 u$ implies $u_1 = u_2$); it is not an isomorphism unless f is onto.

We now take a look at homological algebra concepts in the category of Hilbert Γ-modules $\mathcal{C}_\Gamma$. A Hilbert Γ-chain complex $C = (C_*, c_*)$ is a sequence of Hilbert Γ-modules

$$\ldots \xrightarrow{c_{n+1}} C_n \xrightarrow{c_n} C_{n-1} \xrightarrow{c_{n-1}} \ldots$$

such that $c_{n+1}c_n = 0$ holds for $n \in \mathbb{Z}$. The homology of C is defined by

$$H_n(C) = \mathrm{Ker}(c_n)/\mathrm{Im}(c_{n+1})$$

and the reduced homology of C is defined by

$$\mathcal{H}_n(C) = \mathrm{Ker}(c_n)/\overline{\mathrm{Im}(c_{n+1})}$$

C is called exact at C_n if $\mathrm{Ker}(c_n) = \mathrm{Im}(c_{n+1})$ while C is called weakly exact at C_n

if $\mathrm{Ker}(c_n) = \overline{\mathrm{Im}(c_{n+1})}$. Analogous statements are used to define the corresponding concepts for cohomology.

We now introduce the notions of Γ-trace and Γ-dimension of a Hilbert Γ-module. The *standard trace* $\mathrm{tr}_\Gamma : \mathcal{N}(\Gamma) \longrightarrow \mathbb{C}$ of the group von Neumann algebra $\mathcal{N}(\Gamma)$ is defined by:

$$\alpha = \sum_{g\in\Gamma} \alpha_g g \longmapsto \alpha_1$$

where $1 \in \Gamma$ is the unit element of the group Γ. This trace extends to matrices:

$$\mathrm{tr}_\Gamma : \mathfrak{M}_{n\times n}\big(\mathcal{N}(\Gamma)\big) \longrightarrow \mathbb{C}$$

by sending a matrix to the sum of the traces of the diagonal entries.

Definition 4.3. Let V be a Hilbert Γ-module of finite type. The Γ-*dimension* of V is defined by:

$$\dim_\Gamma(V) = \mathrm{tr}_\Gamma(p) \in [0, \infty)$$

where $p : \oplus_{i=1}^n \ell^2(\Gamma) \longrightarrow \oplus_{i=1}^n \ell^2(\Gamma)$ is any orthogonal $\mathcal{N}(\Gamma)$-projection whose image is isometrically $\mathcal{N}(\Gamma)$-isomorphic to V.

It is not hard to check that the definition above is independent of the choice of the projection. We next list some properties of the Γ-dimension that show why this invariant is called a dimension. The next lemma is well-known; for a proof see Theorem 1.12 of [21] or [17].

Lemma 4.4. *Let U, V and W be Hilbert Γ-modules of finite type. Then:*

i) Faithfulness:

$$dim_\Gamma(U) = 0 \text{ iff } U = 0$$

ii) Monotonicity:

$$\text{If } U \subseteq V \text{ then } dim_\Gamma(U) \leq dim_\Gamma(V)$$

iii) Continuity: If $U_1 \supseteq U_2 \supseteq \dots$ is a nested sequence of Hilbert Γ-modules of U then:

$$dim_\Gamma\Big(\bigcap_{n=1}^{\infty} U_n\Big) = \lim_{n\to\infty} dim_\Gamma(U_n)$$

iv) Additivity: If

$$0 \to U \xrightarrow{j} V \xrightarrow{q} W \to 0$$

is weakly exact then

$$dim_\Gamma(V) = dim_\Gamma(U) + dim_\Gamma(W)$$

v) Multiplicativity: If H is a subgroup of finite index m in Γ then

$$dim_H(U) = m\, dim_\Gamma(U)$$

We give some examples. First, we have $\dim_\Gamma(\ell^2(\Gamma)) = 1$ and $\dim_\Gamma(\oplus_{i=1}^n \ell^2(\Gamma)) = n$. If Γ is a finite group of order $|\Gamma|$ and U is a Hilbert Γ-module of finite type then $\dim_\Gamma(U) = \frac{1}{|\Gamma|}\dim U$ (where $\dim U$ denotes the usual dimension of a finite dimensional vector space).

If F is a finite subgroup of Γ, then $\ell^2(\Gamma/F)$, the space of square summable functions on Γ/F, can be identified with the subspace of $\ell^2(\Gamma)$ consisting of the square summable functions on Γ which are constant on each coset. This subspace is clearly closed and Γ-invariant; hence, $\ell^2(\Gamma/F)$ is a Hilbert Γ-module. We have $\dim_\Gamma(\ell^2(\Gamma/F)) = 1/|F|$ (see 3.2.13 on page 16 of [13]).

In the case $\Gamma = \mathbb{Z}$ we have a good understanding of the Γ-dimension because of the existence of the Fourier transform. The Fourier transform is an isomorphism of $\ell^2(\Gamma)$ with $L^2(S^1)$ where $n \in \mathbb{Z}$ acts on $L^2(S^1)$ by $f(z) \mapsto z^n f(z)$. Consider on the unit circle $S^1 \subset \mathbb{C}$ the Lebesgue measure renormalized such that $m(S^1) = 1$. If $A \subset S^1$ is measurable then the subspace E_A of functions in $L^2(S^1)$ that vanish outside A is closed and invariant. Then one can show $\dim_\Gamma(E_A) = m(A)$. As a consequence, measure theory on the circle is a particular case of Γ-dimension theory.

ℓ^2-homology and ℓ^2-Betti numbers

Let G be a discrete group. A *G-complex* is a CW complex X together with a cellular action of G on X. This means that each $g \in G$ permutes the cells of X. A G-complex is called a *geometric G-complex* if the action is proper (i.e., each cell stabilizer is finite; if σ is an i-cell of X we denote by G_σ the stabilizer of σ) and cocompact (i.e., X/G

is compact). A CW complex X is called *regular* if the characteristic map of each cell is an embedding (so that the boundary of each cell is an embedded sphere).

Let Y be a geometric G-complex. Define $C_i^{(2)}(Y)$ to be the vector space of (infinite) cellular i-chains on Y with square summable coefficients. Equivalently,

$$C_i^{(2)}(Y) = C_i(Y) \otimes_{\mathbb{Z}[G]} \ell^2(G) = C_i(Y; \ell^2(G))$$

where $C_i(Y)$ denotes the ordinary i-chains on Y. As a $\mathbb{Z}[G]$-module, $C_i(Y)$ can be decomposed as

$$C_i(Y) = \sum \mathbb{Z}[G/G_\sigma]$$

where the sum runs over G-orbits of i-cells σ and $\mathbb{Z}[G/G_\sigma]$ denotes the permutation module $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G_\sigma]} \mathbb{Z}$. Similarly, $C_i^{(2)}(Y)$ can be decomposed as

$$C_i^{(2)}(Y) = \sum \ell^2(G/G_\sigma)$$

where the sum runs over G-orbits of i-cells σ and $\ell^2(G/G_\sigma) = \ell^2(G) \otimes_{\mathbb{C}[G_\sigma]} \mathbb{C}$ can be thought as the Hilbert space of all l^2-functions on G/G_σ. Since there are a finite number of such orbits, $C_i^{(2)}(Y)$ is the direct sum of a finite number of such spaces. Hence, $C_i^{(2)}(Y)$ is a Hilbert G-module.

Since dual Hilbert spaces are canonically isomorphic, there is a canonical isomorphism between ℓ^2-chains and ℓ^2-cochains:

$$C_i^{(2)}(Y) \cong C^i_{(2)}(Y)$$

Define the boundary map $d_i : C_i^{(2)}(Y) \to C_{i-1}^{(2)}(Y)$ and the coboundary map $\delta^i : C_i^{(2)}(Y) \to C_{i+1}^{(2)}(Y)$ by the usual formulae. Then the boundary and the coboundary maps are G-equivariant, bounded linear maps. The coboundary map δ^i can be identified with d_{i+1}^* (the adjoint of d_{i+1}).

Thus, $(C_*^{(2)}(Y), d_*)$ is a G-Hilbert chain complex and $(C_*^{(2)}(Y), \delta^*)$ is a G-Hilbert cochain complex. The reduced ℓ^2-homology of Y is defined as the reduced ℓ^2-homology of $(C_*^{(2)}(Y), d_*)$ and is denoted by $\mathcal{H}_i(Y)$. The cohomological version is defined similarly and is denoted $\mathcal{H}^i(Y)$. Since we will make no use of the unreduced ℓ^2-homology we drop the word reduced but we always mean reduced when we talk about ℓ^2-homology or cohomology.

The subspaces of $C_i^{(2)}(Y)$: Ker d_i, Ker δ^i, Im d_{i+1} and Im δ^{i-1} consist of ℓ^2-cycles, ℓ^2-cocycles, ℓ^2-boundaries and ℓ^2-coboundaries, respectively.

If X is a G-stable subcomplex of Y, then (Y, X) is called a pair of geometric G-complexes. The (reduced) ℓ^2-homology (or ℓ^2-cohomology) $\mathcal{H}_i(Y, X)$ is defined in the usual manner. Versions of most of the Eilenberg-Steenrod Axioms such as functoriality, weak exact sequence of a pair, excision, Mayer-Vietoris sequences and the Künneth Formula hold for $\mathcal{H}_i(Y, X)$. Similar results hold for the contravariant ℓ^2-cohomology functor. We refer the reader to [21] for a detailed explanation of all these properties. In the next chapter we state some of these results in our notation.

CHAPTER 5

ℓ^2-HOMOLOGY OF BUILDINGS

In this chapter we recall definitions, notations and basic results related to ℓ^2-homology of buildings. Davis and Okun started in [13] the study of ℓ^2-homology of right-angled buildings of constant thickness 2. Almost all material in this chapter is a direct reformulation of definitions, notations and basic results from [13] in the more general case of right-angled buildings of thickness $\mathbf{t} = (t_i)_{i \in I}$.

