Criteria for asphericity of polyhedral products: corrigenda to "Right-angularity, flag complexes, asphericity"

Michael W. Davis   Peter H. Kropholler

January 12, 2015

Abstract

Given a simplicial complex L with vertex set I and a family $\mathbf{A} = \{(A(i), B(i))\}_{i \in I}$ of pairs of spaces with base points $*_i \in B(i)$, there is a definition of the "polyhedral product" $\mathbf{A}^L$ of $\mathbf{A}$ with respect to L. Sometimes this is called a "generalized moment angle complex". This note concerns two refinements to earlier work of the first author. First, when L is infinite, the definition of polyhedral product needs clarification. Second, the earlier paper omitted some subtle parts of the necessary and sufficient conditions for polyhedral products to be aspherical. Correct versions of these necessary and sufficient conditions are given in the present paper.

AMS classification numbers. Primary: 20F65, 57M07, 57M10. Secondary: 20F36, 20F55, 20J05.

Keywords: aspherical space, graph product, polyhedral product, right-angled Coxeter group, right-angled building.

The definition of a polyhedral product Let L be a simplicial complex. Suppose that $\mathbf{A} = \{(A(i), B(i))\}_{i \in I}$ is a collection of pairs of spaces, with base points $*_i \in B(i)$, indexed by the vertex set I of L. As a set, the *polyhedral product* of $\mathbf{A}$ with respect to L is the subset $\mathbf{A}^L$ of the product $\prod_{i \in I} A(i)$ consisting of the points $(x_i)_{i \in I}$ satisfing the following two conditions:

(I) $x_i = *_i$ for all but finitely many i,

(II) $\{i \in I \mid x_i \notin B(i)\}$ is a simplex of L.

For pairs $M < M'$ of subcomplexes of L, the base points determine natural inclusions $\mathbf{A}^M \hookrightarrow \mathbf{A}^{M'}$. When L is finite, the polyhedral product inherits its topology from the product $\prod_{i \in I} A(i)$. In general $\mathbf{A}^L$ acquires the topology of the colimit $\varinjlim \mathbf{A}^M$ as M runs through the finite subcomplexes.

The above definition is given in [5, §4.1]. When L is finite, condition (I) is vacuous and the definition reduces to the one in [4, §2.1]. The restriction to finite L in [4] is implicit but inessential.

In [1] and [4], the notation $\mathcal{Z}_L(\mathbf{A}, \mathbf{B})$ was used for the polyhedral product, here denoted simply by $\mathbf{A}^L$.

It is assumed that all the spaces $A(i)$ and $B(i)$ are CW-complexes. In particular, simply connected aspherical spaces are contractible.

Asphericity of polyhedral products We assume each $A(i)$ is path connected. Let G_i denote the fundamental group $\pi_1(A(i), *_i)$ and let $E_i = \pi_1(A(i), B(i), *_i)$, so that E_i is a G_i-set. Define a subset I'' of I, by

$$I'' = \{i \in I \mid \operatorname{Card}(E_i) \neq 1\} \tag{1}$$

and let L'' be the full subcomplex of L with vertex set I''.

In the statement and proof of [4, Theorem 2.22] two implicit assumptions are made:

(a) for each path component of $B(i)$, the induced map of fundamental groups $\pi_1(B(i)) \to \pi_1(A(i))$ is injective, and

(b) when $B(i)$ is path connected, the image of $\pi_1(B(i))$ is a proper subgroup of $\pi_1(A(i))$.

Under assumptions (a) and (b), it is shown in [4] that $\mathbf{A}^L$ is aspherical if and only if the following three conditions hold:

- Each $A(i)$ is aspherical.
- If a vertex i is not joined to every other vertex by an edge, then $(A(i), B(i))$ is an aspherical pair. (See Definition 2, below, for the meaning of "aspherical pair.")
- L is a flag complex.

