

Determining the action dimension of an Artin group by using its complex of abelian subgroups

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ABSTRACT

Suppose that (W, S) is a Coxeter system with associated Artin group A and with a simplicial complex L as its nerve. We define the notion of a ‘standard abelian subgroup’ in A . The poset of such subgroups in A is parameterized by the poset of simplices in a certain subdivision L_{\circlearrowright} of L . This complex of standard abelian subgroups is used to generalize an earlier result from the case of right-angled Artin groups to case of general Artin groups, by calculating, in many instances, the smallest dimension of a manifold model for BA . (This is the ‘action dimension’ of A denoted $\text{actdim } A$.) If $H_d(L; \mathbb{Z}/2) \neq 0$, where $d = \dim L$, then $\text{actdim } A \geq 2d + 2$. Moreover, when the $K(\pi, 1)$ -Conjecture holds for A , the inequality is an equality.

Introduction

Given a discrete, torsion-free group π , its *geometric dimension*, $\text{gd } \pi$, is the smallest dimension of a CW-complex model for its classifying space $B\pi$; its *action dimension*, $\text{actdim } \pi$, is the smallest dimension of a manifold model for $B\pi$. From general principles, $\text{actdim } \pi \leq 2 \text{gd } \pi$. Taking the universal cover of such a manifold model, we see that $\text{actdim } \pi$ can alternatively be defined as the smallest dimension of a contractible manifold which admits a proper π -action. The basic method for calculating $\text{actdim } \pi$ comes from work of Bestvina, Kapovich and Kleiner [3]. They show that if there is a finite complex K such that the 0-skeleton of the cone (of infinite radius) on K coarsely embeds in π and if the van Kampen obstruction for embedding K in S^m is nonzero, then $\text{actdim } \pi \geq m + 2$ (such a complex K is an *m-obstructor*).

A *Coxeter matrix* (m_{st}) on a set S is an $(S \times S)$ -symmetric matrix with 1’s on the diagonal and with each off-diagonal entry either an integer ≥ 2 or the symbol ∞ . Associated to a Coxeter matrix there is a *Coxeter group* W defined by the presentation with set of generators S and with relations: $(st)^{m_{st}} = 1$. The pair (W, S) is a *Coxeter system*. For a subset $T \leq S$, the subgroup generated by T is denoted W_T and called a *special subgroup*. The pair (W_T, T) also is a Coxeter system. The subset T is *spherical* if W_T is finite. Let $\mathcal{S}(W, S)$ denote the poset of spherical subsets of S (including the empty set). There is a simplicial complex, denoted by $L(W, S)$ (or simply by L) called the *nerve* of (W, S) . Its simplices are the elements of $\mathcal{S}(W, S)_{>\emptyset}$. In other words, the vertex set of L is S and its simplices are precisely the nonempty spherical subsets of S . Note that a subset $\{s, t\} < S$ of cardinality 2 is an edge of L if and only if $m_{st} \neq \infty$.

Given a Coxeter matrix (m_{st}) , there is another group A called the associated *Artin group*. It has a generator a_s for each vertex $s \in S$ and for each edge $\{s, t\}$ of L , an *Artin relation*:

$$\underbrace{a_s a_t \dots}_{m_{st} \text{ terms}} = \underbrace{a_t a_s \dots}_{m_{st} \text{ terms}}. \quad (0.1)$$

The *special subgroup* A_T is the subgroup generated by $\{a_s \mid s \in T\}$; it can be identified with the Artin group associated to (W_T, T) . The subgroup A_T is *spherical* if T is spherical.

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A spherical Coxeter group W_T acts as a group generated by linear reflections on \mathbb{R}^n , where $n = \text{Card } T$. After tensoring with \mathbb{C} , it becomes a linear reflection group on \mathbb{C}^n . Deligne [14] proved that the complement of the arrangement of reflecting hyperplanes in \mathbb{C}^n is aspherical. Since W_T acts freely on this complement, it follows that its quotient by W_T is a model for BA_T . Salvetti [25] described a specific n -dimensional CW complex, $\text{Sal } T$, called the *Salvetti complex*, which is a model for this quotient of the hyperplane arrangement complement. Thus, $\text{gd } A_T \leq \dim(\text{Sal } T) = \text{Card } T$. (In fact, since the cohomological dimension, $\text{cd } A_T$, is $\geq \text{Card } T$ the previous inequality is an equality.) More generally, one can glue together the complexes $\text{Sal } T$, with T spherical, to get a CW complex $\text{Sal } S$ with fundamental group $A (= A_S)$. The dimension of $\text{Sal } S$ is $\dim L + 1$. The $K(\pi, 1)$ -Conjecture for A is the conjecture that $\text{Sal } S$ is a model for BA . This conjecture is true in many cases, for example, whenever L is a flag complex, cf. [7]. Thus, if the $K(\pi, 1)$ -Conjecture holds for A , then $\text{gd } A = \dim L + 1$.

A Coxeter matrix (m_{st}) is *right-angled* if all its nondiagonal entries are either 2 or ∞ . The associated Artin group A is a *right-angled Artin group* (abbreviated RAAG). The next theorem is one of our main results. For RAAGs, it was proved in [1].

MAIN THEOREM 1 (Theorem 5.2). *Suppose that L is the nerve of a Coxeter system (W, S) and that A is the associated Artin group. Let $d = \dim L$. If $H_d(L; \mathbb{Z}/2) \neq 0$, then $\text{actdim } A \geq 2d + 2$. So, if the $K(\pi, 1)$ -Conjecture holds for A , then $\text{actdim } A = 2d + 2$.*

The justification for the final sentence of this theorem is that if $K(\pi, 1)$ -Conjecture holds for A , then $\text{gd } A = d + 1$; so, the largest possible value for the action dimension is $2d + 2$.

For any simplicial complex K , there is another simplicial complex OK of the same dimension, defined by ‘doubling the vertices’ of K (cf. subsection 2.3). The complex OK is called the ‘octahedralization’ of K . In the right-angled case, the Main Theorem was proved by using OL for an obstructor, where $L = L(W, S)$.

The principal innovation of this paper concerns a certain subdivision L_\circlearrowleft of L , called the ‘complex of standard abelian subgroups’ of A . The complex L_\circlearrowleft plays the same role for a general Artin group as L does in the right-angled case; its simplices parameterize the ‘standard’ free abelian subgroups in A . We think of OL_\circlearrowleft as the boundary of the union of the standard flat subspaces in the universal cover of BA . The Main Theorem is proved by showing that (1) $\text{Cone } OL_\circlearrowleft$ coarsely embeds in A , and (2) when $H_d(L; \mathbb{Z}/2) \neq 0$, the van Kampen obstruction for OL_\circlearrowleft in degree $2d$ is not zero.

When A is a RAAG, a converse to the Main Theorem also was proved in [1]: if $H_d(L; \mathbb{Z}/2) = 0$ (and $d \neq 2$), then $\text{actdim } A \leq 2d + 1$. For a general Artin group A for which the $K(\pi, 1)$ -Conjecture holds, this converse, under a slightly stronger hypothesis, recently has been proved in the PhD thesis of Le [18, 19]: if $H^d(L; \mathbb{Z}) = 0$ (and $d \neq 2$), then $\text{actdim } A \leq 2d + 1$.

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1. Preliminaries

1.1. Coxeter systems

Let (W, S) be a Coxeter system.

A *reflection* in W is the conjugate of an element of S . Let R denote the set of all reflections in W . For any subset $T < S$, R_T denotes the set of reflections of W that are contained in W_T , that is, $R_T = R \cap W_T$. Actually, R_T coincides with the set of reflections of W_T , see, for example, [9, Lemma 4.2.3].

The *Coxeter diagram* \mathbf{D} of (W, S) is a labeled graph which records the same information as does (m_{st}) . The vertex set of \mathbf{D} is S and there is an edge between s and t whenever $m_{st} > 2$.

If $m_{st} = 3$, the edge is left unlabeled, otherwise it is labeled m_{st} . (The notation $\mathbf{D}(W, S)$ or $\mathbf{D}(S)$ also will be used for \mathbf{D} .)