Definitions, notation and the ℓ^2-Euler characteristic

Let L be a flag complex. Associated to L we have a right-angled Coxeter system (W_L, S) and two other simplicial complexes, K_L and the Davis complex Σ_L. Given $\mathbf{t} = (t_i)_{i \in S}$, where t_i are positive integers, we have constructed a group G_L and the Davis realization $\Sigma(\mathbf{t}, L)$ of a building of thickness $\mathbf{t}$ (whose apartments are copies of Σ_L).

$$\mathfrak{h}_i^{\mathbf{t}}(L) := \mathcal{H}_i(\Sigma(\mathbf{t}, L))$$

If A is a full subcomplex of L

$$\mathfrak{h}_i^{\mathbf{t}}(A) := \mathcal{H}_i(G_L\Sigma(\mathbf{t}, A))$$

If (L, A) is a pair of flag complexes

$$\mathfrak{h}_i^{\mathbf{t}}(L, A) := \mathcal{H}_i(\Sigma(\mathbf{t}, L), G_L\Sigma(\mathbf{t}, A))$$

The ℓ^2-Betti numbers are defined as follows:

$$\beta_i^{\mathbf{t}}(L) := \dim_{G_L}(\mathfrak{h}_i^{\mathbf{t}}(L))$$

$$\beta_i^{\mathbf{t}}(A) := \dim_{G_L}(\mathfrak{h}_i^{\mathbf{t}}(A))$$

$$\beta_i^{\mathbf{t}}(L, A) := \dim_{G_L}(\mathfrak{h}_i^{\mathbf{t}}(L, A))$$

The notation in the previous two definitions is not confusing since $\mathcal{H}_i(G_L\Sigma(\mathbf{t}, A)$ is the induced representation from $\mathcal{H}_i(\Sigma(\mathbf{t}, A)$ (see 2.4.5 and 6.1.4 of [13]) and (see 3.2.12 of [13]) $\dim_{G_L}(\mathcal{H}_i(G_L\Sigma(\mathbf{t}, A))) = \dim_{G_A}(\mathcal{H}_i(\Sigma(\mathbf{t}, A)))$.

The ℓ^2-Euler characteristic is defined as expected:

$$\chi_{\mathbf{t}}(L) := \sum (-1)^i \beta_i^{\mathbf{t}}(L)$$

The following result is well-known.

Theorem 5.1. *(Atiyah's Formula) Let L be a flag complex and let G the group associated to $\Sigma(\mathbf{t}, L)$. Then*

$$\chi_{\mathbf{t}}(L) = \sum_{\sigma} \frac{(-1)^{dim(\sigma)}}{|G_\sigma|}$$

where the sum runs over all orbits of cells in $\Sigma(\mathbf{t}, L)$ *and* G_σ *denotes the stabilizer of the cell* σ*.*

Proof. A standard homological algebra argument shows that

$$\sum_i (-1)^i \beta_i^{\mathbf{t}}(L) = \sum_i (-1)^i c_i^{\mathbf{t}}(L)$$

where $c_i^{\mathbf{t}}(L) = \dim_G(C_i^{(2)}(\Sigma(\mathbf{t}, L))) = \dim_G(\oplus_\sigma \ell^2(G/G_\sigma))$(the last sum runs over orbits of i-cells). Since $\dim_G(\ell^2(G/G_\sigma)) = 1/|G_\sigma|$ the result follows. $\square$

A surprising connection between the ℓ^2-Euler characteristic and the growth series of W_L was proved by Dymara in [16].

Theorem 5.2. *Let* L *be a flag complex and denote by* $W_L(\mathbf{t})$ *the growth series of the group* W_L*. Then*

$$\chi_{\mathbf{t}}(L) = \frac{1}{W_L(\mathbf{t})}$$

Proof. We look at the formula for $\chi_{\mathbf{t}}(L)$ given by the previous theorem. By Remark 3.2 we have $|G_\sigma| = W_{S(\sigma)}(\mathbf{t})$ for each simplex σ in K. But

$$\sum_\sigma \frac{(-1)^{\dim(\sigma)}}{W_{S(\sigma)}(\mathbf{t})} = \sum_{T \subset S} \left(\sum_{\sigma : S(\sigma) = T} \frac{(-1)^{\dim(\sigma)}}{W_T(\mathbf{t})} \right)$$

by grouping together simplices σ that have the same $S(\sigma)$ and summing over $T \subset S$. On the other hand we have

$$\sum_{\sigma : S(\sigma) = T} \frac{(-1)^{\dim(\sigma)}}{W_T(\mathbf{t})} = \frac{1}{W_T(\mathbf{t})} \sum_{\sigma : S(\sigma) = T} (-1)^{\dim(\sigma)} = \frac{1 - \chi(\partial K_T)}{W_T(\mathbf{t})}$$

So we get

$$\chi_{\mathbf{t}}(L) = \sum_{T \in \mathcal{S}} \frac{1 - \chi(\partial K_T)}{W_T(\mathbf{t})} = \frac{1}{W_L(\mathbf{t})}$$

where the last equality is given by Lemma 3.4. $\square$

Properties

The first four theorems are reformulations of properties from [13] which were proved there in the case $\mathbf{t} = 1$.

Theorem 5.3. *(The exact sequence of a pair) Let L be a flag complex and A a full subcomplex of L. The sequence*

$$\to \mathfrak{h}_i^{\mathbf{t}}(A) \to \mathfrak{h}_i^{\mathbf{t}}(L) \to \mathfrak{h}_i^{\mathbf{t}}(L, A) \to$$

is weakly exact.

Theorem 5.4. *(Excision) Let L be a flag complex and A a full subcomplex of L. Let T be a set of vertices of A such that the open star of any vertex in T is contained in the interior of A. Then,*

$$\mathfrak{h}_i^{\mathbf{t}}(L, A) \cong \mathfrak{h}_i^{\mathbf{t}}(L - T, A - T)$$

Theorem 5.5. *(The Mayer-Vietoris sequence) Suppose $L = L_1 \cup L_2$, where L is flag complex and L_1 and L_2 (and therefore, $L_1 \cap L_2$) are full subcomplexes of L.*

i) The Mayer-Vietoris sequence

$$\to \mathfrak{h}_i^{\mathbf{t}}(L_1 \cap L_2) \to \mathfrak{h}_i^{\mathbf{t}}(L_1) \oplus \mathfrak{h}_i^{\mathbf{t}}(L_2) \to \mathfrak{h}_1^{\mathbf{t}}(L) \to$$

is weakly exact.

ii) $\mathfrak{h}_i^{\mathbf{t}}(L, L_1 \cap L_2) \cong \mathfrak{h}_i^{\mathbf{t}}(L_1, L_1 \cap L_2) \oplus \mathfrak{h}_i^{\mathbf{t}}(L_2, L_1 \cap L_2)$

iii) $\chi_{\mathbf{t}}(L) = \chi_{\mathbf{t}}(L_1) + \chi_{\mathbf{t}}(L_2) - \chi_{\mathbf{t}}(L_1 \cap L_2)$

Lemma 5.6. *(The Künneth Formula) Let L_1 and L_2 be flag complexes and denote by $L_1 * L_2$ their join. Then,*

$$\beta_k^{\mathbf{t}}(L_1 * L_2) = \sum_{i+j=k} \beta_i^{\mathbf{t}}(L_1)\beta_j^{\mathbf{t}}(L_2)$$

Lemma 5.7. *(0-dimensional Betti number) Let L be a flag complex.*

i) If W_L is finite then $\beta_0^{\mathbf{t}}(L) = \dfrac{1}{W(\mathbf{t})}$ and $\beta_i^{\mathbf{t}}(L) = 0$, for $i \neq 0$.

ii) If W_L is infinite then $\beta_0^{\mathbf{t}}(L) = 0$.

Proof. i) We give a direct proof of this fact. If W_L (and therefore G_L) is finite the complex $\Sigma(\mathbf{t}, L))$ is finite. Being contractible the only non-trivial ℓ^2-Betti number is in dimension zero and equal to $1/|G_L|$. By Remark 3.2 $|G_L| = W(\mathbf{t})$.

ii) This follows from 2.5.1 of [13]. $\square$

Lemma 5.8. *i) When $L = \emptyset$ we have*

$$\beta_i^{\mathbf{t}}(\emptyset) = \begin{cases} 1 & \text{if } i = 0, \\ \\ 0 & \text{if } i \neq 0. \end{cases}$$

ii) When $L = P_1$ is one point we have

$$\beta_i^{\mathbf{t}}(P_1) = \begin{cases} \dfrac{1}{t+1} & \text{if } i = 0, \\ \\ 0 & \text{if } i \neq 0. \end{cases}$$

iii) Given a k-simplex σ we have

$$\beta_i^{\mathbf{t}}(\sigma_k) = \begin{cases} \prod_{j=0}^{k} \dfrac{1}{t_j+1} & \text{if } i = 0, \\ \\ 0 & \text{if } i \neq 0. \end{cases}$$

Proof. Since $W_\emptyset$ is trivial, $W_{P_1} = \mathbb{Z}_2$ and $W_{\sigma_k} = (\mathbb{Z}_2)^{k+1}$ the result follows immediately from i) of the previous lemma and Remark 3.2. $\square$

Lemma 5.9. *Suppose that $L = L_1 \cup L_2$ and $L_1 \cap L_2$ equals $\emptyset$ or P_1 or σ_1. Then:*

(i) For $i \geq 2$ we have $\beta_i^{\mathbf{t}}(L) = \beta_i^{\mathbf{t}}(L_1) + \beta_i^{\mathbf{t}}(L_2)$

(ii) For $i = 1$, if L_1 and L_2 are not simplices then

$$\beta_1^{\mathbf{t}}(L) = \begin{cases} \beta_1^{\mathbf{t}}(L_1) + \beta_1^{\mathbf{t}}(L_2) + 1 & \text{if } L_1 \cap L_2 = \emptyset, \\ \\ \beta_1^{\mathbf{t}}(L_1) + \beta_1^{\mathbf{t}}(L_2) + \dfrac{1}{t_1+1} & \text{if } L_1 \cap L_2 = P_1, \\ \\ \beta_1^{\mathbf{t}}(L_1) + \beta_1^{\mathbf{t}}(L_2) + \dfrac{1}{(t_1+1)(t_2+1)} & \text{if } L_1 \cap L_2 = \sigma_1 \end{cases}$$

Proof. i) This follows easily from the Mayer-Vietoris sequence and the fact that $\beta_i^{\mathbf{t}}(L_1 \cap L_2) = 0$ for $i \geq 1$.

ii) This follows from the Mayer-Vietoris sequence and the previous lemma. $\square$

Lemma 5.10. *Let P_2 denote the disjoint union of 2 points. Then*

$$\beta_i^{\mathbf{t}}(P_2) = \begin{cases} \dfrac{t_1 t_2 - 1}{(t_1+1)(t_2+1)} & \textit{if } i = 1, \\ 0 & \textit{if } i \neq 1. \end{cases}$$

Proof. Since $P_2 = P_1 \amalg P_1$ we have $W_{P_2} \cong \mathbb{Z}_2 * \mathbb{Z}_2$ which is infinite. So $\beta_0^{\mathbf{t}}(P_2) = 0$. The only possible non-trivial ℓ^2-Betti number is in dimension one and equal to $-\chi_{\mathbf{t}}(P_2)$. But $\chi_{\mathbf{t}}(P_2) = \chi_{\mathbf{t}}(P_1) + \chi_{\mathbf{t}}(P_1) - \chi_{\mathbf{t}}(\emptyset)$ and the formula follows. $\square$

