

Complements of hyperplane arrangements as posets of spaces

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Abstract

The complement of an arrangement, \mathcal{A} , of a finite number of affine hyperplanes in \mathbf{C}^n , has the structure of a poset of spaces indexed by the intersection poset, $L(\mathcal{A})$. The space corresponding to $G \in L(\mathcal{A})$ is homotopy equivalent to the complement of the hyperplanes in the central arrangement \mathcal{A}_G normal to G . This poset of spaces structure can be used to repair a spectral sequence argument in [6], [7] for computing certain cohomology groups of arrangement complements. Similarly, toric hyperplane arrangements have the structure of a diagram of spaces and this structure can be used fix a spectral sequence argument in [11].

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1 Introduction

Suppose \mathcal{A} is an arrangement of affine hyperplanes in \mathbf{C}^n . Its *singular set* $\Sigma(\mathcal{A})$ and its *complement* $\mathcal{M}(\mathcal{A})$ are the subsets of \mathbf{C}^n defined by

$$\Sigma(\mathcal{A}) := \bigcup_{H \in \mathcal{A}} H \quad \text{and} \quad \mathcal{M}(\mathcal{A}) := \mathbf{C}^n - \Sigma(\mathcal{A}).$$

A *subspace* of \mathcal{A} is a nonempty intersection of a collection of elements of \mathcal{A} . (The entire space \mathbf{C}^n is the subspace corresponding to the intersection

of the empty set of hyperplanes.) The *intersection poset* $L(\mathcal{A})$ is the set of subspaces partially ordered by reverse inclusion. N.B. The partial order is *reverse inclusion*. An arrangement is *central* if $L(\mathcal{A})$ has a maximum element. For any $G \in L(\mathcal{A})$ there is a corresponding central arrangement,

$$\mathcal{A}_G := \{H \in \mathcal{A} \mid G \subseteq H\}.$$

There also is an arrangement \mathcal{A}^G of hyperplanes in G called the restriction of \mathcal{A} to G and defined by

$$\mathcal{A}^G := \{G \cap H \mid H \in \mathcal{A} - \mathcal{A}_G \text{ and } G \cap H \neq \emptyset\}.$$

The purpose of this note is to show that $\mathcal{M}(\mathcal{A})$ has the structure of a poset of spaces with indexing poset $L(\mathcal{A})$, where the subspace M_G corresponding to $G \in L(\mathcal{A})$ is homotopy equivalent to $\mathcal{M}(\mathcal{A}_G)$. One reason for proving this is to repair an argument in the proofs of main results of [6], [7] and [11]. In each of these papers the main theorem asserts that the cohomology groups of $\mathcal{M}(\mathcal{A})$ with certain local coefficients vanish except in the top degree or that the ℓ^2 -Betti numbers vanish except in the top degree. (The top degree is the rank of the arrangement.) A corrected version of this argument can be based on a spectral sequence from [10] for posets of spaces where the local cohomology groups satisfy a certain condition (Z').

The argument in [6, 7] used a cover of $\mathcal{M}(\mathcal{A})$ by open sets, each of which was the complement of a central arrangement. The nerve of this cover was *not* the order complex of the intersection poset and it did *not* satisfy condition (Z'). The E_2 page of the resulting Mayer-Vietoris spectral sequence is similar to the one in [10] in that it breaks up as a direct sum, indexed by $L(\mathcal{A})$, of cohomology groups with locally constant coefficients. However, since the set-up was not the same as in [10], one cannot directly conclude that the coefficients are constant. In fact, Graham Denham pointed out to us that in similar situations, such as in the case of projective arrangements, the coefficients can be twisted. In [13] Denham, Suciu and Yuzvinsky have given a different argument for some of the results of [7].

The main result of this paper shows that complements of affine hyperplane arrangements fit precisely into the context of [10]. It follows that the coefficients in the spectral sequences of [6, 7] are untwisted and the theorems in these papers are correct as stated.

In the case of arrangements of toric hyperplanes in $(\mathbf{C}^*)^n$, the results in [11] also are correct; however, the argument requires further modifications.

These modifications are accomplished in Section 4 by passing to the universal cover of $(\mathbf{C}^*)^n$ and lifting the toric arrangement to a periodic arrangement of affine hyperplanes in \mathbf{C}^n . One shows that the complement of periodic arrangement has the structure of a poset of spaces with a \mathbf{Z}^n -action and then uses a spectral sequence associated to this situation.

Next we define our cover of $\mathcal{M}(\mathcal{A})$ by a poset of spaces $\{M_G\}_{G \in L(\mathcal{A})}$. Choose a base point $b \in M(\mathcal{A})$. Let $R_b(H)$ be the real $(2n - 1)$ -dimensional affine subspace in \mathbf{C}^n spanned by b and H . Let $E_b(H)$ be the open half-space in $R_b(H)$ bounded by H and not containing b . So,

$$E_b(H) := \{x + t(x - b) \mid x \in H \text{ and } t \in (0, \infty)\} \quad (1.1)$$

For each $x \in H$, let $\text{Ray}(b, x)$ denote the ray starting from b and passing through x and let $(x, \infty) := \text{Ray}(b, x) - [b, x]$. Thus, $E_b(H)$ is the union of the (x, ∞) , for $x \in H$. In other words, $E_b(H)$ is the “shadow” of H from b . For each $G \in L(\mathcal{A})$, put

$$M_{G,b} := \mathcal{M}(\mathcal{A}) - \bigcup_{H \in \mathcal{A} - \mathcal{A}_G} E_b(H). \quad (1.2)$$

In particular,

$$M_{\mathbf{C}^n,b} := \mathcal{M}(\mathcal{A}) - \bigcup_{H \in \mathcal{A}} E_b(H).$$

2 Posets of spaces

2.1 Posets

Suppose \mathcal{P} is a poset and $|\mathcal{P}|$ ($= \text{Flag}(\mathcal{P})$) is its geometric realization (also called its “order complex”). A k -simplex σ in $|\mathcal{P}|$ is a chain $(p_0 < p_1 < \cdots < p_k)$. Regard \mathcal{P} as the objects of a category – there is a morphism from p to q if and only if $p \leq q$.