The Coxeter system is *irreducible* if \mathbf{D} is connected. A *component* T of (W, S) is the vertex set of a connected component of \mathbf{D} . Thus, if T_1, \dots, T_k are the components of (W, S) , then \mathbf{D} is the disjoint union of the induced graphs on T_i and $W = W_{T_1} \times \dots \times W_{T_k}$. So, the diagram shows us how to decompose W as a direct product. In this paper, we will only be concerned with the diagrams of spherical Coxeter groups.

DEFINITION 1.1. Suppose T is a spherical subset and that T_1, \dots, T_k are the vertex sets of the components of $\mathbf{D}(T)$. Then $\{T_1, \dots, T_k\}$ is the *decomposition* of T (into irreducibles).

LEMMA 1.2 (cf. [4, Chapter IV, Exercise 22, p. 40]). Suppose (W_T, T) is a spherical Coxeter system. Then there is a unique element w_T of longest length in W_T . It has the following properties:

- w_T is an involution;
- w_T conjugates T to itself;
- $\ell(w_T) = \text{Card } R_T$.

REMARK 1.3. Let w_T be the element of longest length in a spherical special subgroup W_T . Let $\iota_T : W_T \rightarrow W_T$ denote inner automorphism by w_T . By Lemma 1.2, ι_T restricts to a permutation of T and this permutation induces an automorphism of the Coxeter diagram $\mathbf{D}(T)$. It follows that w_T belongs to the center of W_T if and only if the restriction of ι_T to T is the trivial permutation. Suppose (W_T, T) is irreducible. It turns out that the center of W_T is trivial if and only if the permutation $\iota_T|_T$ is nontrivial (this is because this condition is equivalent to the condition that w_T does not act as the antipodal map, -1 , on the canonical representation). If this is the case, we say W_T is *centerless*. In particular, when W_T is centerless, its Coxeter diagram admits a nontrivial involution. Using this, the question of the centerlessness of W_T easily can be decided, as follows. When $\text{Card } T = 2$, W_T centerless if and only if it is a dihedral group $\mathbf{I}_2(p)$, with p odd. When $\text{Card } T > 2$, the groups of type $\mathbf{A}_n, \mathbf{D}_n$ with n odd, and \mathbf{E}_6 are centerless, while those of type $\mathbf{B}_n, \mathbf{D}_n$ with n even, $\mathbf{H}_3, \mathbf{H}_4, \mathbf{F}_4, \mathbf{E}_7$ and \mathbf{E}_8 are not (cf. [4, Appendices I–IX, pp. 250–275] or [11, Remark 3.1.2, p. 125]).

REMARK 1.4. It is proved in [24] that if W_T is irreducible and not spherical, then its center is trivial.

For later use we record the following technical lemma.

LEMMA 1.5 (The Highest Root Lemma). Suppose (W, S) is spherical. Then there is a reflection $r \in R$ such that $r \notin W_T$ for any proper subset $T < S$.

Proof. It suffices to prove this when (W, S) is irreducible. So, suppose this. If the diagram (W, S) is not type $\mathbf{H}_3, \mathbf{H}_4$ or $\mathbf{I}_2(p)$ with $p = 5$ or $p > 6$, then (W, S) is associated with a root system in \mathbb{R}^n . Each root ϕ can be expressed as an integral linear combination of the simple roots $\{\phi_s\}_{s \in S}$:

$$\phi = \sum_{s \in S} n_s \phi_s,$$

where the coefficients n_s are either all ≥ 0 or all ≤ 0 ; the root ϕ is said to be *positive* or *negative*, respectively. Moreover, there is a positive root ϕ_r for each reflection $r \in R$. It is proved in [4, Chapter VI §1.8, Proposition 25, p. 178] that there is always a highest root ϕ_r which

dominates all other positive roots ϕ , in the sense that the coefficients of $\phi_r - \phi$ are all ≥ 0 . In particular, since ϕ_r dominates the simple roots, the coefficients n_s of ϕ_r are $\neq 0$. On the other hand, if $r \in R_T$, then $n_s = 0$ for all $s \in S - T$. Hence, if ϕ_r is the highest root, then $r \notin R_T$ for any proper subset $T < S$.

When the diagram is type \mathbf{H}_3 , \mathbf{H}_4 or $\mathbf{I}_2(p)$, one still has a ‘root system’, with the properties in the previous paragraph, except that the coefficients need not be integers. One then can prove the lemma directly in each of these three cases by simply writing down a reflection r which does not lie in any of the R_T . \square

Let S_\emptyset denote the set of irreducible nonempty spherical subsets of S . In other words,

$$S_\emptyset := \{T \in \mathcal{S}(W, S)_{>\emptyset} \mid \mathbf{D}(T) \text{ is connected}\}. \quad (1.1)$$

By Lemma 1.5, one can choose a function $r : S_\emptyset \rightarrow R$, denoted $T \mapsto r(T)$, such that

$$r(T) \in R_T - \bigcup_{T' < T} R_{T'}, \quad (1.2)$$

where the union is over all proper subsets of T . (When $\mathbf{D}(T)$ is type \mathbf{A}_n , it turns out that there is exactly one reflection in $R_T - \bigcup R_{T'}$; however, in other cases, it can contain more than one reflection.)

LEMMA 1.6. *The function $r : S_\emptyset \rightarrow R$ is injective.*

Proof. According to [4, Chapter IV, §1.8, Proposition 7, p. 12; 9, Proposition 4.4.1, p. 44], for any given $w \in W$, there is a well-defined subset $S(w) \leq S$, such that the set of letters used in any reduced expression for w is precisely $S(w)$. By construction, $S(r(T)) = T$. Hence, if $r(T_1) = r(T_2)$, then $T_1 = T_2$, that is, r is injective. \square

1.2. The subdivision L_\emptyset of L

We shall define the subdivision L_\emptyset of $L (= L(W, S))$. First, the vertex set of L_\emptyset is the set S_\emptyset defined in (1.1), that is, S_\emptyset is the set of (vertex sets of) irreducible spherical subdiagrams of $\mathbf{D}(W, S)$. Note that S_\emptyset is a subposet of $\mathcal{S}(W, S)$. We regard $T \in S_\emptyset$ as being the barycenter of the corresponding simplex of L . Given a subset α of S_\emptyset , define its *support*:

$$sp(\alpha) = \bigcup_{T \in \alpha} T.$$

Next we give an inductive definition of what it means for a subset of S_\emptyset to be ‘nested’.

DEFINITION 1.7. Let α be a subset of S_\emptyset such that $sp(\alpha)$ is spherical. By definition, the empty set is *nested*. (This is the base case of the inductive definition.) Let $\{T_1, \dots, T_k\}$ be the set of maximal elements in α . Then α is *nested* if all of the following hold:

- $sp(\alpha)$ is spherical;
- $\{T_1, \dots, T_k\}$ is the decomposition of $sp(\alpha)$ into irreducibles;
- $\alpha_{<T_i}$ is nested. (This is defined by induction since $\text{Card}(\alpha_{<T_i}) < \text{Card} \alpha$.)

Elements T_1, T_2 of S_\emptyset are *orthogonal* if their diagrams $\mathbf{D}(T_1)$ and $\mathbf{D}(T_2)$ are distance ≥ 2 apart in $\mathbf{D}(S)$; in other words, if $\mathbf{D}(T_1)$ and $\mathbf{D}(T_2)$ lie in different components of the full subgraph of $\mathbf{D}(S)$ spanned by $\mathbf{D}(T_1) \cup \mathbf{D}(T_2)$. Elements T_1, T_2 are *comparable* if either $T_1 < T_2$ or $T_2 < T_1$. Elements T_1, T_2 are *transverse* if they are neither orthogonal nor comparable. It is easy to see that a pair $\{T_1, T_2\}$ of S_\emptyset is nested if and only if T_1 and T_2 are either orthogonal or comparable.

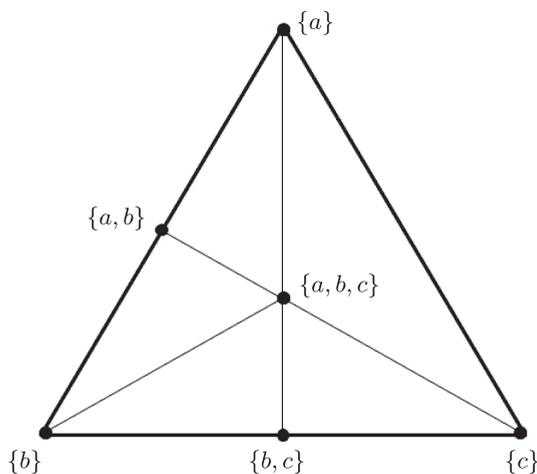


FIGURE 1. Subdivision for $\mathbf{D} = \overset{\bullet}{a} \quad \overset{\bullet}{b} \quad \overset{\bullet}{c}$.