Lemma 5.11. *(The ℓ^2-Betti numbers of a suspension) Let L be a flag complex and let SL denote the suspension over L. Then*

$$\beta_i^{\mathbf{t}}(SL) = \frac{t_1 t_2 - 1}{(t_1+1)(t_2+1)} \beta_{i-1}^{\mathbf{t}}(L)$$

Proof. Note that $SL = P_2 * L$. The result follows from the Kuneth Formula and the previous lemma. $\square$

As an example, let $\square = SP_2$ denote a 4-gon. Then

$$\beta_i^{\mathbf{t}}(\square) = \begin{cases} \dfrac{(t_1 t_3 - 1)(t_2 t_4 - 1)}{(t_1+1)(t_2+1)(t_3+1)(t_4+1)} & \text{if } i = 2, \\ 0 & \text{if } i \neq 2. \end{cases}$$

Lemma 5.12. *Let P_k denote the disjoint union of k points, $k \geq 2$. Then all ℓ^2-Betti numbers are trivial except in dimension* 1*, where*

$$\beta_1^{\mathbf{t}}(P_k) = -1 + \sum_{i=1}^{k} \frac{t_i}{t_i + 1}$$

Proof. Since the group is infinite and $\Sigma(\mathbf{t}, P_k)$ is one-dimensional, the only non-trivial ℓ^2-Betti number is in dimension 1 and equals $-\chi_{\mathbf{t}}(P_k)$. An easy inductive argument shows the formula to be true (since P_k is obtained by a disjoint union of P_{k-1} and P_1). For $k = 2$ it easily seen that this formula agrees with the previously obtained formula for $\beta_1^{\mathbf{t}}(P_2)$. The result follows in general since $\chi^t(P_k) = \chi^t(P_{k-1}) + \chi^t(P_1) - \chi^t(\emptyset)$ and

$$\chi^t(P_1) - \chi^t(\emptyset) = \frac{1}{t_k + 1} - 1 = -\frac{t_k}{t_k + 1}$$

□

Remark 5.13. Elementary algebraic manipulations show that $\beta_1^{\mathbf{t}}(P_k)$ can also be written as

$$\beta_1^{\mathbf{t}}(P_k) = k - 1 - \sum_{i=1}^{k} \frac{1}{t_i + 1}$$

Lemma 5.14. *(The ℓ^2-Betti numbers of a cone) Let L be a flag complex and let CL denote the cone over L. Then*

$$\beta_i^{\mathbf{t}}(CL) = \frac{1}{t_1 + 1} \beta_i^{\mathbf{t}}(L)$$

Proof. As $CL = P_1 * L$, the result follows from the Künneth Formula and the ℓ^2-Betti numbers of P_1. $\square$

CHAPTER 6

LOW DIMENSIONAL RESULTS

This chapter contains the most important contributions of this thesis.

The f-polynomial, h-polynomial and ℓ^2-Euler characteristic

In this section we introduce a several variables version of the f-polynomial and h-polynomial. In this more general context the "h-polynomial" is actually a rational function but has many properties that show that indeed this is the correct version of the h-polynomial in several variables. We establish connections with the ℓ^2-Euler characteristic. When L is a flag triangulation of a sphere we obtain a new formula for the ℓ^2-Euler characteristic.

Let us recall the definitions of the f-polynomial and h-polynomial. Suppose L is a finite simplicial complex of dimension $m-1$, that f_i is the number of i-simplices in L and that $f_{-1}=1$. The *f-vector* of L is the m-tuple $(f_{-1}, f_0, \ldots, f_{m-1})$ and the *h-vector* $(h_0, \ldots, h_m)$ is defined by the equation :

$$\sum_{i=0}^{m} f_{i-1}(t-1)^{m-i} = \sum_{i=0}^{m} h_i t^{m-i}.$$

The *f-polynomial* $f(t) = f_L(t)$ and the *h-polynomial* $h(t) = h_L(t)$ are defined by :

$$f(t) = \sum_{i=0}^{m} f_{i-1} t^i$$

$$h(t) = \sum_{i=0}^{m} h_i t^i$$

Remark 6.1. It is immediate (in the equation defining the h-vector replace t by t^{-1} and multiply both sides by t^m) that the relation between the f-polynomial and h-polynomial can be written as:

$$h(t) = (1-t)^m f\Big(\frac{t}{1-t}\Big)$$

or, replacing t by $-t$:

$$h(-t) = (1+t)^m f\Big(\frac{-t}{1+t}\Big)$$

It is $h(-t)$ that is of interest to topology.

As an example, let's compute the f-polynomial and h-polynomial for a triangulation of a 1-sphere as a 4-gon. Then :

$$f(t) = 1 + 4t + 4t^2$$

$$h(t) = t^2 + 2t + 1$$

We now proceed to define the corresponding concept of f-polynomial and h-polynomial in several variables. Given a finite simplicial complex L as above, denote by $\mathcal{S}(L)$ the set of simplices in L together with the empty set $\emptyset$. Let $v_1, v_2, ..., v_n$ be the vertices of L and $\mathbf{t} = (t_1, t_2, ..., t_n)$. If $\sigma \in \mathcal{S}(L)$ let $I(\sigma) = \{i | v_i \in \sigma\}$. We define the monomials:

$$\mathbf{t}_\sigma = \prod_{i \in I(\sigma)} t_i \textit{ and } \mathbf{t}_\emptyset = 1$$

Similarly,

$$\left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right)_\sigma = \prod_{i \in I(\sigma)} \frac{-t_i}{1+t_i}$$

In this several variables context the correct definition of the f-polynomial $f(\mathbf{t}) = f_L(\mathbf{t})$ is defined by:

$$f(\mathbf{t}) = \sum_{\sigma \in \mathcal{S}(L)} \mathbf{t}_\sigma$$

while the "h-polynomial" $H(\mathbf{t}) = H_L(\mathbf{t})$ is defined by:

$$H(\mathbf{t}) = f\left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right)$$

As an example, if L is a 4-gon then $\mathbf{t} = (t_1, t_2, t_3, t_4)$ and

$$f(\mathbf{t}) = 1 + t_1 + t_2 + t_3 + t_4 + t_1 t_2 + t_2 t_3 + t_3 t_4 + t_4 t_1$$

while

$$H(t_1, t_2, t_3, t_4) = f\left(\frac{-t_1}{1+t_1}, \frac{-t_2}{1+t_2}, \frac{-t_3}{1+t_3}, \frac{-t_4}{1+t_4}\right)$$

Remark 6.2. When $t_1 = t_2 = ... = t_m = t$ we obtain the one variable version of the f-polynomial but

$$H(t) = \frac{h(-t)}{(1+t)^m}$$

Theorem 6.3. *Let L be a flag complex and $\Sigma(\mathbf{t}, L)$ the Davis realization of a building of thickness $\mathbf{t}$. Then:*

$$\chi_{\mathbf{t}}(L) = f_L\left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right)$$

Proof. The following formula is given by Lemma 3.3:

$$\frac{1}{W(\mathbf{t})} = \sum_{T \in \mathcal{S}} \frac{(-1)^{\mathrm{Card}(T)}}{W_T(\mathbf{t}^{-1})}$$

where W denotes the right-angled Coxeter group associated to $\Sigma(\mathbf{t}, L)$ and $\mathcal{S}$ denotes the poset of spherical subsets of S.

The left hand side of the formula above equals $\chi_{\mathbf{t}}(L)$ by Theorem 5.2. We look at the right hand side of the formula above. The spherical subsets T can be viewed as simplices in L. Fix T and suppose T corresponds to σ. We have $I(\sigma) = T$ and

$$W_T(\mathbf{t}) = \prod_{i \in T}(t_i + 1)$$

Since

$$\frac{1}{(t_1^{-1}+1)(t_2^{-1}+1)...(t_k^{-1}+1)} = \frac{t_1 t_2 ... t_k}{(t_1+1)(t_2+1)...(t_k+1)}$$

we get

$$\frac{(-1)^{\mathrm{Card}(T)}}{W_T(\mathbf{t}^{-1})} = \prod_{i\in T}\frac{-t_i}{1+t_i} = \left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right)_\sigma$$

Summing over all $T \in \mathcal{S}$ we obtain the sought for formula. $\square$

We now restrict our attention to triangulations of spheres. We prove another formula for the ℓ^2-Euler characteristic but here we take a "dual" approach.

Let P^n be an n-dimensional simple convex polytope (an n-dimensional convex polytope is simple if the number of codimension-one faces meeting at each vertex is n; equivalently, P^n is simple if the dual of its boundary complex is an $(n-1)$-dimensional simplicial complex). The f-polynomial and h-polynomial (as well as their several variables analogue) associated to a finite simplicial complex were defined in this chapter. Similarly, for an n-dimensional simple convex polytope the associated f-polynomial and h-polynomial are defined as those associated to the dual of its boundary complex.

Let P^n be an n-dimensional simple convex polytope. Let F_1, F_2,... , F_m be the codimension-one faces of P^n (also called facets). Let $\mathcal{F}$ denote the set of all faces of P^n and $\mathbf{t} = (t_1, t_2, ..., t_m)$. If $F \in \mathcal{F}$ then $I(F) = \{i | F \subset F_i\}$. We have:

$$\mathbf{t}_F = \prod_{i\in I(F)} t_i \;\; and \;\; \mathbf{t}_P = 1$$

Similarly,

$$\left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right)_F = \prod_{i\in I(F)} \frac{-t_i}{1+t_i}$$

We now proceed to introduce a different formula for the h-polynomial in several variables. The vertices and edges of a polytope form in an obvious way a non-oriented graph. Following [2] (pp 93-96) we introduce an orientation on the edges of P^n using an admissible vector. A vector $w \in \mathbb{R}^n$ is called admissible for P^n if $< x, w > \neq < y, w >$ for any two vertices x and y of P^n. Geometrically, this means that no hyperplane in $\mathbb{R}^n$ with w as a normal contains more than one vertex of P^n. It is shown in [2] (Theorem 15.1, page 93) that the set of admissible vectors is dense in $\mathbb{R}^n$. Any vector w which is admissible for P^n induces an orientation of the edges of P^n according to the following rule: An edge determined by vertices x and y is oriented towards x (and away from y) if

$$< x, w > \geq < y, w >$$

For each vertex $v \in P^n$, denote by $F_v^{in} \in \mathcal{F}$ the face determined by the inward-pointing edges at v and by $F_v^{out} \in \mathcal{F}$ the face determined by the outward-pointing edges at v. We have $I(v) = \{i | v \in F_i\}$. Moreover:

$$I(F_v^{in}) = \{i | F_v^{in} \subset F_i\} \ \ and \ \ I(F_v^{out}) = I(v) - I(F_v^{in})$$

Using an admissible vector w we now define:

$$H_w(\mathbf{t}) = \sum_v \frac{(-\mathbf{t})_{F_v^{out}}}{(\mathbf{1}+\mathbf{t})_v} = \sum_v \frac{\prod_{i\in I(F_v^{out})}(-t_i)}{\prod_{i\in I(v)}(1+t_i)}$$

As an example, if P^2 is a 4-gon then

$$\begin{aligned} H_w(t_1,t_2,t_3,t_4) &= \frac{(-t_1)(-t_2)}{(1+t_1)(1+t_2)} + \frac{(-t_2)}{(1+t_2)(1+t_3)} \\ &\quad + \frac{(-t_1)}{(1+t_1)(1+t_4)} + \frac{1}{(1+t_3)(1+t_4)} \end{aligned}$$

which can be simplified to

$$H_w(t_1,t_2,t_3,t_4) = \frac{(1-t_1t_3)(1-t_2t_4)}{(1+t_1)(1+t_2)(1+t_3)(1+t_4)}$$

To prove the next theorem we need the following combinatorial lemma.