A *poset of spaces* is a functor Y from \mathcal{P} to the category **Top** of topological spaces. So, for each $p \in \mathcal{P}$ one is given a space Y_p and for $p < q$ a map $f_{pq} : Y_p \rightarrow Y_q$. A poset of spaces is a special case of a “diagram of spaces,” where \mathcal{P} is replaced by an arbitrary category.

There is a generalization of the mapping cylinder construction called the *homotopy pushout* or *homotopy colimit* of Y given by

$$\text{hocolim } Y = \left(\coprod_{p \in \mathcal{P}} Y_p \times |\mathcal{P}_{\geq p}| \right) / \sim,$$

where the equivalence relation \sim is defined by the obvious identifications using the f_{pq} (cf. [19, Prop. 4G.2]).

Here is a related concept.

Definition 2.1. (cf. [10, p. 497]) Suppose X is a space. A *poset of subspaces in X over \mathcal{P}* is a cover $\mathcal{V} = \{X_p\}_{p \in \mathcal{P}}$ of X by open subsets (or by subcomplexes when X is a CW complex) so that the elements of the cover are indexed by \mathcal{P} and so that

- (a) $p < q \implies X_p \subset X_q$, and
- (b) the vertex set $\text{Vert}(\sigma)$ of each simplex of $N(\mathcal{V})$ has a greatest lower bound $\wedge \sigma$ in \mathcal{P} , and
- (c) \mathcal{V} is closed under taking finite nonempty intersections, i.e., for any simplex σ of $N(\mathcal{V})$,

$$\bigcap_{p \in \sigma} X_p = X_{\wedge \sigma}.$$

If $\mathcal{V} = \{X_p\}_{p \in \mathcal{P}}$ is a poset of subspaces in X , then $p \mapsto X_p$ gives a functor from \mathcal{P} to **Top** whose homotopy colimit is homotopy equivalent to X (cf. [19]). Conversely, given a functor Y from \mathcal{P} to the category of CW complexes, put $X = \text{hocolim } Y$ and $X_p = \pi^{-1}(|\mathcal{P}_{\leq p}|)$. Then $\{X_p\}_{p \in \mathcal{P}}$ is a poset of subcomplexes in X .

2.2 A spectral sequence

A *poset of coefficients* is a contravariant functor from \mathcal{P} to the category of abelian groups. Suppose $\mathcal{V} = \{X_p\}_{p \in \mathcal{P}}$ is a poset of subspaces in X . Let A be a system of local coefficients on X . For each integer $j \geq 0$ we get a poset of coefficients $\mathcal{H}^j(p)$ defined by $\mathcal{H}^j(p) = H^j(X_p; A)$.

In [10] we considered the following condition:

- (Z') if $p < q$, then for all j (including $j = 0$), $H^j(X_q; A) \rightarrow H^j(X_p; A)$ is the 0-map.

Notation. In everything that follows, when taking the cohomology of the order complex of a poset \mathcal{P} , we will drop $|\cdot|$ from our notation and write $H^*(\mathcal{P})$ instead of $H^*(|\mathcal{P}|)$.

Theorem 2.2. (Davis–Okun [10]). *Suppose (Z') holds. There is a spectral sequence converging to $H^*(X; A)$ whose E_2 page decomposes as a direct sum:*

$$E_2^{i,j} = \bigoplus_{p \in \mathcal{P}} H^i(\mathcal{P}_{\geq p}, \mathcal{P}_{> p}; H^j(X_p; A)).$$

Moreover, in each summand, the coefficients are constant.

Sketch of proof. There is Mayer-Vietoris spectral sequence for any poset of spaces. After taking the vertical differentials, the E_1 -page decomposes as a direct sum:

$$E_1^{i,j} = \bigoplus_{p \in \mathcal{P}} C^i(\mathcal{P}_{\geq p}, \mathcal{P}_{> p}; H^j(X_p; A)).$$

In general, the horizontal differentials do not respect this direct sum decomposition; however, when condition (Z') holds, they do. So, we get decomposition of the E_2 page as indicated. In this direct sum decomposition, for each simplex $\sigma \in \mathcal{P}_{\geq p}$ with $p = \wedge \sigma$ and for each integer j , we associate exactly one group, namely, $H^j(X_p; A)$. So, the coefficients in the summand corresponding to p are constant, not just locally constant. \square

3 Arrangement complements

Let \mathcal{A} be an arrangement of affine hyperplanes in \mathbf{C}^n and let $L(\mathcal{A})$ be its intersection poset. Given $F, G \in L(\mathcal{A})$, their *meet* (or greatest lower bound) in $L(\mathcal{A})$ is the element

$$F \wedge G = \bigcap \{H \in \mathcal{A} \mid F \cup G \subseteq H\}.$$

If $\sigma = \{G_0, \dots, G_k\}$ is a finite subset of $L(\mathcal{A})$, put $\wedge \sigma = G_0 \wedge \dots \wedge G_k$.

If $F \cap G \neq \emptyset$, then $F \cap G$ is a least upper bound for $\{F, G\}$ called the *join* and denoted $F \vee G := F \cap G$. So, the poset $L(\mathcal{A})$ is a “meet semi-lattice.” (However, it is a lattice only when \mathcal{A} is a central arrangement.)

We also assume \mathbf{C}^n is equipped with a Hermitian inner product. For any affine subspace $F \subset \mathbf{C}^n$, let F_{lin} be the parallel vector subspace and F^\perp its orthogonal complement.

3.1 The intersection of a shadow and a subspace

Choose a base point $b \in \mathbf{C}^n$. For each $G \in L(\mathcal{A})$, let b_G be the point in G which is closest to b . So, the vector $b_G - b$ lies in G^\perp .

Given a subspace G in $L(\mathcal{A})$, a base point $a \in G$, and a hyperplane $H \cap G$ in \mathcal{A}^G , let $E_a(H \cap G; G)$ denote the shadow from a of $H \cap G$ in G .

Lemma 3.1. *Suppose H, H' are hyperplanes in \mathcal{A} which are not parallel. Then*

$$H \cap E_b(H') = E_{b_H}(H' \cap H; H).$$

(in other words, $H \cap E_b(H')$ is the shadow of $H' \cap H$ in H .)