The simplicial complex L_\circlearrowleft is defined as follows: the simplices of L_\circlearrowleft are the nested subsets of S_\circlearrowleft . It is clear that L_\circlearrowleft is a subdivision of L . (The subdivision of an irreducible 2-simplex is shown in Figure 1.) This same subdivision plays a role in [10, 11].

REMARK 1.8. Here is a more geometric description of L_\circlearrowleft . Suppose T is a spherical subset and that σ is the corresponding geometric simplex in L . Since L_\circlearrowleft is defined by applying the subdivision procedure to each simplex of L , it suffices to describe σ_\circlearrowleft . Suppose $\{T_1, \dots, T_k\}$ is the decomposition of T into irreducibles and that σ_i is the geometric simplex in L corresponding to T_i . This gives a join decomposition: $\sigma = \sigma_1 * \dots * \sigma_k$. Suppose by induction on $\dim \sigma$ that the subdivision has been defined for each proper subcomplex of σ . The subdivision σ_\circlearrowleft is then defined by using one of the following two rules.

- If $k = 1$, then T is irreducible and the barycenter b_T of σ will be a vertex of the subdivision. Define $\sigma_\circlearrowleft := (\partial\sigma)_\circlearrowleft * b_T$. (In other words, σ_\circlearrowleft is formed by coning off $(\partial\sigma)_\circlearrowleft$ to b_T .)
- If $k > 1$, then $\sigma_\circlearrowleft := (\sigma_1)_\circlearrowleft * \dots * (\sigma_k)_\circlearrowleft$.

The following lemma will be used in subsection 3.2. Its proof is straightforward.

LEMMA 1.9 (cf. [11, Corollary 3.5.4]). *The subdivision L_\circlearrowleft is a flag complex.*

REMARK 1.10. Assuming this lemma, one can define L_\circlearrowleft without using induction. First, there is a direct description of the 1-skeleton of L_\circlearrowleft : the edges are the 2-element subsets $\{T_1, T_2\}$ of S_\circlearrowleft such that T_1 and T_2 are either orthogonal or comparable. Second, L_\circlearrowleft is the flag complex determined by this 1-skeleton.

1.3. Artin groups

Suppose A is the Artin group associated to a Coxeter system (W, S) and that $\{a_s\}_{s \in S}$ is its standard set of generators. There is a canonical epimorphism $p : A \rightarrow W$ defined by $a_s \mapsto s$. The *pure Artin group* PA is the kernel of p .

There is a map $i : w \mapsto a_w$ from W to A , which is a set theoretic section for p . It is defined as follows. If $s_1 \cdots s_l$ is a reduced expression for w , then

$$a_w = a_{s_1} \cdots a_{s_l}. \tag{1.3}$$

N.B. the map $i : W \rightarrow A$ is not a homomorphism. The fact that the element a_w does not depend on the choice of reduced expression for w is a consequence of Tits' solution to word problem for Coxeter groups. (This asserts that any two reduced expressions for an element of W differ by 'braid moves', corresponding to Artin relations of type (0.1), for example, see [9, §3.4, p. 40].)

REMARK 1.11. For subsets T, T' subsets of S , we have

$$W_T \cap W_{T'} = W_{T \cap T'}, \quad A_T \cap A_{T'} = A_{T \cap T'}, \quad PA_T \cap PA_{T'} = PA_{T \cap T'}.$$

The first formula is proved in [4, Chapter IV.8, Theorem 2, p. 12], the second in [20]. The third follows from the first two.

1.4. Complements of hyperplane arrangements and pure Artin groups

When T is a spherical subset of S , W_T is a finite reflection group on \mathbb{R}^n (where n is the cardinality of T) and so, by complexification, also a reflection group on \mathbb{C}^n . Let M_T denote the complement of the union of reflecting hyperplanes in \mathbb{C}^n . Deligne [14] proved that $\pi_1(M_T)$ is the pure Artin group PA_T and that M_T is a model for BPA_T . Even when W_T is infinite it still has a geometric representation as a reflection group on \mathbb{R}^n so that W_T acts properly on the interior of a certain convex cone (the 'Tits cone'). Hence, W_T acts properly on some convex open subset $\Omega_T < \mathbb{C}^n$. We also denote the complement of the union of reflecting hyperplanes in Ω_T by M_T . It is proved in [20] that $\pi_1(M_T) \cong PA_T$. The original version of the $K(\pi, 1)$ -Conjecture was that $M_T \sim BPA_T$. (For more details, see [7] or [20].)

When $T' < T$, there is an open convex subset in Ω_T which only intersects the reflecting hyperplanes of $W_{T'}$ and which, therefore, can be identified with the product of $\Omega_{T'}$ and a disk. Taking complements of hyperplanes we get an inclusion $M_{T'} \hookrightarrow M_T$. On the level of fundamental groups, this induces the standard inclusion $PA_{T'} \hookrightarrow PA_T$.

Next, we consider the abelianization PA^{ab} of PA as well as the abelianization of the special subgroups PA_T . Since PA is the fundamental group of the hyperplane arrangement complement M_S , we have $PA^{ab} = H_1(M_S)$. There is one reflecting hyperplane for each reflection $r \in R$. Since each such hyperplane is the intersection of a linear hyperplane with an open convex subset of \mathbb{C}^n , $H_1(M_S)$ is generated by loops around these hyperplanes; in fact, $H_1(M_S)$ is the free abelian group on R , denoted \mathbb{Z}^R (for example, see [23]). So, we have proved the following.

LEMMA 1.12. PA^{ab} is the free abelian group \mathbb{Z}^R . Similarly, for any $T < S$, $(PA_T)^{ab} = \mathbb{Z}^{R_T}$.

Let $\{e_r\}_{r \in R}$ be the standard basis for \mathbb{Z}^R . For each fundamental reflection $s \in S$, the element $(a_s)^2 \in PA$ is a loop around the reflecting hyperplane corresponding to S , that is, e_s is represented by $(a_s)^2$, where a_s is a standard Artin generator. Any reflection $r \in R$ can be written as $r = wsw^{-1}$ for some $s \in S$ and $w \in W$. Put

$$\varepsilon_r := (a_w)(a_s)^2(a_w)^{-1}, \tag{1.4}$$

where a_w is the element of A defined by (1.3) (any other element of A lying above w would serve as well). Note that $\varepsilon_r \in PA$ and that it projects to $e_r \in PA^{ab}$. Thus, $\{\varepsilon_r\}_{r \in R}$ is the standard basis for $\mathbb{Z}^R = PA^{ab}$. (Usually we will write the group operation in PA^{ab} additively.)

2. Configurations of standard abelian subgroups

2.1. The center of a spherical Artin group

Suppose A_T is a spherical Artin group. Define $\Delta_T \in A$ by

$$\Delta_T = a_{w_T}, \tag{2.1}$$

where $w_T \in W_T$ is the element of longest length defined in Lemma 1.2 and a_{w_T} is defined by (1.3). For any $t \in T$, we have $w_T t = \iota(t)w_T$, where $\iota : T \rightarrow T$ is the permutation induced by w_T . Moreover, there is a reduced expression for w_T ending with the letter t as well as another reduced expression beginning with $\iota(t)$. So, $a_{w_T}(a_t)^{-1} = (a_{\iota(t)})^{-1}a_{w_T}$. Hence,

$$\Delta_T a_t = a_{\iota(t)} \Delta_T.$$

It follows that $(\Delta_T)^2$ lies in the center, $Z(A_T)$, of A_T and that $\Delta_T \in Z(A_T)$ if and only if $\iota : T \rightarrow T$ is the identity permutation. If T is spherical and irreducible, put

$$\delta_T := \begin{cases} \Delta_T, & \text{if } \iota = \text{id}; \\ (\Delta_T)^2, & \text{otherwise.} \end{cases}$$

Note that $(\Delta_T)^2 \in PA_T$. The next lemma follows from [2, Theorem 4.7, p. 294].