Lemma 6.4. *Let X be an index set with n elements. Then:*

$$\sum_{A\subset X}(-1)^{n-|A|}(\mathbf{1}+\mathbf{t})_A = (-\mathbf{t})_X$$

Proof. The following identity is well-known:

$$(\mathbf{x}-\mathbf{u})_X = \sum_{A\subset X} x^{n-|A|}(-\mathbf{u})_A$$

where $\mathbf{x} = (x, ..., x)$. Let $\mathbf{u} = 1 + \mathbf{t}$ and evaluate the above identity when $x = 1$. $\square$

Theorem 6.5. *Let P^n be an n-dimensional simple convex polytope and denote by L the dual of its boundary complex. Suppose L is a flag complex and let w be an admissible vector for P^n. Then*

$$\chi_{\mathbf{t}}(L) = H_w(\mathbf{t})$$

Proof. To prove this formula we use a different "cell" structure on $\Sigma(\mathbf{t}, L)$ obtained with the help of an admissible vector. We refer the reader to [11] for more details on this construction. There is one orbit of (open) "cells" for each vertex $v \in P^n$. The dimension of C_v is $\mathrm{d}im(F_v^{in})$. C_v is constructed as follows. Let $\widehat{F_v^{in}}$ denote the union of the relative interiors of those faces $F' \in \mathcal{F}$ which are contained in F_v^{in} and contain v. The "cell" C_v consists of $\widehat{F_v^{in}}$ and all its translates under $G_{F_v^{in}}$. To prove our formula we have to show that the contribution c_v of C_v to the ℓ^2-Euler characteristic is exactly

$$\frac{(-\mathbf{t})_{F_v^{out}}}{(\mathbf{1}+\mathbf{t})_v}$$

For a face F we denote by G_F its stabilizer. If I is an index set put $|G_I| = \prod_{i|\in I}(t_i+1)$. Then $|G_F| = |G_{I(F)}|$. We have:

$$c_v = \sum_{F' \subset \widehat{F_v^{in}}} \frac{(-1)^{(dim(F')}}{|G_{F'}|} = \sum_{F' \subset \widehat{F_v^{in}}} \frac{(-1)^{(dim(F')} \frac{|G_v|}{|G_{F'}|}}{|G_v|} = \sum_{F' \subset \widehat{F_v^{in}}} \frac{(-1)^{(dim(F')}|G_{I(v)-I(F')}|}{|G_v|}$$

Since $|G_v| = (\mathbf{1}+\mathbf{t})_v$ the proof is complete if

$$\sum_{F' \subset \widehat{F_v^{in}}} (-1)^{(dim(F')}|G_{I(v)-I(F')}| = (-\mathbf{t})_{F_v^{out}}$$

Written explicitly, the above formula coincides with the identity proved in the previous lemma. Summing over vertices of P^n, the proof is completed. □

Remark 6.6. It follows from the previous theorem that $H_w(\mathbf{t})$ does not depend on the choice of the admissible vector w.

Remark 6.7. Let L be a flag complex, P^n its dual simple polytope and w an admissible vector for P. Then the ℓ^2-Euler characteristic, the growth series of the associated Coxeter group W, the associated f and "h"-polynomials are related as follows:

$$\chi_{\mathbf{t}}(L) = \frac{1}{W(\mathbf{t})} = f\left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right) = H_w(\mathbf{t}) = H(\mathbf{t})$$

This follows from Theorems 5.2, 6.3 and 6.5

Remark 6.8. Writing the corollary above in one variable we get:

$$\chi_t(L) = \frac{1}{W(t)} = f\left(\frac{-t}{1+t}\right) = \frac{h_w(-t)}{(1+t)^n} = \frac{h(-t)}{(1+t)^n}$$

where $h_w(t) = \sum_v t^{ind(v)}$ where $ind(v)$ denotes the index of the vertex v with respect to w in P.

Are the roots of the h-polynomial real ?

In this section we look at a conjecture due to Januszkiewicz and its connection with The Flag Complex Conjecture.

As an example, let's compute the h-polynomial for a triangulation of a 1-sphere as a flag complex. This means we have a p-gon with $p \geq 4$. Then :

$$h(-t) = t^2 - (p-2)t + 1$$

It is easy to see that this polynomial has real roots if and only if $(p-2)^2 - 4 \geq 0$ i.e $p \geq 4$.

For a second example, if L^2 is a triangulation of a 2-sphere we have :

$$h(-t) = -t^3 + \tfrac{f-2}{2}t^2 - \tfrac{f-2}{2}t + 1$$

where f denotes the number of 2-simplices of L^2. This can be simplified to:

$$h(-t) = (1-t)(t^2 + \tfrac{4-f}{2}t + 1)$$

and this polynomial has real roots if and only if $(\frac{4-f}{2})^2 - 4 \geq 0$ i.e $f(f-8) \geq 0$.

Following [6] (page 136), if L^{m-1} is a triangulation of a $(m-1)$-sphere then :

$$h(t) = t^m h(t^{-1})$$

This formula (which means that $h_i = h_{m-i}$) is equivalent to the Dehn-Sommerville Relations. It also implies that if α is a root of $h(t)$ then α^{-1} is a root, as well.

We now state the two conjectures that we look at in this section.

Conjecture 6.9. (Januszkiewicz) *If L^{m-1} is a triangulation of a $(m-1)$-sphere as a flag complex then the roots of $h(-t)$ are real numbers.*

Conjecture 6.10. (The flag complex conjecture)(Charney, Davis) *Suppose L^{2n-1} is a generalized homology sphere. If K is a flag complex, then $(-1)^n h(-1) \geq 0$.*

The Flag Complex Conjecture was shown to be true for $m = 2n = 4$ in [13], page 47, Theorem 11.2.1.

The following lemma is a purely elementary algebraic result.

Lemma 6.11. *Let $p(t) = t^4 - at^3 + bt^2 - at + 1$ be a polynomial with $a, b \in \mathbb{R}_+$. Then all the roots of $p(t)$ are real if :*

$$a \geq 4\ ,\ p(1) = -2a + b + 2 \geq 0 \textit{ and } a^2 - 4b + 8 \geq 0$$

Proof. Suppose $\alpha, \alpha^{-1}, \beta, \beta^{-1}$ are the roots of $p(t)$. Then :

$$p(t) = (t - \alpha)(t - \alpha^{-1})(t - \beta)(t - \beta^{-1})$$

which can be rewritten as

$$\begin{aligned} p(t) &= (t^2 - \alpha' t + 1)(t^2 - \beta' t + 1) \\ &= t^4 - (\alpha' + \beta')t^3 + (2 + \alpha'\beta')t^2 - (\alpha' + \beta')t^2 + 1 \end{aligned}$$

where $\alpha' = \alpha + \alpha^{-1}$ and $\beta' = \beta + \beta^{-1}$. Identifying the coefficients of $p(t)$ we obtain $\alpha' + \beta' = a$ and $\alpha'\beta' = b - 2$ and therefore (as roots of a quadratic equation)

$$\alpha' = \frac{a + \sqrt{a^2 - 4b + 8}}{2} \textit{ and } \beta' = \frac{a - \sqrt{a^2 - 4b + 8}}{2}$$

So, a first condition for the roots of $p(t)$ to be real is $a^2 - 4b + 8 \geq 0$.

We now look at α. Since $\alpha + \alpha^{-1} = \alpha'$ we get a quadratic equation in α which has real roots if and only if

$$(\alpha')^2 - 4 = \Big(\frac{a + \sqrt{a^2 - 4b + 8} - 4}{2}\Big)\Big(\frac{a + \sqrt{a^2 - 4b + 8} + 4}{2}\Big) \geq 0$$

This is obviously the case if $a \geq 4$.

Similarly, for β, since $\beta + \beta^{-1} = \beta'$ we get a quadratic equation in β which has real roots if and only if

$$(\beta')^2 - 4 = \Big(\frac{a - \sqrt{a^2 - 4b + 8} - 4}{2}\Big)\Big(\frac{a - \sqrt{a^2 - 4b + 8} + 4}{2}\Big) \geq 0$$

If $a \geq 4$ we have

$$a - \sqrt{a^2 - 4b + 8} - 4 \geq 0 \Longleftrightarrow$$

$$a - 4 \geq \sqrt{a^2 - 4b + 8} \iff$$

$$a^2 - 8a + 16 \geq a^2 - 4b + 8 \iff$$

$$4(-2a + b + 2) \geq 0$$

which gives the third inequality needed to ensure that the roots of $p(t)$ are real numbers. □

Theorem 6.12. *The Januszkiewicz Conjecture is true for $m \leq 4$.*

Proof. For $m = 1$ the h-polynomial is $h(-t) = -t + 1$. The cases $m = 2$ and $m = 3$ follow from the first two examples of this section. In the case $m = 3$ we mention that a flag complex has at least 8 2-simplices by Lemma A.2.