Proof. Put $G = H \cap H'$. Let $R_{b_H}(G; H)$ be the real affine hyperplane in H generated by G and a vector $a - b_H$ for any $a \in G$. (The vector $a - b_H$ points into $E_{b_H}(G; H)$.) The vector $b_G - b$ points into $E_b(G)$ at b_G and hence, into $E_b(H')$. Therefore, its orthogonal projection onto H points into $H \cap E_b(H')$. But this orthogonal projection is just $b_G - b_H$, so $E_{b_H}(G; H) = H \cap E_b(H')$. \square

Corollary 3.2. *Suppose $G \in L(\mathcal{A})$. If $H' \in \mathcal{A}^G$, then $G \cap E_b(H') = E_{b_G}(H' \cap G; G)$.*

3.2 Generic base points

We want to give a condition on the base point b which prevents nesting of shadows, $E_b(H) \subseteq E_b(H')$ (and similarly for shadows of smaller subspaces in subspaces).

If H, H' are parallel hyperplanes in a complex vector space V , then there is a linear form $\phi : V \rightarrow \mathbf{C}$ such that H and H' are defined by $\phi(z) = c$ and $\phi(z) = c'$, respectively. Notice that if H and H' are parallel, then the real hyperplanes $R_b(H)$ and $R_b(H')$ in V are equal if and only if $\phi(b)$ is on the real line $L_{c,c'} \subset \mathbf{C}$, joining c to c' (where $L_{c,c'} = \{tc + (1-t)c' \mid t \in \mathbf{R}\}$). When this happens then, depending on whether or not $\phi(b)$ lies between c and c' , either $E_b(H) \cap E_b(H')$ is the region in $R_b(H)$ between H and H' , or $E_b(H) \subset E_b(H')$, or $E_b(H) \supset E_b(H')$. In light of these remarks, a point $b \in \mathbf{C}^n$ is *generic with respect to* $\{H, H'\}$ if $\phi(b) \notin L_{c,c'} - [c, c']$. Given any two hyperplanes H and H' in an arrangement \mathcal{A} , a base point b which is on neither is *generic* for $\{H, H'\}$ if either H and H' are not parallel or, if they are parallel, then $\phi(b) \notin L_{c,c'} - [c, c']$.

Definition 3.3. Given an arrangement \mathcal{A} of affine hyperplanes, a base point $b \in \mathcal{M}(\mathcal{A})$ is *generic* if for any $G \in L(\mathcal{A})$ and for any two hyperplanes K, K' in \mathcal{A}^G , the point b_G is generic with respect to $\{K, K'\}$. (K and K' are hyperplanes in G .) The set R of all such generic b is called the *region of genericity*.

Example 3.4. A hyperplane arrangement in \mathbf{C} is just a collection of points $\{c_1, \dots, c_k\}$ and any two such points are parallel hyperplanes. For $i \neq j$, let $L_{i,j}$ be the line through c_i and c_j . The region R of genericity is essentially the complement of $\bigcup L_{i,j}$ except that we should allow points in the open intervals (c_i, c_j) to be in R . Thus,

$$R = \mathbf{C} - \bigcup_{i \neq j} [L_{i,j} - (c_i, c_j)].$$

More generally, suppose $\phi : V \rightarrow \mathbf{C}$ is a linear form and $\{H_1, \dots, H_k\}$ is a family of parallel hyperplanes in V defined by $\phi(H_i) = c_i$. The region of genericity is then $\phi^{-1}(R)$, where R is defined above.

Lemma 3.5. *The set of points b which are generic for an arrangement \mathcal{A} of affine hyperplanes in \mathbf{C}^n is contained in the complement of a finite collection of real hyperplanes (of real codimension one) in \mathbf{C}^n . In particular, the region of genericity for b is open and dense in \mathbf{C}^n .*

Proof. If b is generic, then, for each $G \in L(\mathcal{A})$, the orthogonal projection onto $\pi_G : \mathbf{C}^n \rightarrow G$ must take b to a point b_G which is generic with respect to the complex hyperplanes in \mathcal{A}^G . By Example 3.4, b_G lies in the complement of a finite union of real hyperplanes in G . Since $b \in \pi_G^{-1}(b_G)$, the point b lies in the complement of finitely many real hyperplanes in \mathbf{C}^n . \square

Question. If b and b' are generic base points for \mathcal{A} and $G \in L(\mathcal{A})$, then is $M_{G,b}$ homeomorphic to $M_{G,b'}$?

It seems unlikely that the answer is always affirmative.. On the other hand, it is plausible that if the line segment from b to b' is through generic points then the answer is affirmative.

3.3 Decomposing the complement

In (1.2) we defined, for each $G \in L(\mathcal{A})$, an open subset $M_{G,b}$ of $\mathcal{M}(\mathcal{A})$.

Theorem 3.6. *Suppose b is generic for \mathcal{A} . Then $\mathcal{V} = \{M_{G,b}\}_{G \in L(\mathcal{A})}$ is a poset of subspaces in $\mathcal{M}(\mathcal{A})$.*

For any $z \in \mathcal{M}(\mathcal{A})$, put

$$\mathcal{A}(z) := \{H \in \mathcal{A} \mid z \in E_b(H)\}, \quad \text{and} \quad G(z) := \bigcap \mathcal{A}(z). \quad (3.1)$$

Theorem 3.6 follows from the next lemma.

Lemma 3.7. *Suppose b is generic for \mathcal{A} . Then*

- (i) *For $F, G \in L(\mathcal{A})$, we have $M_{F,b} \cap M_{G,b} = M_{F \wedge G,b}$.*
- (ii) *Any point $z \in \mathcal{M}(\mathcal{A})$ lies in the subspace $M_{G(z),b}$, where $G(z) \in L(\mathcal{A})$ is defined by (3.1).*
- (iii) $\mathcal{M}(\mathcal{A}) = \bigcup_{G \in L(\mathcal{A})} M_{G,b}$.

Proof. (i) We have that $M_{F,b} \cap M_{G,b}$ contains $E_b(H)$ for H in $(\mathcal{A} - \mathcal{A}_F) \cap (\mathcal{A} - \mathcal{A}_G) = \mathcal{A} - (\mathcal{A}_F \cup \mathcal{A}_G)$. Since $\mathcal{A} - (\mathcal{A}_F \cup \mathcal{A}_G) = \mathcal{A} - \mathcal{A}_{F \wedge G}$, statement (i) follows.