LEMMA 2.1. *Suppose (W_T, T) is spherical and irreducible. Then $Z(A_T)$ is infinite cyclic with generator δ_T . Similarly, $Z(PA_T)$ is infinite cyclic with generator $(\Delta_T)^2$.*

If (W_T, T) has more than one component, define δ_T to be the central element given by product of the appropriate choice of $\Delta_{T'}$ or $(\Delta_{T'})^2$ for each component T' .

We also can define corresponding central elements in PA_T and $(PA_T)^{ab}$ by

$$\begin{aligned} \varepsilon_T &:= (\Delta_T)^2 \in PA_T, \quad \text{and} \\ e_T &:= \text{image of } (\Delta_T)^2 \text{ in } (PA_T)^{ab}. \end{aligned}$$

LEMMA 2.2. *Suppose T is spherical. Then $e_T \in (PA_T)^{ab}$ is the sum of the standard basis elements:*

$$e_T = \sum_{r \in R_T} e_r.$$

Proof. Consider the hyperplane arrangement complement, M_T . If Δ_T is defined by (2.1), then the central element $(\Delta_T)^2 \in \pi_1(M_T) = PA_T$ is represented by the ‘Hopf fiber’. (More precisely, the Hopf fiber is \mathbb{C}^* and $(\Delta_T)^2$ is represented by $S^1 < \mathbb{C}^*$.) The Hopf fiber links each reflecting hyperplane once. (To see this, consider the complement in \mathbb{C}^n of a single linear hyperplane, in which case it is obvious.) Hence, the class of the Hopf fiber in $H_1(M_T)$ is $\sum_{r \in R_T} e_r$. \square

Here is an alternative algebraic proof of the above lemma.

Alternate proof of Lemma 2.2. Let $F(T)^+$ be the free monoid on T . For any word $f = t_1 \cdots t_l$ which is a reduced expression for the corresponding element w_f of W_T , let a_f be the corresponding element of A_T , that is, $a_f = a_{w_f} = a_1 \cdots a_l$, where a_i is the Artin generator corresponding to t_i . Pick a reduced expression $f \in F(T)^+$ for the element w_T of longest length in W_T . Let \bar{f} denote the reverse of f , that is, if $f = t_1 t_2 \cdots t_k$, then $\bar{f} = t_k \cdots t_2 t_1$. Note that

- since $f \cdot \bar{f} = 1$ in W_T and since $w_f = w_T$ is an involution (by Lemma 1.2), $w_f = w_{\bar{f}}$; since both f and \bar{f} are reduced expressions for w_T , the words f and \bar{f} represent the same element in A_T , namely, Δ_T ;
- the subset of W_T represented by $\{(t_1 \cdots t_{i-1}) \cdot t_i \cdot (t_1 \cdots t_{i-1})^{-1}\}_{1 \leq i \leq k}$ is exactly the set of reflections in W_T (cf. [4, Lemma 2, p. 6]).

A simple calculation shows that $\varepsilon_T = (\Delta_T)^2 = a_f \cdot a_{\bar{f}} = x_k x_{k-1} \cdots x_1$ where $x_i = (a_1 \cdots a_{i-1}) \cdot (a_i)^2 \cdot (a_1 \cdots a_{i-1})^{-1}$ for $1 \leq i \leq k$ (cf. (1.4)). By the discussion in subsection 1.4, the images of the x_i 's in $(PA_T)^{ab}$ are exactly the e_r 's with $r \in R_T$. The lemma follows. \square

Let $j : \mathbb{Z}^{S_\circ} \rightarrow \mathbb{Z}^R = PA^{ab}$ be the homomorphism defined by sending T to e_T and extending linearly. By combining Lemmas 1.6 and 2.2, we get the following.

LEMMA 2.3. *The homomorphism $j : \mathbb{Z}^{S_\circ} \rightarrow \mathbb{Z}^R$ is injective.*

Proof. The standard inner product, $(x, y) \mapsto x \cdot y$ on \mathbb{Z}^R is defined on standard basis elements by $(e_r, e_{r'}) \mapsto \delta_{r, r'}$. For each $T \in S_\circ$ let $r(T)$ be as in (1.2) and Lemma 1.6. Note that for any $T \in S_\circ$ and $r \in R$, we have $e_T \cdot e_r = 1$ if $r \in R_T$ and is 0 otherwise. It follows that for $T, T' \in S_\circ$, we have

$$e_{T'} \cdot e_{r(T)} = \begin{cases} 1, & \text{if } T \leq T', \\ 0, & \text{otherwise.} \end{cases}$$

Using this one sees that the e_T are linearly independent in \mathbb{Z}^R . Indeed, if $\sum x_T e_T = 0$, after taking the inner product with $e_{r(T)}$, we get $x_T = 0$ whenever T is a maximal element of S_\circ . Let S'_\circ be S_\circ with its maximal elements removed. Repeat the argument for maximal elements of S'_\circ . After finitely many iterations, we get that $x_T = 0$ for all $T \in S_\circ$. \square

2.2. Standard abelian subgroups

Suppose α is a d -simplex in L_\circ . We are going to define a subgroup $H_\alpha < A$ which is free abelian of rank $d + 1$. Suppose $T, T' \in \text{Vert } \alpha$ (where $\text{Vert } \alpha$ means the vertex set of α). By Remark 1.10, T and T' are either orthogonal or comparable. In either case δ_T and $\delta_{T'}$ commute. (This is obvious if T and T' are orthogonal. If T and T' are comparable, then without loss of generality $T < T'$, in which case, $\delta_{T'}$ centralizes $A_T < A_{T'}$.)

Define H_α to be the subgroup of $A_{sp(\alpha)}$ generated by $\{\delta_T\}_{T \in \text{Vert } \alpha}$. By the previous paragraph, H_α is abelian. It is called the *standard abelian subgroup* associated to α . Similarly, define PH_α to be the subgroup of finite index in H_α generated by $\{(\Delta_T)^2\}_{T \in \text{Vert } \alpha}$. Also, denote by J_α the image of PH_α in $(PA_{sp(\alpha)})^{ab}$.

LEMMA 2.4. *The natural map $\mathbb{Z}^{\text{Vert } \alpha} \rightarrow H_\alpha$ is an isomorphism; so, H_α is free abelian of rank $(d + 1)$.*

Proof. It suffices to show that the homomorphism $\mathbb{Z}^{\text{Vert } \alpha} \rightarrow (PH_\alpha)^{ab}$ induced by $T \mapsto e_T$ is an isomorphism. This is immediate from Lemma 2.3. \square

COROLLARY 2.5. *If $\dim L = d$, then $\text{cd } A \geq d + 1$. If the $K(\pi, 1)$ -Conjecture holds for A , then $\text{gd } A = d + 1$.*

Proof. If α is a d -simplex in L_\circ , then $H_\alpha \cong \mathbb{Z}^{d+1}$, which has cohomological dimension $d + 1$. Since $\text{cd } A \geq \text{cd } H_\alpha$, this proves the first sentence. If the $K(\pi, 1)$ -Conjecture holds for A , then its Salvetti complex is a model for BA and consequently, $d + 1 \geq \text{gd } A \geq \text{cd } A$. \square

Let $j : \mathbb{Z}^{S_\circ} \rightarrow \mathbb{Z}^R = PA^{ab}$ be the homomorphism defined in the sentence preceding Lemma 2.3. Let $J := \text{Im } j < PA^{ab}$ denote its image. By Lemma 2.3, $j : \mathbb{Z}^{S_\circ} \rightarrow J$ is an isomorphism. So, for any simplex α of L_\circ , $J_\alpha = j(\mathbb{Z}^{\text{Vert } \alpha})$. The next result is a key lemma.