In proving the necessity of these conditions, assumption (a) leads to a conclusion that certain pairs $(A(i), B(i))$ must be aspherical; without (a), the $(A(i), B(i))$ are only "nearly aspherical" (cf. Definition 2, below). Assumption (b) implies that $I = I''$ and hence $L = L''$. When (b) does not hold further modifications are needed in [4, Theorem 2.22] (cf. Example 1 below).

Example 1. (cf. [4, Example 2.1]). If for each $i \in I$, $(A(i), B(i)) = ([0,1], \{1\})$, then the polyhedral product $([0,1], \{1\})^L$ is elsewhere called a *chamber* and denoted $K(L)$. Since $K(L)$ is isomorphic to the cone on the barycentric subdivision of L, the space $([0,1], \{1\})^L$ is contractible for any L.

Example 2. More generally, if, for each i, $A(i)$ is contractible and $B(i)$ is acyclic, then $\mathbf{A}^L$ is simply connected and the the natural map $\mathbf{A}^L \to ([0,1], \{1\})^L$ induces an isomorphism on homology; hence, $\mathbf{A}^L$ is contractible for any L.

Example 3. If L is a simplex, then $\mathbf{A}^L = \prod_{i \in I} A(i)$. Hence, $\mathbf{A}^L$ is aspherical if and only if each $A(i)$ is aspherical – no condition is needed on the $B(i)$.

Definition 1. (cf. [4, p. 249]). A vertex of a simplicial complex L is *conelike* if it is joined by an edge to every other vertex of L

Definition 2. (cf. [4, Definition 2.21]). Suppose (A, B) is a pair of spaces with A path connected. Let $\widetilde{A}$ denote the universal cover of A and let $\widetilde{B}$ be the inverse image of B in $\widetilde{A}$. Then (A, B) is an *aspherical pair* if $\widetilde{A}$ is contractible and each component of $\widetilde{B}$ is contractible (in other words, A is aspherical, each path component of B is aspherical and the fundamental group of each such component maps injectively into $\pi_1(A)$). The pair (A, B) is *nearly aspherical* if $\widetilde{A}$ is contractible and if each path component of $\widetilde{B}$ is acyclic.

For any subset $J \leq I$, write $\langle J \rangle$ for the full subcomplex of L spanned by J. Given a field $\mathbb{F}$ define disjoint subsets of $I - I''$:

$$I_1 := \{i \in I - I'' \mid \widetilde{B}(i) \text{ is acyclic with coefficients in } \mathbb{Z}\}$$
$$I_2 := \{i \in I - I'' \mid \widetilde{B}(i) \text{ is not acyclic with coefficients in } \mathbb{F}\},$$

Put

$$I' = I_2 \cup I'' \quad \text{and} \quad L' = \langle I' \rangle. \tag{2}$$

From now on we assume the following:

(∗) *for some field* $\mathbb{F}$, $I_1 \cup I_2$ *is a partition of* $I - I''$.

Although this assumption seems fairly mild, without it there is a problem finding a clean statement of our theorem (cf. Example 4 below).

The corrected statement of Theorem 2.22 in [4] is the following.

Theorem 1. *Suppose that* I *is the vertex set of a simpliciial complex* L, *that* $\mathbf{A} = \{(A(i), B(i))\}_{i \in I}$ *is a family of pairs and that condition* (∗) *holds. The following four conditions are necessary and sufficient for* $\mathbf{A}^L$ *to be aspherical.*

(i) *Each* $A(i)$ *is aspherical.*

(ii) *If* $i \in I'$ *is not conelike in* L', *then* $(A(i), B(i))$ *is nearly aspherical.*

(iii) *The vertices of* I_2 *are conelike in* L'. *Moreover,* $\langle I_2 \rangle$ *is a simplex and* L' *is the join* $\langle I_2 \rangle * L''$

(iv) L' *is a flag complex.*

Remark. When $I = I''$, Theorem 2.22 of [4] is almost correct as stated, except that in condition (ii) the phrase "aspherical pair" should be replaced by "nearly aspherical pair." Condition (iii) of Theorem 2.22 reads L instead of L' (as in condition (iv) above). The point is that for $\mathbf{A}^L$ to be aspherical it is not necessary that L be a flag complex but only that L' be a flag complex. This should have been obvious from Example 2.1 in [4] (which is essentially Example 1, above).