(ii) For any $H \in \mathcal{A}(z)$,

$$z \in \left[E_b(H) - \bigcup_{H' \in \mathcal{A} - \mathcal{A}(z)} E_b(H') \right] \subseteq M_{G(z),b}.$$

(iii) This follows from (ii). \square

Proof of Theorem 3.6. We need to check the conditions in Definition 2.1. Statement (iii) of Lemma 3.7 means that \mathcal{V} is an open cover of $\mathcal{M}(\mathcal{A})$. As for conditions (a), (b) and (c) in the definition, (a) is immediate, (b) holds since every subset of $L(\mathcal{A})$ has a greatest lower bound, and (c) follows from part (i) of the lemma. \square

3.4 Identifying the open subsets as complements of central arrangements

Theorem 3.8. *If b is a generic base point for \mathcal{A} and $G \in L(\mathcal{A})$, then the open subset $M_{G,b}$ is homotopy equivalent to the complement of the central arrangement $\mathcal{M}(\mathcal{A}_G)$.*

Suppose $a \in G - \Sigma(\mathcal{A}^G)$ is a nonsingular point of G . The line segment from a to b is denoted $[a, b]$. Let $D(\varepsilon)$ be a closed convex neighborhood of $[a, b]$ in \mathbf{C}^n contained in some ε -neighborhood of $[a, b]$, where ε is small enough so that $D(\varepsilon)$ does not intersect any hyperplane in $\mathcal{A} - \mathcal{A}_G$. (For example $D(\varepsilon)$ could be the region bounded by an ellipsoid with foci at a and b .) For $c = a$ or b , let $\rho_c : \mathbf{C}^n \rightarrow D(\varepsilon)$ be the radial deformation retraction defined by

$$\rho_c(z) = \begin{cases} z, & \text{if } z \in D(\varepsilon); \\ [c, z] \cap \partial D(\varepsilon), & \text{if } z \in \mathbf{C}^n - D(\varepsilon). \end{cases} \quad (3.2)$$

Put $\hat{D} = D(\varepsilon) - \Sigma(\mathcal{A}_G)$. Then

$$\hat{D} = D(\varepsilon) \cap \mathcal{M}(\mathcal{A}_G) = D(\varepsilon) \cap M_{G,b}.$$

(The second equation follows from the fact that $M_{G,b}$ doesn't meet any shadow $E_b(H)$ for $H \notin \mathcal{A}_G$.)

Theorem 3.8 follows immediately from the next lemma.

Lemma 3.9.

- (i) *The map ρ_b restricts to a deformation retraction $M_{G,b} \rightarrow \hat{D}$.*
- (ii) *The map ρ_a restricts to a deformation retraction $\mathcal{M}(\mathcal{A}_G) \rightarrow \hat{D}$.*

Proof. The map ρ_b takes $M_{G,b}$ onto \hat{D} and ρ_a takes $\mathcal{M}(\mathcal{A}_G)$ onto \hat{D} . To check that ρ_b restricts to a deformation retraction on $M_{G,b}$ we need to show that the straight line homotopy between the identity and ρ_b is well-defined, i.e., that for any $z \in M_{G,b}$ the line segment $[z, b] \subset M_{G,b}$. This is precisely the reason we removed the shadows. Similarly, it is clear that ρ_a restricts to a deformation retract on $\mathcal{M}(\mathcal{A}_G)$. \square

3.5 Folkman's Theorem

There is the following well-known theorem of Folkman. (Also, see [22].)

Theorem 3.10. (Folkman [16]). *Suppose \mathcal{A} is an arrangement of affine hyperplanes and that $r(\mathcal{A}) = r$. Then $\Sigma(\mathcal{A})$ is homotopy equivalent to a wedge of $(r - 1)$ -spheres.*

In fact, Folkman shows that the order complex of $L(\mathcal{A})_{>\mathbf{C}^n}$ is a wedge of spheres and that $\Sigma(\mathcal{A})$ is homotopy equivalent to it. Let $\beta(\mathcal{A})$ denote the rank of $\overline{H}^{r-1}(\Sigma(\mathcal{A}))$ (i.e., $\beta(\mathcal{A}) = |\chi(M(\mathcal{A}))|$).

If \mathcal{A} is a central arrangement, then the Hopf fibration provides a homeomorphism, $\mathcal{M}(\mathcal{A}) \cong \mathbf{C}^* \times \mathcal{M}(\mathcal{A}'')$, where \mathcal{A}'' is an arrangement of projective hyperplanes. Its complement in $\mathbf{C}P^{n-1}$ is homeomorphic to the complement of an affine hyperplane arrangement \mathcal{A}' of rank $r - 1$. Put $\beta'(\mathcal{A}) := \beta(\mathcal{A})$.

Given $G \in L(\mathcal{A})$, let r^G and r_G denote the ranks of \mathcal{A}^G and \mathcal{A}_G , respectively. The next lemma is a corollary to Theorem 3.10

Lemma 3.11. *Suppose $G \in L(\mathcal{A})$. Put $\mathcal{P} = L(\mathcal{A})$ and $p = G$. Let $T = \mathbf{C}^n$ denote the minimum element of \mathcal{P} .*

- (i) $H^*(\mathcal{P}_{\geq p}, \mathcal{P}_{>p})$ concentrated in degree r^G , where it is a free abelian group of rank $\beta(\mathcal{A}^G)$.
- (ii) $H^*(\mathcal{P}_{<p}, \mathcal{P}_{(T,p)})$ is concentrated in degree $r_G - 1$, where it is free abelian of rank $\beta'(\mathcal{A}_G)$.

Note that both $|\mathcal{P}_{\geq p}|$ and $|\mathcal{P}_{<p}|$ are cones.

3.6 Remarks concerning trivial coefficients

We drop the base point b from our notation and write M_G instead of $M_{G,b}$.