LEMMA 2.6. *If α and β are simplices of L_\circ , then $H_\alpha \cap H_\beta = H_{\alpha \cap \beta}$.*

Proof. Since PH_α has finite index in H_α , the lemma will follow once we show that $PH_\alpha \cap PH_\beta = PH_{\alpha \cap \beta}$. Let $h : PA \rightarrow PA^{ab}$ be the abelianization homomorphism. Recall that $h(PH_\alpha) = J_\alpha$ and $h|_{PH_\alpha}$ is injective (Lemma 2.4). It follows that if $PH_\alpha \cap PH_\beta \supsetneq PH_{\alpha \cap \beta}$, then $J_\alpha \cap J_\beta \supsetneq J_{\alpha \cap \beta}$. So it suffices to show $J_\alpha \cap J_\beta = J_{\alpha \cap \beta}$. But this is immediate from the equality:

$$\mathbb{Z}^{\text{Vert } \alpha} \cap \mathbb{Z}^{\text{Vert } \beta} = \mathbb{Z}^{\text{Vert}(\alpha \cap \beta)},$$

after using the isomorphism j to transport this equality to an equality involving subgroups of J . \square

We record the following lemma concerning the geometry of standard abelian subgroups, even though we do not need it in the sequel.

LEMMA 2.7. *For any simplex $\alpha \subset L_\mathcal{O}$, H_α is quasi-isometrically embedded in A .*

Proof. First, consider the case when A is spherical. Charney [6] proved that spherical Artin groups are biautomatic and Gersten and Short [16] proved that biautomatic groups are semihyperbolic. The Algebraic Flat Torus Theorem from [5, p. 475] states that every monomorphism of a finitely generated abelian group to a semihyperbolic group is a quasi-isometric embedding. The general case follows from the spherical case, since any standard abelian subgroup is contained in a spherical special subgroup and special subgroups are convex in A [8]. \square

2.3. Configurations of flats

Octahedralization. For a finite set V , let $O(V)$ denote the boundary complex of the octahedron (or cross polytope) on V . In other words, $O(V)$ is the simplicial complex with vertex set $V \times \{\pm 1\}$ such that a subset $\{(v_0, \varepsilon_0), \dots, (v_k, \varepsilon_k)\}$ of $V \times \{\pm 1\}$ spans a k -simplex if and only if its first coordinates v_0, \dots, v_k are distinct. Projection to the first factor $V \times \{\pm 1\} \rightarrow V$ gives a simplicial projection $p : O(V) \rightarrow \sigma(V)$, where $\sigma(V)$ means the full simplex on V . Any finite simplicial complex K with vertex set V is a subcomplex of $\sigma(V)$. Its *octahedralization* OK is defined to be the inverse image of K in $O(V)$:

$$OK := p^{-1}(K) \leq O(V).$$

Thus, each simplex of OK is of the form (σ, ε) , where σ is a simplex of K and $\varepsilon : \text{Vert } \sigma \rightarrow \{\pm 1\}$ is a function. (This terminology comes from [1] or [13, § 8].) We also say that OK is the result of *doubling the vertices of K* .

Given a finite simplicial complex or metric space K , put

$$\text{Cone}(K) := K \times [0, \infty) / K \times \{0\}. \quad (2.2)$$

If K is a metric space, then there is an induced metric on $\text{Cone}(K)$ which gives it the structure of a *euclidean cone* (cf. [5, p. 60]). The idea is to use the formula for the euclidean metric on \mathbb{R}^n in polar coordinates — the coordinate in K is the ‘angle’ (and the metric on K gives the angular distance) and $[0, \infty)$ gives the radial coordinate. For example, if K is the round $(n-1)$ -sphere, then $\text{Cone}(K)$ is isometric to \mathbb{E}^n ; if K is a spherical $(n-1)$ -simplex, then $\text{Cone}(K)$ is isometric to the corresponding sector in \mathbb{E}^n .

A spherical $(n-1)$ -simplex σ is *all right* if its edges all have length $\pi/2$. This means that σ is isometric to the intersection of the unit sphere in \mathbb{R}^n with the positive orthant $[0, \infty)^n$. Its octahedralization $O\sigma$ is a triangulation of S^{n-1} and with the induced all right piecewise spherical metric, it is isometric to the round sphere. Hence, $\text{Cone}(O\sigma)$ is isometric to euclidean space.

Suppose K is given its all right piecewise spherical structure in which each edge has length $\pi/2$. Then $\text{Cone}(K)$ is isometric to a configuration of the positive orthants $\text{Cone}(\sigma)$ and $\text{Cone}(OK)$ is isometric to a configuration of euclidean spaces all passing through the cone point. (There is one euclidean space for each $\sigma \in K$.) The link of the cone point in $\text{Cone}(OK)$ is OK . By Gromov's Lemma (cf. [9, Appendix I.6, p. 516]), $\text{Cone}(OK)$ is CAT(0) if and only if OK is a flag complex.

Coordinate subspace arrangements. Given a finite simplicial complex K , one can define an arrangement of coordinate subspaces in the euclidean space $\mathbb{R}^{\text{Vert } K}$. Let $\mathcal{S}(K)$ denote the poset of simplices in K . For each $\sigma \in \mathcal{S}(K)$, there is the subspace $\mathbb{R}^{\text{Vert } \sigma} \leq \mathbb{R}^{\text{Vert } K}$. The *coordinate subspace arrangement* corresponding to K is the collection of subspaces, $\mathcal{A}(K) := \{\mathbb{R}^{\text{Vert } \sigma}\}_{\sigma \in \mathcal{S}(K)}$. The intersection of these subspaces with the integer lattice $\mathbb{Z}^{\text{Vert } K}$ gives the *coordinate arrangement of free abelian subgroups* associated to K ,

$$\mathcal{A}_{\mathbb{Z}}(K) := \{\mathbb{Z}^{\text{Vert } \sigma}\}_{\sigma \in \mathcal{S}(K)}. \quad (2.3)$$

The union of the elements of $\mathcal{A}(K)$ or $\mathcal{A}_{\mathbb{Z}}(K)$ is denoted \mathbb{E}^{OK} or \mathbb{Z}^{OK} , respectively:

$$\mathbb{E}^{OK} := \bigcup_{\sigma \in \mathcal{S}(K)} \mathbb{R}^{\text{Vert } \sigma}, \quad \mathbb{Z}^{OK} := \bigcup_{\sigma \in \mathcal{S}(K)} \mathbb{Z}^{\text{Vert } \sigma}. \quad (2.4)$$

Let $\{b_v\}_{v \in \text{Vert } K}$ be the standard basis for $\mathbb{R}^{\text{Vert } K}$ so that $\{b_v\}_{v \in \text{Vert } \sigma}$ is the standard basis for $\mathbb{R}^{\text{Vert } \sigma}$. The positive cone spanned by standard basis $\{b_v\}_{v \in \text{Vert } \sigma}$ is the *positive orthant*, $[0, \infty)^{\text{Vert } \sigma} < \mathbb{R}^{\text{Vert } \sigma}$. Similarly, if (σ, ε) is a simplex of OK , where ε is a choice of signs $v \mapsto \varepsilon_v \in \{\pm 1\}$, we get the orthant spanned by $\{\varepsilon_v b_v\}_{v \in \text{Vert } \sigma}$.

The proof of the next lemma is immediate from the definitions.

LEMMA 2.8. *Let $\{h_v\}_{v \in \text{Vert } K}$ be a collection of elements in some group π indexed by vertices of a finite simplicial complex K . Suppose h_v and $h_{v'}$ commute whenever v and v' are joined by an edge. For a simplex σ in K , let $H_\sigma \leq \pi$ denote the image of the homomorphism $\mathbb{Z}^{\text{Vert } \sigma} \rightarrow \pi$ induced by $v \mapsto h_v$. Suppose that*

- *the homomorphism $\mathbb{Z}^{\text{Vert } \sigma} \rightarrow \pi$ is injective;*
- *the conclusion of Lemma 2.6 holds, that is, $H_\sigma \cap H_\tau = H_{\sigma \cap \tau}$ for all σ, τ in K .*

Then the collection of free abelian subgroups $\mathcal{A} = \{H_\sigma\}_{\sigma \in \mathcal{S}(K)}$ is isomorphic to the coordinate arrangement of abelian groups in \mathbb{Z}^{OK} in the obvious sense.

By Lemmas 2.4 and 2.6, the lemma above applies to the simplicial complexes $K = L_\emptyset$ and $OK = OL_\emptyset$. For each $\alpha \in L_\emptyset$, let

$$\mathcal{B}_\alpha = \{\delta_T\}_{T \in \text{Vert } \alpha}$$

be the standard basis for H_α and let $\mathbb{R}^{\text{Vert } \alpha}$ be the euclidean space with basis \mathcal{B}_α , that is, $\mathbb{R}^{\text{Vert } \alpha} = H_\alpha \otimes \mathbb{R}$.