The case $m = 4$. In this case the h-polynomial is

$$h(-t) = t^4 - at^3 + bt^2 - at + 1$$

where $a = f_0 - 4$ and $b = 6 + f_3 - 2f_0$. By the previous lemma is enough to check three inequalities. We have $a - 4 = f_0 - 8 \geq 0$ because a flag complex that triangulates a 3-sphere has at least 8 vertices by Lemma A.2. Also, $a^2 - 4b + 8 = (f_0 - 4)^2 - 4(6 + f_3 - 2f_0) + 8 = f_0^2 - 4f_3 \geq 0$ because of the following argument. It is enough to show that $f_3 \leq f_0^2/4$. The Upper Bound Theorem (Theorem 18.1, page 113 in [2]) guarantees that $f_3 \leq 3f_0 - 10$. But $3f_0 - 10 \leq f_0^2/4$ (since $3x - 10 \leq x^2/4$ is equivalent to $x^2 - 12x + 40 \geq 0$ which is obviously true for all real x) so the third condition is satisfied.

Finally, $h(-1) \geq 0$ by the Flag Complex Conjecture which is true for $m = 4$. $\square$

Our next result gives a sufficient condition for The Januszkiewicz Conjecture to hold in dimension 5, i.e. for $m = 5$.

Lemma 6.13. *If L^4 is a triangulation of a 4-sphere as a flag complex and*

$$f_4 - 8f_0 + 48 \geq 0$$

then The Januszkiewicz Conjecture is true for $m = 5$.

Proof. In this case the h-polynomial is:

$$h(-t) = (1 - t)(t^4 - at^3 + bt^2 - at + 1)$$

where $a = f_0 - 6$ and $b = 10 - 2f_0 + \frac{1}{2}f_4$. By the previous Lemma is enough to check three inequalities. We have $a - 4 = f_0 - 10 \geq 0$ because a flag complex that triangulates a 4-sphere has at least 10 vertices. Also, $a^2 - 4b + 8 = (f_0 - 6)^2 - 4(10 - 2f_0 + \frac{1}{2}f_4) + 8 = (f_0 - 2)^2 - 2f_4 \geq 0$ because of the following argument. It is enough to show that $f_4 \leq (f_0 - 2)^2/2$. The Upper Bound Theorem (Theorem 18.1, page 113 in [2]) guarantees that $f_4 \leq 4f_0 - 18$. But $4f_0 - 18 \leq (f_0 - 2)^2/2$ (since $4x - 18 \leq (x - 2)^2/2$ is equivalent to $x^2 - 12x + 40 \geq 0$ which is obviously true for all x) so the third condition is satisfied.

Thus, the only remaining condition is $-2a + b + 2 = -2(f_0 - 6) + (10 - 2f_0 + \frac{1}{2}f_4) + 2 = \frac{1}{2}(f_4 - 8f_0 + 48) \geq 0$ i.e.

$$f_4 - 8f_0 + 48 \geq 0$$

□

Remark 6.14. Analyzing the results we obtained thus far we see a pattern emerging.If L^{m-1} is a triangulation as a flag complex of a $(m-1)$-sphere and $h(t)$ denotes its h-polynomial then a sufficient condition for The Januszkiewicz Conjecture to hold is the following:

Case 1. For $m = 2n$ is enough to check that $(-1)^n h(-1) \geq 0$ i.e The Flag Complex Conjecture holds.

Case 2. For $m = 2n - 1$ is enough to have that $(-1)^n h_1(-1) \geq 0$ where $h_1(t)$ is the polynomial defined by $h_1(-t) = (1-t)^{-1} h(-t)$ (since $t = 1$ is a root of $h(-t)$

when m is an odd number because $h(-t) = (-t)^m h(-t^{-1})$). It is easy to see that this condition is equivalent to $(-1)^n h'(-1) \geq 0$ where $h'(t)$ denotes the derivative of $h(t)$(if $p(t) = (t-1)p_1(t)$ then $p'(t) = p_1(t) + (t-1)p_1'(t)$ and therefore $p'(1) = p_1(1)$).

We are now in position to state the following Conjecture.

Conjecture 6.15. *If L^{m-1} is a triangulation as a flag complex of a $(m-1)$-sphere , $h(t)$ denotes its h-polynomial and $h'(t)$ denotes its derivative then a sufficient condition for The Januszkiewicz Conjecture to hold is one of the following:*

$$(-1)^n h(-1) \geq 0 \quad \textit{if} \quad m = 2n$$

or

$$(-1)^n h'(-1) \geq 0 \quad \textit{if} \quad m = 2n-1$$

A reformulation of our previous results gives the following Lemma.

Lemma 6.16. *Conjecture 6.7 is true for $m \leq 5$*

Next, we want to discuss the case $m = 6$. Again, the corresponding technical lemma is purely elementary.

Lemma 6.17. *Let $p(t) = t^6 - at^5 + bt^4 - ct^3 + bt^2 - at + 1$ be a polynomial with $a, b, c \in \mathbb{R}_+$. Then all the roots of $p(t)$ are real if :*

$$a \geq 6$$

$$-4a + b + 9 \geq 0$$

$$a^2 - 3b + 9 \geq 0$$

$$p(1) = -2a + 2b - c + 2 \leq 0$$

$$(-2a^3 + 9ab + 27a - 27c) + 2(a^2 - 3b + 9)^{3/2} \geq 0$$

$$(-2a^3 + 9ab + 27a - 27c) - 2(a^2 - 3b + 9)^{3/2} \leq 0$$

Proof. Suppose $\alpha, \alpha^{-1}, \beta, \beta^{-1}, \gamma, \gamma^{-1}$ are the roots of $p(t)$. Then :

$$p(t) = (t - \alpha)(t - \alpha^{-1})(t - \beta)(t - \beta^{-1})(t - \gamma)(t - \gamma^{-1})$$

which can be rewritten as

$$\begin{aligned}
p(t) &= (t^2 - \alpha' t + 1)(t^2 - \beta' t + 1)(t^2 - \gamma' t + 1) \\
&= t^6 - (\alpha' + \beta' + \gamma')t^5 + (3 + \alpha'\beta' + \beta'\gamma' + \gamma'\alpha')t^4 - \\
&\quad -[2(\alpha' + \beta' + \gamma') + \alpha'\beta'\gamma']t^3 + \\
&\quad +(3 + \alpha'\beta' + \beta'\gamma' + \gamma'\alpha')t^2 - (\alpha' + \beta' + \gamma')t + 1
\end{aligned}$$

where $\alpha' = \alpha + \alpha^{-1}$, $\beta' = \beta + \beta^{-1}$ and $\gamma' = \gamma + \gamma^{-1}$. Identifying the coefficients of $p(t)$ we obtain

$$\alpha' + \beta' + \gamma' = a$$

$$\alpha'\beta' + \beta'\gamma' + \gamma'\alpha' = b - 3$$

$$\alpha'\beta'\gamma' = c - 2a$$

We now form the polynomial $q(t)$ which has the roots α', β' and γ':

$$q(t) = t^3 - at^2 + (b-3)t - (c-2a)$$

If the roots of this polynomial are all real and greater than 2 then the roots of $p(t)$ are all real. A set of sufficient conditions such that the roots of $q(t)$ are real and greater than 2 consists of:

1) $q(2) \leq 0$

2) The roots of $q'(t)$ are real and greater than 2.

3) The value of $q(t)$ at the smallest root of $q'(t)$ is non-negative and the value of $q(t)$ at the largest root of $q'(t)$ is non-positive.

We now analyze these conditions:

1) $q(2) = -2a + 2b - c + 2 = p(1)$, so a first condition is

$$-2a + 2b - c + 2 \leq 0$$

2) The derivative of $q(t)$ is:

$$q'(t) = 3t^2 - 2at + (b-3)$$

and its roots are:

$$t_1 = \frac{a - \sqrt{a^2 - 3b + 9}}{3} \; and \; t_2 = \frac{a + \sqrt{a^2 - 3b + 9}}{3}$$

A first obvious condition is $a^2 - 3b + 9 \geq 0$ while, if $a \geq 6$ then the condition $t_1 \geq 2$ is equivalent to $-4a + b + 9 \geq 0$. So, 2) is implied by the following three conditions:

$$a \geq 6$$

$$a^2 - 3b + 9 \geq 0$$

$$-4a + b + 9 \geq 0$$

3) Using Maple to compute and simplify the values of $p(t)$ at t_1 and t_2 we obtain:

$$p(t_1) = \frac{1}{27}(-2a^3 + 9ab + 27a - 27c) + \frac{2}{27}(a^2 - 3b + 9)^{3/2}$$

and

$$p(t_2) = \frac{1}{27}(-2a^3 + 9ab + 27a - 27c) - \frac{2}{27}(a^2 - 3b + 9)^{3/2}$$

We note that:

$$\frac{1}{27}(-2a^3 + 9ab + 27a - 27c) = q(\frac{a}{3})$$

and

$$a^2 - 3b + 9 = -3q'(\frac{a}{3})$$

Multiplying by 27 the inequalities $p(t_1) \geq 0$ and $p(t_2) \leq 0$ we obtain:

$$(-2a^3 + 9ab + 27a - 27c) + 2(a^2 - 3b + 9)^{3/2} \geq 0$$

$$(-2a^3 + 9ab + 27a - 27c) - 2(a^2 - 3b + 9)^{3/2} \leq 0$$

$\square$

Remark 6.18. Some of the conditions in the previous Lemma are automatically satisfied for the h-polynomial of a triangulation as a flag complex of a 5-sphere but some reduce to stronger versions of The Upper Bound Theorem and The Lower Bound Theorem.

In this case the h-polynomial is:

$$h(-t) = t^6 - at^5 + bt^4 - ct^3 + bt^2 - at + 1$$

where $a = f_0 - 6$, $b = f_1 - 5f_0 + 15$ and $c = 20 - 10f_0 + 4f_1 - f_2$

The first condition of the previous Lemma is $a \geq 6$, which is equivalent to $f_0 \geq 12$. But a flag complex that triangulates a 5-sphere has at least 12 vertices so the first condition is satisfied.

The second condition of the previous Lemma is $-4a+b+9 \geq 0$, which is equivalent to $f_1 \geq 9f_0 - 48$. The corresponding inequality given by The Lower Bound Theorem is $f_1 \geq 6f_0 - 48$. It is obvious that our condition is not a consequence of this later condition.

The third condition of the previous Lemma is $a^2 - 3b + 9 \geq 0$, which is equivalent to $f_0^2 + 3f_0 - 3f_1 \geq 0$. The corresponding inequality given by The Upper Bound Theorem is $f_0^2 - f_0 - 2f_1 \geq 0$. It is obvious that our condition is not a consequence of this later condition.

Computation of ℓ^2-Betti numbers

In this section we attempt to begin a systematic study of ℓ^2-Betti numbers for buildings for which L is a 1-dimensional flag complex. The first case we analyze is when L is a tree.