The decomposition

$$\mathcal{M}(\mathcal{A}) = \bigcup_{G \in L(\mathcal{A})} M_G$$

fits well with the computation of the homology of $\mathcal{M}(\mathcal{A})$ by Brieskorn [3, Lemme 3, p. 27] and Goresky-MacPherson [17, III.1.3. Theorem A]. For each $G \in L(\mathcal{A})$, put $M_{<G} = \bigcup_{F < G} M_F$. Let $r_G := r(\mathcal{A}_G)$ be the rank of the corresponding central arrangement. Put $L_k(\mathcal{A}) := \{G \in L(\mathcal{A}) \mid r_G = k\}$. When not written explicitly the coefficients of a homology group are taken with \mathbf{Z} coefficients with trivial action of π_1 . Here are the main facts about $H_*(M(\mathcal{A}))$.

- (1) (Brieskorn [3]). $H_*(\mathcal{M}(\mathcal{A}))$ is free abelian in each degree. Moreover,

$$H_k(M(\mathcal{A})) = \bigoplus_{G \in L_k(\mathcal{A})} H_k(M_G).$$

(2) For $G \in L(\mathcal{A})$, we have

$$H_i(M_G, M_{<G}) = \begin{cases} 0, & \text{if } i \neq r_G, \\ H_i(M_G), & \text{if } i = r_G. \end{cases}$$

(3) (Goresky-MacPherson [17]). $H_{r_G}(M_G) = H_{r_G-1}(L(\mathcal{A})_{<G}, L(\mathcal{A})_{(\mathbf{C}^n, G)})$. Moreover, the geometric realization of $L(\mathcal{A})_{<G}$ is a cone and the geometric realization of $L(\mathbf{C}^n, G)$ has the homotopy type of a wedge of $(r_G - 2)$ -spheres.

What does this all mean? Property (1) states that in degree k the homology of $\mathcal{M}(\mathcal{A})$ is a sum of homology groups coming from central arrangements corresponding to subspaces of codimension k . Property (2) means that the relative homology of $(M_G, M_{<G})$ is the same as that of a disjoint union pairs $(D, \partial D)$ of r_G -cells relative to their boundaries. So, for $M_k := \bigcup_{G \in L_k(\mathcal{A})} M_G$, the filtration:

$$\mathcal{M}(\mathcal{A}) = M_r \supset M_{r-1} \cdots \supset M_0 = M_{\mathbf{C}^n}$$

looks like the filtration of a CW complex by its skeleta. The meaning of (1) (or of (2)) is that this “cell structure” is perfect in the sense of Morse Theory.

Remark 3.12. Complements of affine arrangements are homotopy equivalent to “minimal” CW complexes, cf. [1, ?]. (A CW complex is *minimal* if the number of its i -cells is equal to its i^{th} Betti number.) Facts (1), (2), (3) above suggest that $\mathcal{M}(\mathcal{A})$ should be homotopy equivalent to a minimal CW complex X with a poset of spaces structure $\{X_G\}_{G \in L(\mathcal{A})}$ adapted to the M_G . In particular, X_G should be a r_G -dimensional subcomplex homotopy equivalent to M_G and number of its top dimensional cells should be the number $\beta'(\mathcal{A}_G)$ from Lemma 3.11(ii).

3.7 Applying the spectral sequence

We begin this subsection by recalling some facts from [20] about ℓ^2 -Betti numbers. Given a group π , let $\mathcal{N}\pi$ denote its group von Neumann algebra. As explained in [20], there is a function, $\{\mathcal{N}\pi\text{-modules}\} \rightarrow [0, \infty]$, denoted by $N \mapsto \dim_{\mathcal{N}\pi} N$, so that any $\mathcal{N}\pi$ -modules N decomposes into its “torsion” and “projective parts”: $N = T(N) \oplus P(N)$ with $\dim_{\mathcal{N}\pi} N = \dim_{\mathcal{N}\pi} P(N)$ and $\dim_{\mathcal{N}\pi} T(N) = 0$. The class of torsion $\mathcal{N}\pi$ -modules is a Serre class, $\mathcal{C}(\pi)$

(cf. [6, Section 6]). For a space X with $\pi_1(X) = \pi$, take cohomology with local coefficients in $\mathcal{N}\pi$ and then define the ℓ^2 -Betti numbers of X by

$$\ell^2 b^k(X) := \dim_{\mathcal{N}\pi} H^k(X; \mathcal{N}\pi).$$

We can recover the results of [6, 7] by applying the spectral sequence of Theorem 2.2 to the case where the poset of spaces is $\{M_G\}_{G \in L(\mathcal{A})}$. Let $\pi = \pi_1(\mathcal{M}(\mathcal{A}))$ and $\pi_G = \pi_1(M_G) = \pi_1(\mathcal{M}(\mathcal{A}_G))$.

Theorem 3.13. (cf. [6]). *Suppose \mathcal{A} is an affine arrangement of rank r . Then, for $i \neq r$, the i^{th} ℓ^2 -Betti number of $\mathcal{M}(\mathcal{A})$ is 0, while $\ell^2 b^r(\mathcal{M}(\mathcal{A})) = \beta(\mathcal{A})$.*

Proof. Here we are interested in the cohomology of $\mathcal{M}(\mathcal{A})$ with local coefficients in the von Neumann algebra $\mathcal{N}\pi$. The $\mathcal{N}\pi$ coefficient system on M_G is induced from coefficients in $\mathcal{N}\pi_G$. For a nontrivial central arrangement such as \mathcal{A}_G , with $G \neq \mathbf{C}^n$, cohomology with coefficients in its von Neumann algebra is an $\mathcal{N}\pi_G$ torsion module in each degree. So, for $G > \mathbf{C}^n$, we have $H^j(M_G; \mathcal{N}\pi) = H^j(M_G; \mathcal{N}\pi_G) \otimes_{\pi_G} \mathcal{N}\pi$ which has $\mathcal{N}\pi$ -dimension 0 for all j . For $G = \mathbf{C}^n$, the only nonzero term is $H^0(M_{\mathbf{C}^n}) \otimes \mathcal{N}\pi = H^0(\text{point}) \otimes \mathcal{N}\pi$. Using the spectral sequence modulo the Serre class $\mathcal{C}(\pi)$, it follows as in [6, p. 308] that $\dim_{\mathcal{N}\pi} E_2^{i,j} \neq 0$ only for $(i, j) = (r, 0)$ and that $\ell^2 b^r(\mathcal{M}(\mathcal{A})) = \dim_{\mathcal{N}\pi} E_2^{r,0}$, which is equal to

$$\begin{aligned} & \dim_{\mathcal{N}\pi} [H^r(L(\mathcal{A})_{\geq \mathbf{C}^n}, L(\mathcal{A})_{> \mathbf{C}^n}) \otimes \mathcal{N}\pi] \\ &= \dim_{\mathbf{Q}} H^r(L(\mathcal{A})_{\geq \mathbf{C}^n}, L(\mathcal{A})_{> \mathbf{C}^n}; \mathbf{Q}) = b^r(\mathbf{C}^n, \Sigma(\mathcal{A})) = \beta(\mathcal{A}). \end{aligned}$$