REMARK 2.9. These notions also make sense when K is infinite and \mathcal{A} is an infinite arrangement of free abelian subgroups in π . If \mathcal{A} is invariant under conjugation, then the conjugation action induces an action π on OK . In several cases, such action plays an important role of understanding the group π . For example, if π is the mapping class group of closed hyperbolic surface and \mathcal{A} is the collection of abelian subgroups generated by Dehn twists, then K is the curve complex of the underlying surface. If π is a right-angled Artin group and \mathcal{A} is the collection standard abelian subgroups and their conjugations, then K is the flag complex of the ‘extension graph’, which was introduced by Kim and Koberda [17].

3. The obstructor dimension

We review some definitions from [3].

3.1. The meaning of ‘ $K \subset \partial\pi$ ’

For a finite simplicial complex K , define $\text{Cone}(K)$ as in (2.2). We give $\text{Cone}(K)$ the euclidean cone metric after choosing a piecewise spherical or piecewise euclidean metric on K . Next we triangulate $\text{Cone}(K)$ so that

- for each simplex σ of K , $\text{Cone}(\sigma)$ is a subcomplex;
- the edge-length metric on the 0-skeleton, $\text{Cone}(K)^0$, is bi-Lipschitz equivalent to the metric on $\text{Cone}(K)^0$ induced from the euclidean cone metric.

DEFINITION 3.1. Suppose π is a discrete, finitely generated group equipped with a word metric. As in [3, p. 228], write ‘ $K \subset \partial\pi$ ’ to mean that there is a proper, expanding, Lipschitz map, $\text{Cone}(K)^0 \rightarrow \pi$.

The terms ‘proper’ and ‘Lipschitz’ have their usual meanings. The term ‘expanding’ needs further explanation, which is given below.

Two functions $h_1 : A_1 \rightarrow B$ and $h_2 : A_2 \rightarrow B$ to a metric space B *diverge* from one another if for every $D > 0$ there are compact subsets $C_i \leq A_i$, $i = 1, 2$, such that $h_1(A_1 - C_1)$ and $h_2(A_2 - C_2)$ are of distance $> D$ apart. Following [3], a proper map $h : \text{Cone}(K) \rightarrow B$ is *expanding* if for any two disjoint simplices σ and τ in K , the maps $h|_{\text{Cone}(\sigma)}$ and $h|_{\text{Cone}(\tau)}$ diverge. Similarly, if $\text{Cone}(K)^0$ is equipped with the restriction of an edge-length metric as above, then $h : \text{Cone}(K)^0 \rightarrow B$ is *expanding* if for disjoint simplices σ, τ in K , $h|_{\text{Cone}(\sigma)^0}$ and $h|_{\text{Cone}(\tau)^0}$ diverge.

Note that the meaning of ‘ $K \subset \partial\pi$ ’ in Definition 3.1 does not depend on choices we made in the first paragraph of this subsection.

It is shown in [3, p. 228] that if $B\pi$ is a finite complex and if its universal cover $E\pi$ has a \mathcal{Z} -structure (that is, a \mathcal{Z} -set compactification) with boundary denoted $\partial\pi$ and if $K \subset \partial\pi$, then there is a proper, expanding, Lipschitz map $\text{Cone}(K) \rightarrow E\pi$. In other words, $K \subset \partial\pi \Rightarrow$ ‘ $K \subset \partial\pi$ ’.

REMARK 3.2. A not necessarily continuous function i between two metric spaces X and Y is a *coarse embedding* if there exist nondecreasing functions ρ_- and $\rho_+ : [0, \infty) \rightarrow [0, \infty)$ such that $\rho_-(d(x, y)) \leq d(i(x), i(y)) \leq \rho_+(d(x, y))$ and $\lim_{t \rightarrow \infty} \rho_-(t) = \infty$ (cf. [22, p. 83]). When π acts properly and cocompactly on a contractible geodesic space E , the condition that ‘ $K \subset \partial\pi$ ’ essentially means that there is a coarse embedding $\text{Cone}(K) \rightarrow E$.

3.2. Obstructors

Let K be a finite simplicial complex. The *deleted simplicial product*, $(K \times K) - \Delta$, is the union of all cells in $K \times K$ of the form $\sigma \times \tau$, where σ and τ are disjoint closed simplices in K . The *configuration space*, $\mathcal{C}(K)$ is the quotient of $(K \times K) - \Delta$ by the free involution which switches the factors. Let $c : \mathcal{C}(K) \rightarrow \mathbb{R}P^\infty$ classify the double cover. If $w \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$ is the generator, then the *van Kampen obstruction* for K in degree m is the cohomology class $\text{vk}_{\mathbb{Z}/2}^m(K) \in H^m(\mathcal{C}(K); \mathbb{Z}/2)$ defined by

$$\text{vk}_{\mathbb{Z}/2}^m(K) := c^*(w^m).$$

It is an obstruction to embedding K in S^m . Moreover, in the case $m = 2 \dim K$ with $\dim K \neq 2$, it (or actually an integral version of it) is the complete obstruction to embedding K in S^m . If $\text{vk}_{\mathbb{Z}/2}^m(K) \neq 0$, then K is an *m-obstructor*.

The main results of [1] concern the van Kampen obstruction $\text{vk}_{\mathbb{Z}/2}^m(OK)$ of an octahedralization. From a nonzero $\mathbb{Z}/2$ -cycle $M \in Z_k(K; \mathbb{Z}/2)$ and a k -simplex Δ in the support of M , one produces a $2k$ -chain $\Omega \in C_{2k}(\mathcal{C}(OK); \mathbb{Z}/2)$ on which one can evaluate the van Kampen obstruction. If the pair (M, Δ) satisfies a certain technical condition, called the ‘*-condition’ in [1, p. 122], then Ω is a cycle and $\text{vk}_{\mathbb{Z}/2}^{2k}(OK)$ evaluates nontrivially on it. This gives the following.

THEOREM 3.3 [1, Theorem 5.2, p. 121]. *Given a simplicial complex K , suppose there is a k -cycle $M \in Z(K; \mathbb{Z}/2)$ and a k -simplex Δ in M such that (M, Δ) satisfies the *-condition. Then $\text{vk}_{\mathbb{Z}/2}^{2k}(OK) \neq 0$, that is, OK is a $2k$ -obstructor.*

It is then remarked in [1] that when $k = \dim K$ and K is a flag complex, the *-condition is automatically satisfied. It is also proved in [1, Theorem 5.1] that when there is no nontrivial cycle on K in the top degree, $\text{vk}_{\mathbb{Z}/2}^{2k}(OK) = 0$. So, the following is a corollary of Theorem 3.3.

THEOREM 3.4 [1, Theorem 5.4, p. 121]. *Suppose K is a d -dimensional flag complex. Then $\text{vk}_{\mathbb{Z}/2}^{2d}(OK) \neq 0$ if and only if $H_d(K; \mathbb{Z}/2) \neq 0$. In other words, OK is a $2d$ -obstructor if and only if $H_d(K; \mathbb{Z}/2) \neq 0$.*

An immediate corollary is the following.

THEOREM 3.5. *Suppose a d -dimensional complex L is the nerve of (W, S) . Then OL_{\circlearrowleft} is a $2d$ -obstructor if and only if $H_d(L; \mathbb{Z}/2) \neq 0$.*

Proof. Theorem 3.4 applies, since by Lemma 1.9, L_{\circlearrowleft} is a flag complex. □

REMARK 3.6. Kevin Schreve has pointed out to us that below the top dimension, it is easier for the *-condition to be satisfied for L_{\circlearrowleft} than for L . Indeed, suppose M is a k -cycle on L and Δ is a k -simplex in M . The subdivision M_{\circlearrowleft} is homologous to M . Moreover, if the barycenter of Δ occurs in $\Delta_{\circlearrowleft}$, then the *-condition holds for $(M_{\circlearrowleft}, \Delta')$ where Δ' is any k -simplex in Δ containing the barycenter.