Proposition 6.19. *(ℓ^2-Betti numbers of a tree) Let T be a tree with at least 2 edges. Then:*

$$\beta_i^{\mathbf{t}}(T) = \begin{cases} -\dfrac{1}{W(\mathbf{t})} & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases}$$

Proof. $\beta_0^{\mathbf{t}}(T) = 0$ because G is infinite. The fact that $\beta_i^{\mathbf{t}}(T) = 0$ for $i \geq 2$ follows by induction on the number of edges of the tree. Any tree T is obtained by one point union of an one simplex σ_1 and another tree T' (with one less edge than T). By Lemma 5.9, for $i \geq 2$ we have $\beta_i^{\mathbf{t}}(T) = \beta_i^{\mathbf{t}}(T') + \beta_i^{\mathbf{t}}(\sigma_1) = 0$. Thus, the only non-zero Betti number is $\beta_1^{\mathbf{t}}(T) = -\chi_{\mathbf{t}}(T) = -\dfrac{1}{W(\mathbf{t})}$. □

When T is a tree with k edges and $\mathbf{t} = t$ we obtain a very simple formula.

Corollary 6.20. *Let T_k be a tree with k edges, $k \geq 2$. Then:*

$$\beta_i^{t}(T_k) = \begin{cases} \dfrac{(k-1)t - 1}{(t+1)^2} & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases}$$

Proof. By the previous lemma we analyze only the case $i = 1$. We proceed by induction on k. For $k = 2$, a tree with two edges is just a cone over P_2. Hence,

$$\beta_1^t(T_2) = \frac{1}{t+1}\frac{t-1}{t+1} = \frac{t-1}{(t+1)^2}$$

T_k is obtained by one point union of an one simplex σ_1 and another tree T_{k-1}. We have $\chi_t(T_k) = \chi_t(T_{k-1}) + \chi_t(\sigma_1) - \chi_t(P_1)$ i.e.

$$\chi_t(T_k) = \frac{-(k-2)t+1}{(t+1)^2} + \frac{1}{(t+1)^2} - \frac{1}{t+1} = -\frac{(k-1)t-1}{(t+1)^2}$$

$\square$

The general case of a tree is treated next.

Lemma 6.21. *Let T be a tree with at least 2 edges. The only non-zero ℓ^2-Betti number of T is given by:*

$$\beta_1^{\mathbf{t}}(T) = -1 + \sum_i \frac{t_i}{1+t_i} - \sum_{(i,j)} \frac{t_i t_j}{(1+t_i)(1+t_j)}$$

where the first sum runs over all vertices of T and the second sum runs over all pairs (i,j) such that $\{t_i, t_j\}$ spans an edge of T.

Proof. Just like for the previous result we proceed by induction on the number of edges of the tree. A tree with two edges is a cone over P_2. It is easily verified that this formula agrees with the formula previously obtained for the ℓ^2-Betti numbers of a cone. In general, whenever we add an edge (corresponding to (t_{k-1}, t_k)) at a vertex (corresponding to t_{k-1}) the contribution to $\chi_{\mathbf{t}}(T)$ is

$$\frac{1}{(t_{k-1}+1)(t_k+1)} - \frac{1}{t_{k-1}+1}$$

which is easily seen to coincide to

$$-\frac{t_k}{t_k+1} + \frac{t_k t_{k-1}}{(t_{k-1}+1)(t_k+1)}$$

$\square$

The next case of interest is when L is n-gon, $n \geq 4$. This case was solved by Dymara in [15] in a more general setting. It states that ℓ^2-Betti numbers are concentrated in one dimension. Recall that $\mathcal{R}$ denotes the region of convergence of $W(\mathbf{t})$.

Lemma 6.22. *(ℓ^2-Betti numbers when L is a n-gon) Let L denote an n-gon, $n \geq 4$. We have:*

(i) If $\mathbf{t}^{-1} \in \mathcal{R}$ then the only non-trivial ℓ^2-Betti number is in dimension 2 and

$$\beta_2^{\mathbf{t}}(L) = 1 - \sum_i \frac{t_i}{1+t_i} + \sum_{(i,j)} \frac{t_i t_j}{(1+t_i)(1+t_j)}$$

where the first sum runs over all vertices of L and the second sum runs over all pairs (i,j) such that $\{t_i, t_j\}$ spans an edge of L, i.e. $(i,j) \in \{(1,2),(2,3),...,(n-1,n),(n,1)\}$.

(ii) If $\mathbf{t}^{-1} \notin \mathcal{R}$ *then the only non-trivial* ℓ^2*-Betti number is in dimension* 1 *and*

$$\beta_1^{\mathbf{t}}(L) = -1 + \sum_i \frac{t_i}{1+t_i} - \sum_{(i,j)} \frac{t_i t_j}{(1+t_i)(1+t_j)}$$

where the first sum runs over all vertices of L *and the second sum runs over all pairs* (i,j) *such that* $\{t_i, t_j\}$ *spans an edge of* L*, i.e.* $(i,j) \in \{(1,2),(2,3),...,(n-1,n),(n,1)\}$.

Proof. When L is an n-gon the only possibly non-trivial ℓ^2-Betti numbers are in dimension 1 and 2. The result follows from Theorems 1 and 2 of [15] and the fact that $\chi_{\mathbf{t}}(L)$ can be rewritten as shown. We include here a sketch of the proof of Dymara's result.

If we determine $\beta_2^{\mathbf{t}}(L)$ then $\beta_1^{\mathbf{t}}(L)$ is completely determined since we can compute $\chi_{\mathbf{t}}(L)$. Therefore, we turn our attention to studying $\beta_2^{\mathbf{t}}(L)$. We have $C^2_{(2)}(\Sigma(\mathbf{t}, L)) \cong \ell^2(G_L)$. There is a bijection between the set of chambers of $\Sigma(\mathbf{t}, L)$ and elements of G_L. $1 \in G_L$ corresponds to the fundamental chamber b of $\Sigma(\mathbf{t}, L)$ under this bijection. Let $p : G_L \mapsto W_L$ be the obvious map. In top dimension we have the following harmonic cochain: for $g \in G_L$ put

$$\phi(g) = (-\mathbf{t}_{p(g)})^{-1}$$

One can calculate that $||\phi||^2 = \sum_{w \in W} (\mathbf{t}_w)^{-1}$. Therefore, $\phi \in \ell^2(G_L)$ if and only if $\mathbf{t}^{-1} \in \mathcal{R}$. Moreover, if $\phi \in \ell^2(G_L)$ then $||\phi||^2 = W(\mathbf{t}^{-1})$. One can show that

$\mathcal{H}^2_{(2)}(\Sigma(\mathbf{t}, L))$ (realized here as the space of square-summable harmonic 2-cochains) is non-zero if and only if $\mathbf{t}^{-1} \in \mathcal{R}$. So, if $\mathbf{t}^{-1} \notin \mathcal{R}$ then $\beta_2^{\mathbf{t}}(L) = 0$.

Suppose $\mathbf{t}^{-1} \in \mathcal{R}$. Define $P \in \mathcal{N}(G_L) \subset \ell^2(G_L)$ by

$$P = \frac{1}{W(\mathbf{t}^{-1})} \sum_{g \in G_L} (-\mathbf{t}_{p(g)})^{-1} g$$

One can show that P is the orthogonal projection on the space of square-summable harmonic 2-cochains. We have

$$\beta_2^{\mathbf{t}}(L) = \mathrm{t}r_{G_L}(P) = \frac{1}{W(\mathbf{t}^{-1})} = \frac{1}{W(\mathbf{t})} = \chi_{\mathbf{t}}(L)$$

This completes the sketch of this proof. $\square$

Again, when $\mathbf{t} = t$ we obtain explicit formulas (as in [15]). For an n-gon

$$W(t) = \frac{(1+t)^2}{1 + (2-n)t + t^2}$$

and the radius of convergence lies between $(n-2)^{-1}$ and $(n-3)^{-1}$. This gives the following:

Corollary 6.23. *Let L denote an n-gon, $n \geq 4$. We have:*

(i) If $t \geq n-2$ then the only non-trivial ℓ^2-Betti number is in dimension 2 and

$$\beta_2^{\mathbf{t}}(L) = 1 - \frac{nt}{(1+t)^2}$$

(ii) If $t < n-2$ then the only non-trivial ℓ^2-Betti number is in dimension 1 and

$$\beta_1^{\mathbf{t}}(L) = -1 + \frac{nt}{(1+t)^2}$$

We are now in position to compute ℓ^2-Betti numbers when L is any one-dimensional flag complex that contains exactly one circuit. We begin with the following lemma.

Lemma 6.24. *Let $L' = L \cup T$ such that $L \cap T = P_1$ (with vertex coresponding to t_{i_1}), where T is a tree and L is not a simplex. Then for $i \neq 2$*

$$\beta_i^{\mathbf{t}}(L') = \beta_i^{\mathbf{t}}(L)$$

and

$$\beta_1^{\mathbf{t}}(L') = \beta_1^{\mathbf{t}}(L) + \sum_{i \neq i_1} \frac{t_i}{1+t_i} - \sum_{(i,j)} \frac{t_i t_j}{(1+t_i)(1+t_j)}$$

where the first runs over all vertices of T except P_1 and the second sum runs over all pairs (i,j) such that $\{t_i, t_j\}$ spans an edge of T.

Proof. This follows from Lemma 5.9 and Lemma 6.21. The terms

$$-1 + \frac{t_{i_1}}{1+t_{i_1}} \text{ and } \frac{1}{1+t_{i_1}}$$

cancel out. $\square$

Let L be a one-dimensional flag complex that contains only one circuit. This means that L is obtained from an n-gon P, $n \geq 4$ by repeated one point unions with trees $T_1, T_2, ..., T_n$ at all vertices of P. Let $\mathcal{R}$ denote the region of convergence of the growth series corresponding to P.

Proposition 6.25. *(ℓ^2-Betti numbers when L contains only one circuit) Suppose L is as above. We have:*

i) If $\mathbf{t}^{-1} \notin \mathcal{R}$ then the only non-trivial ℓ^2-Betti number is in dimension 1 and

$$\beta_1^{\mathbf{t}}(L) = -1 + \sum_i \frac{t_i}{1+t_i} - \sum_{(i,j)} \frac{t_i t_j}{(1+t_i)(1+t_j)}$$

where the first sum runs over all vertices of L and the second sum runs over all pairs (i,j) such that $\{t_i, t_j\}$ spans an edge of L.