□

Theorem 3.14. (cf. [7]). *Suppose \mathcal{A} is an affine arrangement of rank r . Then $H^i(\mathcal{M}(\mathcal{A}); \mathbf{Z}\pi) = 0$ for $i \neq r$ and $H^r(\mathcal{M}(\mathcal{A}); \mathbf{Z}\pi)$ is free abelian (usually of infinite rank).*

Proof. Since \mathcal{A}_G is a central arrangement, we have $\mathcal{M}(\mathcal{A}_G) \cong \mathbf{C}^* \times \mathcal{M}(\mathcal{A}'_G)$, where \mathcal{A}'_G is an affine hyperplane arrangement of rank $r_G - 1$. Using induction on the rank and arguing as in [7], we conclude that $H^j(\mathcal{M}(\mathcal{A}_G); \mathbf{Z}\pi_G) \neq 0$ only for $j = r_G$ and that it is free abelian of rank $\beta'(\mathcal{A}_G)$ for $j = r_G$. If $G, F \in L(\mathcal{A})$ with $G < F$, then $r_G < r_F$; hence, $H^*(M_F; \mathbf{Z}\pi) \rightarrow H^*(M_G; \mathbf{Z}\pi)$ is the zero map for dimensional reasons. (Recall that, by Theorem 3.8,

$M_G \sim \mathcal{M}(\mathcal{A}_G)$.) So, (Z') holds. The $E_2^{i,j}$ term of our spectral sequence is a sum of terms of the form:

$$H^i(L(\mathcal{A})_{\geq G}, L(\mathcal{A})_{>G}; H^j(M_G; \mathbf{Z}\pi)). \quad (3.3)$$

By Lemma 3.11 (i), $H^i(L(\mathcal{A})_{\geq G}, L(\mathcal{A})_{>G}) \neq 0$ only for $i = r^G$ and it is free abelian in that degree. Hence, the expression in (3.3) is nonzero only for $(i, j) = (r^G, r_G)$. Since $r^G + r_G = r$, the $E_2^{i,j}$ terms are concentrated along the isogonal $i + j = r$. The theorem follows. \square

4 Toric arrangements

4.1 The general set-up

Suppose a group Γ acts on a poset \mathcal{P} . To these data one associates a category $\Gamma \ltimes \mathcal{P}$. The objects of $\Gamma \ltimes \mathcal{P}$ are the elements of \mathcal{P} . The morphisms $\text{Hom}(p, p')$ are $\{\gamma \in \Gamma \mid \gamma p \leq p'\}$. The geometric realization (or “nerve”) of the category $\Gamma \ltimes \mathcal{P}$ is Γ -equivariantly homotopy equivalent to $E\Gamma \times |\mathcal{P}|$, where $E\Gamma$ is the universal cover of the classifying space $B\Gamma$ of Γ . There is a quotient poset $\mathcal{Q} := \mathcal{P}/\Gamma$ with projection $f : \mathcal{P} \rightarrow \mathcal{Q}$. The quotient of $\Gamma \ltimes \mathcal{P}$ by Γ is a category $C(\Gamma, \mathcal{Q})$. The objects of $C(\Gamma, \mathcal{Q})$ are the elements of \mathcal{Q} . If $q \leq q'$, then the set of morphisms in $C(\Gamma, \mathcal{Q})$ from q to q' is nonempty and can be identified with the elements of $\gamma \in \Gamma_p$ for any $p \in \pi^{-1}(q)$. Here Γ_p denotes the isotropy subgroup at $p \in \mathcal{P}$. ($C(\Gamma, \mathcal{Q})$ is a “complex of groups” as described in [18].)

As in Definition 2.1, suppose $\{X_q\}_{q \in \mathcal{Q}}$ is a poset of subspaces over a poset \mathcal{Q} . Let X denote the union of the X_q . Suppose each X_q is path connected. Let $\pi : Y \rightarrow X$ be a regular covering space with group of deck transformations Γ . Let \mathcal{P} be the set of path components of the $\pi^{-1}(X_q)$ as q ranges over \mathcal{Q} . For each $p \in f^{-1}(q)$, let Y_p denote the corresponding path component of $\pi^{-1}(X_q)$. The Γ -action on Y induces a Γ -action on \mathcal{P} with quotient poset \mathcal{Q} .

The Γ -action on Y gives the data for a diagram of spaces, i.e., a functor from $\Gamma \ltimes \mathcal{P}$ to the category **Top** of topological spaces. This functor takes an object $p \in \mathcal{P}$ to a space Y_p . Its homotopy colimit can be identified with $E\Gamma \times Y$ (which is Γ -equivariantly homotopy equivalent to Y). This induces a functor F from $C(\Gamma, \mathcal{Q})$ to **Top** with

$$\text{hocolim}(F) = E\Gamma \times_{\Gamma} Y$$

(which is homotopy equivalent to X).

Suppose A is a system of local coefficients on X . There is an induced local coefficient system on Y (also denoted by A). For each $q \in \mathcal{Q}$, choose a representative $s(q) \in f^{-1}(q) \subset \mathcal{P}$. For each nonnegative integer j we have a contravariant functor \mathcal{H}^j from $\Gamma \ltimes \mathcal{P}$ to the category \mathbf{Ab} of abelian groups defined on objects by

$$\mathcal{H}^j(p) = H^j(Y_p; A).$$

(So, \mathcal{H}^j is a “diagram of coefficients” or, as in [18], a “right $\Gamma \ltimes \mathcal{P}$ -module.”) Similarly, there is a functor \mathcal{H}^j from $C(\Gamma, \mathcal{Q})$ to \mathbf{Ab} defined by $\mathcal{H}^j(q) = H^j(Y_{s(q)}; A)$.