3.3. The definition of obstructor dimension

DEFINITION 3.7 [3, p. 225]. The *obstructor dimension* of π , denoted by $\text{obdim } \pi$, is $\geq m + 2$ if there is an m -obstructor K such that ‘ $K \subset \partial\pi$ ’, that is,

$$\text{obdim } \pi := \sup\{m + 2 \mid \exists \text{ an } m\text{-obstructor } K \text{ with } 'K \subset \partial\pi'\}.$$

If there is a proper, expanding, Lipschitz map from $\text{Cone}(K)$ into a contractible $(m + 1)$ -manifold, then $\text{vk}_{\mathbb{Z}/2}^m(K) = 0$. So, if ‘ $K \subset \partial\pi$ ’ and π acts properly on a contractible $(m + 1)$ -manifold, then the van Kampen obstruction of K vanishes in degree m (cf. [3, Theorem 15, p. 226]); hence,

$$\text{obdim } \pi \leq \text{actdim } \pi. \tag{3.1}$$

EXAMPLE 3.8 (RAAGs, cf. [1]). Suppose A is a RAAG with d -dimensional nerve L . The standard model for BA is a union of tori; moreover, it is a locally $\text{CAT}(0)$ cubical complex of dimension $d + 1$. Its universal cover EA is $\text{CAT}(0)$, hence, contractible. Therefore, $\text{gd } A = d + 1$. Consequently, $\text{actdim } A \leq 2(d + 1)$. The lifts of the tori in BA to EA , which contain a given base point, give an isometrically embedded copy of \mathbb{E}^{OL} in EA . Thus, $OL \subset \partial A$. First

suppose that $H_d(L; \mathbb{Z}/2) \neq 0$. Since L is a flag complex, Theorem 3.3 implies that OL is a $2d$ -obstructor; so, $\text{obdim } A \geq 2d + 2$. This gives $\text{actdim } A \leq 2d + 2 \leq \text{obdim } A$. By (3.1), both inequalities are equalities:

$$\text{actdim } A = 2d + 2 = \text{obdim } A.$$

Conversely, if $H_d(L; \mathbb{Z}/2) = 0$, then OL is not a $2d$ -obstructor. So, when $d \neq 2$, OL embeds in S^{2d} . An argument of [1, Proposition 2.2] then shows that the right-angled Coxeter group with nerve OL , denoted W_{OL} , acts properly on a contractible $(2d + 1)$ -manifold. Since $A < W_{OL}$, so does A . So, in this case, $\text{obdim } A \leq \text{actdim } A \leq 2d + 1$.

4. Proper, expanding, Lipschitz maps

4.1. Coarse intersections of subgroups

We will need a lemma from [21]. Before stating it as Lemma 4.1 below, we explain some terminology.

Subspaces A and B of a metric space X are *coarsely equivalent*, denoted $A \stackrel{c}{\sim} B$, if their Hausdorff distance is finite. The subspace A is coarsely contained in B , denoted $A \stackrel{c}{\subset} B$, if there is a positive real number R such that A is contained in the R -neighborhood $N_R(B)$ of B . In this case, we also write $A \subset_R B$. We use $A \cap_R B$ to mean $N_R(A) \cap N_R(B)$. A subspace C is the *coarse intersection* of A and B , denoted by $C = A \overset{c}{\cap} B$, if the Hausdorff distance between C and $A \cap_R B$ is finite for all sufficiently large R . In general, the coarse intersection of A and B may not exist; however, if it exists, then it is unique up to coarse equivalence.

The word metric (with respect to some finite generating set) on a finitely generated group π gives it the structure of a metric space. Different choices of a generating set give rise to bi-Lipschitz equivalent word metrics. So, the various notions of ‘coarseness’ explained in the previous paragraph make sense for subsets of π .

LEMMA 4.1 [21, Lemma 2.2]. *Let H_1, H_2 be subgroups of a finitely generated group π . The coarse intersection of H_1 and H_2 exists and is coarsely equivalent to $H_1 \cap H_2$.*

4.2. Subcomplexes at infinity

Suppose H is a finitely generated subgroup of a finitely generated group π . With respect to the word metrics, the inclusion $H \hookrightarrow \pi$ obviously is Lipschitz. The next lemma, a standard result, implies that $H \hookrightarrow \pi$ is a metrically proper map. For a proof, see [15].

LEMMA 4.2 (cf. [15, Lemma 3.6; 22, Example 1.4.8]). *Suppose H is a finitely generated subgroup of a finitely generated group π . Then there is a function $\rho_- : [0, \infty) \rightarrow [0, \infty)$ with $\lim_{t \rightarrow \infty} \rho_-(t) = +\infty$ such that for any $a, b \in H$, if $d_H(a, b) \geq t$, then $d_\pi(i(a), i(b)) \geq \rho_-(t)$. (Here d_π and d_H denote the word metrics on the corresponding groups.)*

We return to the situation of Subsection 2.3: $\mathcal{A} = \{H_\sigma\}_{\sigma \in \mathcal{S}(K)}$ is an arrangement of free abelian subgroups in a finitely generated group π , indexed by the poset of simplices of a finite simplicial complex K . Put $\mathcal{H} = \bigcup H_\sigma$. Further suppose \mathcal{A} is isomorphic to $\mathcal{A}_{\mathbb{Z}}(K)$, the coordinate arrangement of free abelian groups in \mathbb{Z}^{O_K} , defined by (2.3) and (2.4). The word metrics on the H_σ induce a metric on \mathcal{H} . Since H_σ is bi-Lipschitz to $\mathbb{Z}^{\text{Vert } \sigma}$, the isomorphism $\text{Cone}(OK)^0 \rightarrow \mathcal{H}$ also is bi-Lipschitz.

THEOREM 4.3. *Suppose $\mathcal{A} = \{H_\sigma\}_{\sigma \in \mathcal{S}(K)}$ is an arrangement of free abelian subgroups isomorphic to the coordinate arrangement in \mathbb{Z}^{OK} . Then ‘ $OK \subset \partial\pi$ ’.*

By Lemma 2.8, the arrangement of standard abelian subgroups in A , $\{H_\alpha\}_{\alpha \in \mathcal{S}(L_\emptyset)}$, is isomorphic to the coordinate arrangement $\mathcal{A}_{\mathbb{Z}}(L_\emptyset)$ in $\mathbb{Z}^{OL_\emptyset}$. So, the following theorem is a corollary of Theorem 4.3. It is one of our main results.

THEOREM 4.4. ‘ $OL_\emptyset \subset \partial A$ ’.

Proof of Theorem 4.3. Let $i : \text{Cone}(OK)^0 \cong \mathcal{H} \hookrightarrow \pi$ be the inclusion. As we explained above, i is Lipschitz and proper. The remaining issue is to show that it is expanding. So, suppose $\Delta = (\sigma, \varepsilon)$ and $\Delta' = (\sigma', \varepsilon')$ are disjoint simplices of OK . Let \mathbb{S} and \mathbb{S}' denote the octahedral spheres $O\sigma$ and $O\sigma'$ inside OK . It suffices to show

$$i(\text{Cone}(\Delta)^{(0)}) \cap_r i(\text{Cone}(\Delta')^{(0)}) \text{ is finite for any } r > 0. \quad (4.1)$$

Case 1: $\mathbb{S} = \mathbb{S}'$. In this case $\sigma = \sigma'$ (and since (σ, ε) and (σ', ε') are disjoint, $\varepsilon' = -\varepsilon$). Condition (4.1) follows from Lemma 4.2 applied to the monomorphism $i : \mathbb{Z}^{\text{Vert } \sigma} \rightarrow \pi$. Given any two subcomplexes F_1 and F_2 of $\mathbb{S} = O\sigma$, we also deduce from Lemma 4.2 that for any $r > 0$, there exists $r' > 0$ such that

$$i(\text{Cone}(F_1)^{(0)}) \cap_r i(\text{Cone}(F_2)^{(0)}) \subset_{r'} i(\text{Cone}(F_1 \cap F_2)^{(0)}). \quad (4.2)$$

When $F_1 \cap F_2 = \emptyset$, $\text{Cone}(F_1 \cap F_2)^{(0)}$ is just the cone point.