(ii) If $\mathbf{t}^{-1} \in \mathcal{R}$ then non-trivial ℓ^2-Betti numbers appear in dimension 1 and 2:

$$\beta_1^{\mathbf{t}}(L) = \sum_i \frac{t_i}{1+t_i} - \sum_{(i,j)} \frac{t_i t_j}{(1+t_i)(1+t_j)}$$

where the first sum runs over all vertices of $L - P$, the second sum runs over all pairs (i,j) such that $\{t_i, t_j\}$ spans an edge of some T_i and

$$\beta_2^{\mathbf{t}}(L) = 1 - \sum_i \frac{t_i}{1+t_i} + \sum_{(i,j)} \frac{t_i t_j}{(1+t_i)(1+t_j)}$$

where the first sum runs over all vertices of P and the second sum runs over all pairs (i,j) such that $\{t_i, t_j\}$ spans an edge of P.

Proof. Both parts follow from the previous lemma and Dymara's result on ℓ^2-Betti numbers of an n-gon. $\square$

The above formulas are much simpler when $\mathbf{t} = t$. An interesting case is when $t \geq n-2$. Suppose that the number of edges of L not in P is k. Then

$$\beta_i^t(T_k) = \begin{cases} 1 - n\dfrac{t}{(t+1)^2} & \text{if } i = 2, \\ k\dfrac{t}{(t+1)^2} & \text{if } i = 1, \\ 0 & \text{if } i \neq 1, 2. \end{cases}$$

We begin discussing the case where L contains two circuits. Let P be an n-gon and P' be an m-gon, $n \geq 4$, $m \geq 4$. The first three cases we analyze are disjoint unions, one point unions and union of two polygons that have one edge in common. It seems interesting to note that in all these cases $\beta_1^{\mathbf{t}}(L)$ is non-zero, no matter the value of $\mathbf{t}$.

Lemma 6.26. *(ℓ^2-Betti numbers of a disjoint union of two polygons) Let $L = P \cup P'$ such that $P \cap P' = \emptyset$. Then*

$$\beta_i^{\mathbf{t}}(L) = \begin{cases} \beta_2^{\mathbf{t}}(P) + \beta_2^{\mathbf{t}}(P') & \textit{if } i = 2, \\ \beta_1^{\mathbf{t}}(P) + \beta_1^{\mathbf{t}}(P') + 1 & \textit{if } i = 1, \\ 0 & \textit{if } i \neq 1, 2. \end{cases}$$

Proof. The result follows immediately from the Mayer-Vietoris sequence. $\square$

Lemma 6.27. *(ℓ^2-Betti numbers of a one point union of two polygons) Let $L = P \cup P'$ such that $P \cap P' = P_1$ (with t being the variable corresponding to P_1). Then*

$$\beta_i^{\mathbf{t}}(L) = \begin{cases} \beta_2^{\mathbf{t}}(P) + \beta_2^{\mathbf{t}}(P') & \text{if } i = 2, \\ \beta_1^{\mathbf{t}}(P) + \beta_1^{\mathbf{t}}(P') + \dfrac{1}{t+1} & \text{if } i = 1, \\ 0 & \text{if } i \neq 1, 2. \end{cases}$$

Proof. The result follows immediately from the Mayer-Vietoris sequence. □

Lemma 6.28. *(ℓ^2-Betti numbers of two polygons which have a common edge) Let $L = P \cup P'$ such that $P \cap P' = \sigma_1$ (with (t_1, t_2) being the variables corresponding to σ_1). Then*

$$\beta_i^{\mathbf{t}}(L) = \begin{cases} \beta_2^{\mathbf{t}}(P) + \beta_2^{\mathbf{t}}(P') & \text{if } i = 2, \\ \beta_1^{\mathbf{t}}(P) + \beta_1^{\mathbf{t}}(P') + \dfrac{1}{(t_1+1)(t_2+1)} & \text{if } i = 1, \\ 0 & \text{if } i \neq 1, 2. \end{cases}$$

Proof. The result follows immediately from the Mayer-Vietoris sequence. □

Remark 6.29. In all these cases the process can be iterated and corresponding formulas can be written.

Before analyzing other cases when L contains more then one circuit we discuss ℓ^2-Betti numbers and ℓ^2-homology of pairs (L, A), where L is as before a one-dimensional flag complex and A is a full subcomplex.

Lemma 6.30. *Let I_k denote a triangulation of a 1-disc (with k edges, $k \geq 2$) and let ∂I_k be its boundary. Then:*

i) If $t \geq k-1$, $\beta_2^t(I_k, \partial I_k) = \dfrac{t^2-(k-1)t}{(t+1)^2}$ and all other ℓ^2-Betti numbers are trivial.

i) If $t \leq k-1$, $\beta_1^t(I_k, \partial I_k) = -\dfrac{t^2-(k-1)t}{(t+1)^2}$ and all other ℓ^2-Betti numbers are trivial.

Proof. We compute

$$\chi_t(I_k, \partial I_k) = \chi_t(I_k) - \chi_t(\partial I_k) = \frac{(k-1)t-1}{(t+1)^2} - \frac{t-1}{t+1} = \frac{t^2-(k-1)t}{(t+1)^2}$$

By Corollary 10.4 of [16] we have $\beta_2^t(I_k, \partial I_k) = 0$ for $t \leq k-1$ and $\beta_2^t(I_k, \partial I_k) = \chi_t(I_k, \partial I_k)$ for $t \leq k-1$. The result follows since $\beta_0^t(I_k, \partial I_k) = 0$. □

Lemma 6.31. *Let I denote a triangulation of a 1-disc and let $P = \{v_0, ..., v_n\}$ be a subset of the vertex set of I such that P is a full subcomplex of I and $\partial I = \{v_0, v_n\}$. Denote by I_i the 1-disc that joins two consecutive points, v_{i-1} and v_i of P. Then*

$$\mathfrak{h}_*^t(I, P) = \bigoplus_{i=1}^{n} \mathfrak{h}_*^t(I_i, \partial I_i)$$

Proof. The result follows by induction on n. For $n = 3$, let $L_1 = \{v_0\} \cup I_2$ and $L_2 = \{v_2\} \cup I_1$. We have $L_1 \cup L_2 = I$ and $L_1 \cap L_2 = P$. By *ii*) of Lemma 5.5 we get

$$\mathfrak{h}_*^t(I, P) = \mathfrak{h}_*^t(L_1, P) \oplus \mathfrak{h}_*^t(L_2, P)$$

By Excision, $\mathfrak{h}_*^t(L_1, P) \cong \mathfrak{h}_*^t(I_1, \partial I_1)$ and $\mathfrak{h}_*^t(L_2, P) \cong \mathfrak{h}_*^t(I_2, \partial I_2)$. Thus, the result follows for $n = 3$.

Let n be arbitrary. Let $L_1 = \{v_0\} \cup I_2 \cup ... \cup I_n$ and $L_2 = I_1 \cup \{v_2, ..., v_n\}$. We have $L_1 \cup L_2 = I$ and $L_1 \cap L_2 = P$. By *ii*) of Lemma 5.5 we get

$$\mathfrak{h}_*^t(I, P) = \mathfrak{h}_*^t(L_1, P) \oplus \mathfrak{h}_*^t(L_2, P)$$

By Excision, $\mathfrak{h}_*^t(L_1, P) \cong \mathfrak{h}_*^t(I_2 \cup ... \cup I_n, \{v_1, ..., v_n\})$ and $\mathfrak{h}_*^t(L_2, P) \cong \mathfrak{h}_*^t(I_1, \partial I_1)$. Using the induction hypothesis for the first summand the result follows. □

The lemma above can be used to compute the relative ℓ^2-homology for a pair (S, A) when S is a polygon (i.e a triangulation of a 1-sphere) and A is a full subcomplex consisting of points.

Proposition 6.32. *Let S be a triangulation of a 1-sphere as a flag complex and let $P_k = \{v_1, ..., v_k\}$, $k \geq 2$, be a full subcomplex. Denote by I_i the 1-disc that joins two consecutive points, v_i and v_{i+1} of P_k. Then*

$$\mathfrak{h}_*^t(S, P_k) = \bigoplus_{i=1}^{k} \mathfrak{h}_*^t(I_i, \partial I_i)$$

Proof. For $k = 2$, let $L_1 = I_1$ and $L_2 = I_2$. We have $L_1 \cup L_2 = S$ and $L_1 \cap L_2 = P_k$. By *ii*) of Lemma 5.5 we get

$$\mathfrak{h}_*^t(S, P_2) = \mathfrak{h}_*^t(I_1, \partial I_1) \oplus \mathfrak{h}_*^t(I_2, \partial I_2)$$

When k is arbitrary, let $L_1 = I_1 \cup \{v_3, ..., v_k\}$ and $L_2 = I_2 \cup ... \cup I_k$. We have $L_1 \cup L_2 = S$ and $L_1 \cap L_2 = P_k$. By $ii)$ of Lemma 5.5 we get

$$\mathfrak{h}^t_*(S, P) = \mathfrak{h}^t_*(L_1, P_k) \oplus \mathfrak{h}^t_*(L_2, P_k)$$

By Excision, $\mathfrak{h}^t_*(L_1, P_k) \cong \mathfrak{h}^t_*(I_1, \partial I_1)$. Using the previous lemma to rewrite the second term the result follows. □

The following lemma is a good source of interesting examples. Let P_2 denote, as usual, two distinct points. Attach to P_2 several 1-discs I_i (each I_i contains at least two edges), $i = 1, ..., k$, along the boundary ∂I_i and denote by L_k the one-dimensional flag complex obtained this way.

Lemma 6.33. *With L_k defined as above we have:*

$$\mathfrak{h}^t_*(L_k, P_2) = \bigoplus_{i=1}^{k} \mathfrak{h}^t_*(I_i, \partial I_i)$$

Proof. The result follows by induction on k. The case $k = 2$ coincides with the case $k = 2$ of the previous lemma.

Let k be arbitrary. With $A = L_{k-1}$ and $B = I_k$ we have $A \cup B = L_k$ and $A \cap B = P_2$. By $ii)$ of Lemma 5.5 we get

$$\mathfrak{h}^t_*(L_k, P_2) = \mathfrak{h}^t_*(L_{k-1}, P_2) \oplus \mathfrak{h}^t_*(I_k, \partial I_k)$$

Using the induction hypothesis to rewrite the first term we obtain the desired formula.