Condition (Z') from subsection 2.2 can be restated as follows:

(Z') if $q < q'$, then for all j , $\mathcal{H}^j(q') \rightarrow \mathcal{H}^j(q)$ is the 0-map.

Also consider the following condition:

(T) for all $p \in \mathcal{P}$ and $j \in \mathbf{Z}_+$, the action of Γ_p on $\mathcal{H}^j(p)$ is trivial.

Since the natural map $Y \rightarrow |\mathcal{P}|$ is Γ -equivariant, it induces a map,

$$E\Gamma \times_\Gamma Y \rightarrow E\Gamma \times_\Gamma |\mathcal{P}|.$$

There is corresponding a Leray-Serre spectral sequence:

$$E_2^{i,j} = H^i(E\Gamma \times_\Gamma |\mathcal{P}|; \mathcal{H}^j), \quad (4.1)$$

converging to $H^*(X; A)$ (since $X \sim E\Gamma \times_\Gamma Y$). We then get the following generalization of Theorem 2.2.

Theorem 4.1. *If (Z') holds, then the spectral sequence (4.1) decomposes as a direct sum:*

$$E_2^{i,j} = \bigoplus_{q \in \mathcal{Q}} H^i(E\Gamma \times_{\Gamma_{s(q)}} (\mathcal{P}_{\geq s(q)}, \mathcal{P}_{> s(q)}); H^j(Y_{s(q)}; A)),$$

where the coefficients in each summand are locally constant. Moreover, when (T) holds, the coefficients in each summand are constant.

The E_2 page of the spectral sequence decomposes as a direct sum for the same reason as before. The point is that when $\Gamma_{s(q)}$ acts trivially on $H^j Y_{s(q)}; A$ the coefficients are untwisted.

4.2 Toric shadows

Let T denote the torus $(\mathbf{C}^*)^n$ and let $\pi : \mathbf{C}^n \rightarrow T$ be the projection map of its universal cover. The group of deck transformations Λ is $2\pi i\mathbf{Z}^n$ acting on \mathbf{C}^n by translations. The standard Euclidean metric on \mathbf{C}^n descends to a flat metric on T .

A *toric hyperplane* in T is a subtorus of complex codimension one in T . A finite set \mathcal{T} of toric hyperplanes in T is a *toric hyperplane arrangement*. A *toric subspace* of \mathcal{T} is a connected component of an intersection of toric hyperplanes – such a subspace is isomorphic to a torus $(\mathbf{C}^*)^k$ for some $0 \leq k \leq n$. The *intersection poset* $L(\mathcal{T})$ is the poset of toric subspaces ordered by reverse inclusion. For each $U \in L(\mathcal{T})$, $\mathcal{T}_U := \{V \in \mathcal{T} \mid U \subset V\}$. Each path component of the inverse image of any toric subspace of $L(\mathcal{T})$ is an affine subspace of \mathbf{C}^n . So, the arrangement \mathcal{T} toric hyperplanes in T lifts to a periodic arrangement \mathcal{A} of affine hyperplanes in \mathbf{C}^n .

Let $\Sigma(\mathcal{T})$ be the union of toric hyperplanes in \mathcal{T} and let $\mathcal{R}(\mathcal{T}) := T - \Sigma(\mathcal{T})$ be the complement. Given a toric hyperplane $V \in \mathcal{T}$, let \tilde{V} be a component of $\pi^{-1}(V)$ (so that \tilde{V} is an affine hyperplane in \mathbf{C}^n). Choose a base point $a \in \mathcal{R}(\mathcal{T})$ and a base point $b \in \pi^{-1}(a)$ closest to \tilde{V} . As before, let $R_b(\tilde{V})$ denote the real affine hyperplane spanned by b and \tilde{V} . As in (1.1), let $E_b(\tilde{V})$ be the open half-space in $R_b(\tilde{V})$ bounded by \tilde{V} . If U is a toric subspace of V , let \tilde{U} be a component of $\pi^{-1}(U)$ which lies in \tilde{V} , and let Λ_U be the stabilizer of \tilde{U} in Λ . As in (1.1), define the *toric shadow* of V by $E_a(V) := \pi(E_b(\tilde{V}))$ and as in (1.2), for each $U \in L(\mathcal{T})$ define $N_{U,a} := \pi(M_{\tilde{U},b})$, i.e.,

$$N_{U,a} = \mathcal{R}(\mathcal{T}) - \bigcup_{V \in \mathcal{T} - \mathcal{T}_U} E_a(V). \quad (4.2)$$

(Note that Λ_U stabilizes $M_{\tilde{U},b}$.)

Lemma 4.2. *Notation is as above.*

- (i) *The complement of the affine arrangement complement $\mathcal{M}(\mathcal{A}_{\tilde{U}})$ is a Λ_U -cover of $\mathcal{R}(\mathcal{T}_U)$.*
- (ii) *$M_{\tilde{U},b}$ is a component of $\pi^{-1}(N_{U,b})$ which contains the intersection of $\mathcal{M}(\mathcal{A})$ with a small neighborhood of \tilde{U} .*

Consequently, $\mathcal{R}(\mathcal{T}_U)$ is homotopy equivalent to $N_{U,b}$.

Proof. The last sentence follows from Theorem 3.8. □

4.3 Cohomology of toric arrangements

A proof of the next lemma can be found in [11].

Lemma 4.3. ([11, Lemma 2.1]). *Suppose \mathcal{T} is an essential arrangement of toric hyperplanes in $T = (\mathbf{C}^*)^n$. Then $H_*(T, \Sigma(\mathcal{T}))$ vanishes except in degree n . Moreover, $H_n(T, \Sigma(\mathcal{T}))$ is free abelian.*

Let $\beta(\mathcal{T})$ denote the rank of $H_n(T, \Sigma(\mathcal{T}))$. Set $\pi = \pi_1(\mathcal{R}(\mathcal{T}))$. Our goal is to prove the following theorem of [11].