Case 2: $\mathbb{S} \neq \mathbb{S}'$. By Lemma 4.1,

$$\begin{aligned} i(\text{Cone}(\Delta)^{(0)}) \cap_r i(\text{Cone}(\Delta')^{(0)}) &\subset i(\text{Cone}(\mathbb{S})^{(0)}) \cap_r i(\text{Cone}(\mathbb{S}')^{(0)}) \\ &\subset_{r'} i(\text{Cone}(\mathbb{S} \cap \mathbb{S}')^{(0)}) \end{aligned} \quad (4.3)$$

for some $r' > 0$. Applying (4.2) to subcomplexes of \mathbb{S} , \mathbb{S}' and $\mathbb{S} \cap \mathbb{S}'$ respectively, we deduce that

$$i(\text{Cone}(\Delta)^{(0)}) \cap_r i(\text{Cone}(\mathbb{S} \cap \mathbb{S}')^{(0)}) \subset_{r'} i(\text{Cone}(\Delta \cap \mathbb{S} \cap \mathbb{S}')^{(0)}) \quad (4.4)$$

$$i(\text{Cone}(\Delta')^{(0)}) \cap_r i(\text{Cone}(\mathbb{S} \cap \mathbb{S}')^{(0)}) \subset_{r'} i(\text{Cone}(\Delta' \cap \mathbb{S} \cap \mathbb{S}')^{(0)}) \quad (4.5)$$

$$i(\text{Cone}(\Delta \cap \mathbb{S} \cap \mathbb{S}')^{(0)}) \cap_r i(\text{Cone}(\Delta' \cap \mathbb{S} \cap \mathbb{S}')^{(0)}) \text{ is finite.} \quad (4.6)$$

Condition (4.1) follows from (4.3), (4.4), (4.5) and (4.6). \square

REMARK 4.5. Theorem 4.3 and its proof can be extended to the case of an arrangement of free abelian subgroups \mathcal{A} isomorphic to the intersection of the integer lattice with some real subspace arrangement (not necessarily a coordinate subspace arrangement).

REMARK 4.6. The proof of Theorem 4.3 can be simplified if we know the abelian subgroups are quasi-isometrically embedded. This is indeed the case for standard abelian subgroups in an Artin group (cf. Lemma 2.7).

5. The action dimension of A

First, consider the case when A is spherical. The action dimension of any braid group is computed in [3]. More generally, by using Theorem 4.4, Le [18, 19] computed the action dimension of any spherical Artin group. Here is the result.

THEOREM 5.1 (cf. [3, p. 234; 18, Theorem 4.10; 19, Theorems 15, 16]). *Suppose A is a spherical Artin group associated to (W, S) . Let T_1, \dots, T_k be the irreducible components of $\mathbf{D}(W, S)$ and put $d_i = \text{Card}(T_i) - 1$. Then $\text{actdim } A = k + \sum 2d_i$. In other words, $\text{actdim } A = 2 \text{gd } A - k$.*

Proof. The nerve of (W, S) is a simplex σ of dimension $d = \text{Card}(S) - 1$. First consider the case $k = 1$, that is, the case where (W, S) is irreducible. The barycenter v of σ is then a vertex of σ_\emptyset ; the link of v in σ_\emptyset is $(\partial\sigma)_\emptyset$; and σ_\emptyset is the cone, $(\partial\sigma)_\emptyset * v$. Since $(\partial\sigma)_\emptyset$ is a triangulation of S^{d-1} , it follows from Theorem 4.4 that $O(\partial\sigma_\emptyset)$ is a $(2d - 2)$ -obstructor. By [3, Lemma 9], $O\sigma_\emptyset = O(\partial\sigma_\emptyset) * S^0$ is a $(2d - 1)$ -obstructor. Therefore, $\text{obdim } A \geq 2d + 1$. Since the hyperplane complement in S^{2d+1} is a $(2d + 1)$ -manifold, $\text{actdim } A \geq 2d + 1$. By (3.1), both inequalities are equalities; so, $\text{actdim } A = 2d + 1 = \text{obdim } A$.

Consider the general case, $A = A_{T_1} \times \dots \times A_{T_k}$. By [3, Lemma 6], for any two groups π_1, π_2 , we have $\text{actdim}(\pi_1 \times \pi_2) \leq \text{actdim } \pi_1 + \text{actdim } \pi_2$ and $\text{obdim}(\pi_1 \times \pi_2) \geq \text{obdim } \pi_1 + \text{obdim } \pi_2$. Therefore,

$$\text{actdim } A \leq \sum_{i=1}^k (2d_i + 1) \leq \text{obdim } A,$$

which proves the lemma. □

By using Theorems 3.5 and 4.4, the results about RAAGs in Example 3.8 can be extended to general Artin groups. This gives the following restatement of the Main Theorem in the Introduction.

THEOREM 5.2. *Suppose A is an Artin group associated to a Coxeter system with nerve L of dimension d . If $H_d(L; \mathbb{Z}/2) \neq 0$, then $\text{obdim } A \geq 2d + 2$. So, if the $K(\pi, 1)$ -Conjecture holds for A , then $\text{actdim } A = 2d + 2$.*

Proof. By Theorems 3.5 and 4.4, $\text{obdim } A \geq 2d + 2$. If the $K(\pi, 1)$ -Conjecture holds for A , then, by Corollary 2.5, $\text{gd } A = d + 1$ and consequently, the maximum possible value for its action dimension is $2d + 2$. □

The other statement in Example 3.8 is that, in the case of a RAAG, if $H_d(L; \mathbb{Z}/2) = 0$, $\text{actdim } A \leq 2d + 1$. In her PhD thesis Le [18, 19] proved that the same conclusion holds for general Artin groups under the slightly weaker hypothesis that $H^d(L; \mathbb{Z}) = 0$.

THEOREM 5.3 [18, 19]. *Suppose A is an Artin group associated to a Coxeter system with d -dimensional nerve L . Further suppose that the $K(\pi, 1)$ -Conjecture holds for A . If $H^d(L; \mathbb{Z}) = 0$ (and $d \neq 2$), then $\text{actdim } A \leq 2d + 1$.*

COMMENTS ON THEOREM 5.3. As explained in Example 3.8, when A is a RAAG, if $H_d(L; \mathbb{Z}/2) = 0$, then OL is not a $2d$ -obstructor and (provided $d \neq 2$), OL embeds as a full subcomplex of some flag triangulation J of S^{2d} (cf. the proof of [1, Proposition 2.2]). Since the Davis complex for the right-angled Coxeter group W_J is a contractible $(2d + 1)$ -manifold and since $A < W_{OL} < W_J$, we see that $\text{actdim } A \leq 2d + 1$. Similarly, for a general Artin group A , the vanishing of $H_d(L; \mathbb{Z}/2)$ implies that OL_\emptyset is a full subcomplex of some flag triangulation S^{2d} . However, we do not know how to deduce from this that A acts properly on a contractible $(2d + 1)$ -manifold. Le's argument in [18, 19] is different. The condition $H^d(L; \mathbb{Z}) = 0$ implies $H_d(L; \mathbb{Z}) = 0$ and that $H_{d-1}(L; \mathbb{Z})$ is torsion-free. Standard arguments show that one can attach cells to L to embed it into a contractible complex L' of the same dimension d (provided $d \neq 2$).

We can assume L' is a simplicial complex. For each simplex σ of L , there is a model for BA_σ by a $(2d + 1)$ -manifold with boundary. Le proves Theorem 5.3 by showing that one can glue together copies of these manifolds along their boundaries, in a fashion dictated by L' , to obtain another manifold with boundary M with fundamental group A so that when the $K(\pi, 1)$ -Conjecture holds for A , M is a model for BA .

REMARK 5.4. The Action Dimension Conjecture is the conjecture that if a group π acts properly on a contractible manifold M , then the ℓ^2 -Betti numbers of π vanish above half the dimension of M . In [1, §7] it is explained how the calculations for RAAGs provide evidence for this conjecture. The same discussion applies to the results in this paper for general Artin groups. To wit, when the $K(\pi, 1)$ -Conjecture holds for an Artin group A , it is proved in [12] that the ℓ^2 -Betti number of A in degree $(k + 1)$ is equal to the ordinary (reduced) Betti number of L (or L_\emptyset) in degree k ; in particular, the ℓ^2 -Betti numbers of A vanish in degrees $> d + 1$, when $d = \dim L$.

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