□

Remark 6.34. We discuss in detail, for $\mathbf{t} = t$, the following example. Let S denote a 4-gon and I_k, $k \geq 2$ denote a tringulation of a 1-disc. Let $L = S \cup I_k$ such that $S \cap I_k$ is a disjoint union of two opposite vertices in S. We compute the ℓ^2-Betti numbers of L. Clearly, the only possibly non-trivial ℓ^2-Betti numbers of L are in dimension 1 and 2. Recall that $\beta_i^t(S) = 0$ for $i \neq 2$ and $\beta_i^t(I_k) = 0$ for $i \neq 1$. Consider the following portion of the Mayer-Vietoris sequence:

$$\to \mathfrak{h}_1^t(S \cap I_k) \to \mathfrak{h}_1^t(S) \oplus \mathfrak{h}_1^t(I_k) \to \mathfrak{h}_1^t(L) \to$$

Since $\mathfrak{h}_1^t(S) = 0$, the first map coincides with the map induced by the inclusion $P_2 \hookrightarrow I_k$. But we have seen (see Lemma 6.30) that this map is surjective for $t \geq k-1$ and injective for $t \leq k-1$. Hence, if $t \geq k-1$ we have $\beta_1^t(L) = 0$ while if $t \leq k-1$ we have $\beta_2^t(L) = \beta_2^t(S)$. On the other hand, an easy computation shows that

$$\chi_t(L) = \frac{2t^2 - (k+1)t + 1}{(t+1)^2}$$

Summarizing:

i) If $t \geq k-1$ then

$$\beta_i^t(L) = \begin{cases} \dfrac{t^2 - (k+1)t + 1}{(t+1)^2} & \text{if } i = 2, \\ 0 & \text{if } i \neq 2. \end{cases}$$

ii) If $t \leq k-1$ then

$$\beta_i^t(L) = \begin{cases} \dfrac{t^2-2t+1}{(t+1)^2} & \text{if } i=2, \\ \dfrac{-t^2+(k-1)t}{(t+1)^2} & \text{if } i=1, \\ 0 & \text{if } i \neq 1,2. \end{cases}$$

When L contains more then two circuits, aside from disjoint unions, one point unions and union of several polygons that have one edge in common we can also discuss the case of joins of P_n and P_m. It is interesting to note that in this case the only non-trivial ℓ^2-Betti number is in dimension 2.

Lemma 6.35. *(ℓ^2-Betti numbers of the join of n points and m points) The only non-trivial ℓ^2-Betti number of $P_n * P_m$ is in dimension 2 and is given by*

$$\beta_2^{\mathbf{t}}(P_n * P_m) = \Big(-1+\sum_{i=1}^{n}\frac{t_i}{t_i+1}\Big)\Big(-1+\sum_{j=1}^{m}\frac{t_{n+i}}{t_{n+i}+1}\Big)$$

Proof. Recall that the only non-trivial ℓ^2-Betti number of P_n (and P_m) is in dimension 1. The result follows from The Künneth Formula. □

APPENDIX A

SIMPLICIAL COMPLEXES AND FLAG COMPLEXES

In this section we recall definitions and notations regarding simplicial complexes and flag complexes for the reader's convenience. All this material can be found in [13].

Given a simplicial complex L, denote by $\mathcal{S}(L)$ the set of simplices in L together with the empty set $\emptyset$. It is partially ordered by inclusion. $\mathcal{S}_i(L)$ denotes the subset of $\mathcal{S}(L)$ consisting of the simplices of dimension i. For notational purposes it will be convenient to regard $\emptyset$ as an element of dimension -1 in $\mathcal{S}(L)$. $\mathcal{S}_0(L)$ is the *vertex set* of L.

A subcomplex A of L is a *full subcomplex* if whenever $\sigma \in \mathcal{S}(L)$ is such that the vertex set of σ is contained in $\mathcal{S}(A)$, then $\sigma \in \mathcal{S}(A)$.

Suppose L_1 and L_2 are simplicial complexes. Define a partial order on $\mathcal{S}(L_1) \times \mathcal{S}(L_2)$ by $(\sigma, \tau) \leq (\sigma', \tau')$ if and only if $\sigma \leq \sigma'$ and τ' and $\tau \leq \tau'$. For example, if σ and τ are simplices of dimension i and j, respectively, then $\mathcal{S}(\sigma) \times \mathcal{S}(\tau)$ is isomorphic to the poset of faces of a simplex of dimension $i+j+1$. We denote this simplex by $\sigma * \tau$.

It follows that there is a unique simplicial complex $L_1 * L_2$, called the *join* of L_1 and L_2, characterized by the property that $\mathcal{S}(L_1 * L_2)$ is isomorphic to $\mathcal{S}(L_1) \times \mathcal{S}(L_2)$. The empty element of $\mathcal{S}(L_1 * L_2)$ corresponds to $(\emptyset, \emptyset) \in \mathcal{S}(L_1) \times \mathcal{S}(L_2)$ and the vertex set of $L_1 * L_2$ coresponds to $(\mathcal{S}_0(L_1) \times \{\emptyset\})(\{\emptyset\} \times \mathcal{S}_0(L_2))$. As is well known, the geometric realization of $L_1 * L_2$ is homeomorphic to the space formed from $L_1 \times L_2 \times [-1, 1]$ by identifying points of the form $(x_1, x_2, -1)$ with $(x_1', x_2, -1)$ and those of the form $(x_1, x_2, +1)$ with $(x_1, x_2', +1)$.

The *cone* on a simplicial complex L is the join of L with a single point, say v. We will denote it by CL (or by C_vL whe we wish to distinguish the *cone point* v).

The *suspension* of L, denoted by SL, is the join of L with a 0-sphere $\mathbb{S}^0$.

A symmetric and reflexive relation is an *incidence relation*. Suppose Q is a set equipped with an incidence relation. A *flag* in Q is a nonempty finite subset of pairwise related elements. There is an associated simplicial complex, $\mathrm{Flag}(Q)$, the i-simplices of which are flags of cardinality $i+1$. (The vertex set of $\mathrm{Flag}(Q)$ is Q and two vertices are connected by an edge if and only if they are incident.) An important special case is where the incidence relation is given by symmetrizing the partial order on a poset P. A flag in P is then a nonempty finite totally ordered subset. In this case, $\mathrm{Flag}(P)$ is called the *derived complex* of P. When P is the poset of cells of a regular CW complex X, then $\mathrm{Flag}(P)$ can be identified with the barycentric subdivision of X. As another example, if L is a simplicial complex, then $\mathrm{Flag}(\mathcal{S}(L))$ is the cone on the barycentric subdivision of L. The vertex corresponding to $\emptyset$ is the cone point.

Given a poset P and an element $x \in P$, define a subposet by $P_{\leq x} = \{y \in P \mid y \leq x\}$. Subposets $P_{\geq x}$, $P_{<x}$ and $P_{>x}$ are defined similarly. Associated to any poset P there is a simplicial complex $|P|$, called its geometric realization; its vertex set is P and a nonempty finite subset of P spans a simplex if and only if is totally ordered.

If v is a vertex of L, then L_v, the *link* of v in L, is the union of all simplices σ such that

(a) v is not a vertex of σ and

(b) σ and v span a simplex of L.

The subcomplex L_v is characterized by the condition that

$$\mathcal{S}(L_v) \cong \mathcal{S}(L)_{\geq v}.$$

The *star* of v in L, denoted $\mathrm{St}(v, L)$, is the union of all simplices which contain$v$. Thus, $\mathrm{St}(v, L) = C_v L_v$. The *open star* of v is the complement of L_v in $\mathrm{St}(v, L)$. It is an open subset of L.

Recall that a simplicial complex L is a *flag complex* if any nonempty finite set of vertices which are pairwise connected by edges span a simplex in L. In other words,

L is a flag complex if and only if whenever a subcomplex isomorphic to the 1-skeleton of a simplex is in L, then the entire simplex lies in L. (In [18] Gromov used the terminology that L satisfies the "no Δ condition" for this property.)

If Q is a set with an incidence relation, then $\mathrm{Flag}(Q)$ is a flag complex. Conversely, any flag complex arises from this construction. (Indeed, given a flag complex L, define two vertices in $\mathcal{S}_0(L)$ to be *incident* if they are connected by an edge. Then $L \cong \mathrm{Flag}(\mathcal{S}_0(L))$.)

An m-gon (i.e., a triangulation of a circle into m edges) is a flag complex if and only if $m \geq 4$.

Any full subcomplex of a flag complex is a flag complex.

If v is a vertex of a flag complex L, then its link L_v and its star $\mathrm{St}(v, L)$ are both full subcomplexes. Hence, they are both flag complexes.

The join of two flag complexes is again a flag complex. In particular, the cone on a flag complex is a flag complex and the suspension of a flag complex is a flag complex.

For any set of vertices T of L, let $N(T)$ be the union of all open stars of vertices in T. We will use $L - T$ to denote the complement of $N(T)$ in L. In other words, $L - T$ is the full subcomplex of L spanned by $\mathcal{S}_0(L) - T$. For example,for any vertex

s of L, $L - s$ denotes the complement of the open star of s in L. Similarly, if A is any subcomplex of L, then we will write $L - A$ for $L - \mathcal{S}_0(A)$.

Lemma A.1. *If K^{m-1} is a triangulation of a $(m-1)$-sphere as a flag complex then*

$$\#(\mathcal{S}_0(K^{m-1})) \geq 2m$$

i.e. the number of vertices of K^{m-1} is no less then $2m$.

Proof. We proceed by induction on m. If $m = 2$, as was pointed out above, a p-gon is a flag complex if and only if $p \geq 4$. But p also represents the number of vertices of a p-gon.

Let v be a vertex of K^{m-1}. Its link in K^{m-1}, denoted K_v^{m-1}, is a full subcomplex which triangulates a $(m-2)$-sphere. By the induction hypothesis, K_v^{m-1} contains at least $2(m-2)$ vertices. But, besides v, K^{m-1} should contain at least another vertex. Hence, K^{m-1} contains at least $2(m-1)+1+1 = 2m$ vertices. □

In a similar way we obtain a lower bound for the number of codimension-one simplices in the same setting.

Lemma A.2. *If K^{m-1} is a triangulation of a $(m-1)$-sphere as a flag complex then*

$$\#(\mathcal{S}_{m-1}(K^{m-1})) \geq 2^m$$

i.e. the number of top dimensional simplices of K^{m-1} is no less then 2^m.

Proof. We proceed by induction on m. If $m = 2$, as was pointed out above, a p-gon is a flag complex if and only if $p \geq 4$. But p represents the number of top dimensional simplices of a p-gon. Let v be a vertex of K^{m-1}. Its link in K^{m-1}, denoted K_v^{m-1}, is a full subcomplex which triangulates a $(m-2)$-sphere. By the induction hypothesis, K_v^{m-1} contains 2^{m-1} $(m-2)$-simplices. The star of v in K^{m-1}, denoted $\mathrm{St}(v, K^{m-1})$ is the cone over K_v^{m-1} and hence contains exactly 2^{m-1} top dimensional simplices. But, besides v, K^{m-1} should contain at least another vertex. Hence, K^{m-1} contains at least 2^{m-1} more top dimensional simplices. Therefore, the total number of top dimensional simplices is at least $2^{m-1} + 2^{m-1} = 2^m$ simplices. $\square$

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