Theorem 4.4. *Suppose \mathcal{T} is an essential arrangement of toric hyperplanes in $(\mathbf{C}^*)^n$.*

- (i) ([11, Theorem 5.2]): *$H^*(\mathcal{R}(\mathcal{T}); \mathbf{Z}\pi)$ vanishes in degrees $\neq n$ and $H^n(\mathcal{R}(\mathcal{T}); \mathbf{Z}\pi)$ is free abelian.*
- (ii) ([11, Theorem 5.1]): *Suppose \mathcal{L} is local coefficient system coming from a nonresonant flat line bundle on $\mathcal{R}(\mathcal{T})$. Then $H^*(\mathcal{R}(\mathcal{T}); \mathcal{L})$ vanishes in degrees $\neq n$, while $\dim_{\mathbf{C}} H^n(\mathcal{R}(\mathcal{T}); \mathcal{L}) = \beta(\mathcal{T})$.*
- (iii) ([11, Theorem 5.3]): *For $i \neq n$, the i^{th} ℓ^2 -Betti number of $\mathcal{R}(\mathcal{T})$ is 0, while $\ell^2 b^n(\mathcal{R}(\mathcal{T})) = \beta(\mathcal{T})$.*

A corollary to Lemma 4.3 is the following version of Lemma 3.11 (i).

Lemma 4.5. *Suppose $\tilde{U} \in L(\mathcal{A})$ and $\pi(\tilde{U}) = U \in L(\mathcal{T})$. Put $r^U = \dim \tilde{U}$. Then*

$$H^*(E\Lambda \times_{\Lambda_U} (L(\mathcal{A})_{\geq \tilde{U}}, L(\mathcal{A})_{> \tilde{U}}))$$

vanishes in degrees $\neq r^U$ and is free abelian of rank $\beta(\mathcal{T}^U)$ in degree r^U .

Proof. For the Borel constructions, we have homotopy equivalences:

$$\begin{aligned} E\Lambda \times_{\Lambda_U} |L(\mathcal{A})_{\geq \tilde{U}}| &\sim \tilde{U}/\Lambda_U, \\ E\Lambda \times_{\Lambda_U} |L(\mathcal{A})_{> \tilde{U}}| &\sim \Sigma(\mathcal{A}^{\tilde{U}})/\Lambda_U. \end{aligned}$$

Moreover, $\tilde{U}/\Lambda_U = U$ and $\Sigma(\mathcal{A}^{\tilde{U}})/\Lambda_U = \Sigma(\mathcal{T}^U)$. So, the result is immediate from Lemma 4.3. \square

The next lemma is needed to show that Condition (T) of subsection 4.1 holds.

Lemma 4.6. *Let Λ_G denote the stabilizer of $G \in L(\mathcal{A})$. The Λ_G -action on $\mathcal{M}(\mathcal{A}_G)$ is through homeomorphisms which are homotopic to the identity.*

Proof. The group Λ_G is isomorphic to \mathbf{Z}^{r_G} acting on G by translations. So, the Λ_G -action on G is homotopically trivial. Suppose G^\perp is the orthogonal complement to G in \mathbf{C}^n and that \mathcal{A}^\perp is the arrangement in G^\perp induced by \mathcal{A}_G . The group Λ_G acts trivially on G^\perp and hence, on $\mathcal{M}(\mathcal{A}^\perp)$. Since the complement $\mathcal{M}(\mathcal{A}_G)$ decomposes as $\mathcal{M}(\mathcal{A}_G) = G \times \mathcal{M}(\mathcal{A}^\perp)$, the Λ_G -action on $\mathcal{M}(\mathcal{A}_G)$ also is homotopically trivial. \square

We sketch the proofs of the three parts of Theorem 4.4 below.

Proof of Theorem 4.4. (i). Let $\tilde{U} \in L(\mathcal{A})$. As in Lemma 3.11 (i), Lemma 4.3 shows that

$$H^*(E\Lambda \times_{\Lambda_U} (L(\mathcal{A})_{\geq \tilde{U}}, L(\mathcal{A})_{> \tilde{U}}))$$

vanishes in degrees $\neq r^U$ and is free abelian of rank $\beta(\mathcal{T}^U)$ in degree r^U . Put $\pi_U = \pi_1(\mathcal{M}(\mathcal{A}_{\tilde{U}}))$. As in the proof of Theorem 3.14, since $M_{\tilde{U}} \sim \mathcal{M}(\mathcal{A}_{\tilde{U}})$, the coefficient group $H^j(M_{\tilde{U}}; \mathbf{Z}\pi_U) \neq 0$ only for $j = r_{\tilde{U}}$ and it is free abelian for $j = r_{\tilde{U}}$. Hence, the same is true for $H^j(M_{\tilde{U}}; \mathbf{Z}\pi) = H^j(M_{\tilde{U}}; \mathbf{Z}\pi_U) \otimes_{\pi_U} \mathbf{Z}\pi$. As before, Condition (Z') holds for dimensional reasons and condition (T) holds by Lemma 4.6. By Theorem 4.1, we have a spectral sequence whose $E_2^{i,j}$ term decomposes as a direct sum of terms of the form:

$$H^i(E\Lambda \times_{\Lambda_U} (L(\mathcal{A})_{\geq \tilde{U}}, L(\mathcal{A})_{> \tilde{U}}); H^j(M_{\tilde{U}}; \mathbf{Z}\pi)). \quad (4.3)$$

The expression in (4.3) is nonzero only for $(i, j) = (r^U, r_{\tilde{U}})$. Since $r^U + r_{\tilde{U}} = n$, the $E_2^{i,j}$ terms are concentrated along the isogonal $i + j = n$. Part (i) follows.

(iii). Using Theorem 4.1, part (iii) follows from the arguments in the proof of Theorem 3.13

(ii). The argument in this case is similar, only easier. The point is that since \mathcal{L} is nonresonant and M_G is the complement of a central arrangement, for all $G > \mathbf{C}^n$, we have $H^*(M_G; \mathcal{L}) = 0$ in all degrees, while for $G = \mathbf{C}^n$, the only nonzero term is $H^0(M_{\mathbf{C}^n}; \mathcal{L}) \cong \mathbf{C}$. It follows that $E_2^{i,j} = 0$ except for $(i, j) = (n, 0)$, in which case:

$$E_2^{n,0} = H^n(E\Lambda \times (L(\mathcal{A})_{\geq \mathbf{C}^n}, L(\mathcal{A})_{> \mathbf{C}^n})),$$

which has rank $\beta(\mathcal{T})$. \square

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