An important observation in the proof of Theorem 2.22 of [4] is that if $J < I$, then the natural map $r : \mathbf{A}^L \to \mathbf{A}^{\langle J \rangle}$ induced by the projection, $\prod_{i \in I} A(i) \to \prod_{i \in J} A(i)$ is a retraction (cf. [4, Lemma 2.5]). Applying this to the case where J is a singleton, we see that if $\mathbf{A}^L$ is aspherical, then so is each $A(i)$.

As in [4], Theorem 1 is proved by reducing to the special case where each pair $(A(i), B(i))$ has the form $(\mathrm{Cone}(E_i), E_i)$, for some discrete set E_i, where $\mathrm{Cone}(E_i)$ means the cone on E_i. Put $\mathbf{C} := \{(\mathrm{Cone}(E_i), E_i)\}_{i \in I}$. Consider the polyhedral product $\mathbf{C}^L$ and its universal cover $\widetilde{\mathbf{C}^L}$. A key point is that $\widetilde{\mathbf{C}^L}$ is the standard realization of a right-angled building (abbreviated as RAB) if and only if the following two conditions hold:

- for each $i \in I$, $\mathrm{Card}(E_i) \geq 2$, and
- L is a flag complex.

This is stated in [4, Example 2.8] and a proof is given in [3]. Let W denote the right-angled Coxeter group associated to the 1-skeleton of L and let $\mathcal{C}$ denote the underlying chamber system of the RAB.

One flaw in [4] traces back to the assertion in the proof of Lemma 2.11, when L is a flag complex, the chamber system $\mathcal{C}$ (obtained, as in [3], via a covering of a nonstandard realization of $\prod_{i\in I} E_i$) is a RAB. However, this is only true when each E_i has more than one element, i.e., when the statement after the first bullet point holds.

Let $I'' < I$ be the subset defined by (1). As before, put $L'' = \langle I'' \rangle$. The correct version of Lemma 2.11 in [4] is the following.

Lemma 1. *Let* $\mathbf{C} := \{(\mathrm{Cone}(E_i), E_i)\}_{i\in I}$. *The polyhedral product* $\mathbf{C}^L$ *is aspherical if and only if* L'' *is a flag complex.*

Proof. First observe that the inclusion $\mathbf{C}^{L''} \hookrightarrow \mathbf{C}^L$ is a homotopy equivalence. The reason is that there is a retraction $r\colon \mathbf{C}^L \to \mathbf{C}^{L''}$ induced by the projection $\prod_{i\in I} \mathrm{Cone}(E_i) \to \prod_{i\in I''} \mathrm{Cone}(E_i)$; moreover, r is a deformation retraction. (Since, for each $i \in I - I''$, E_i is a singleton, $\mathrm{Cone}(E_i)$ deformation retracts onto E_i.)

The family $\{E_i\}_{i\in I''}$ and the right-angled Coxeter group W'' associated to the 1-skeleton of L'' define a right-angled RAB with underlying chamber system $\mathcal{C}''$ (defined as in [3] by taking the universal cover of a nonstandard realization of the product building, $\prod_{i\in I''} E_i$). Suppose L'' is a flag complex. As was pointed out above, $\widetilde{\mathbf{C}^{L''}}$ is the standard realization of $\mathcal{C}''$. Therefore, it is CAT(0) and hence contractible (cf. [2, Thm.18.3.1, p.338]). So, $\mathbf{C}^{L''}$ is aspherical and by the previous paragraph, $\mathbf{C}^L$ also is aspherical.

Conversely, suppose L'' is not a flag complex. Then $\widetilde{\mathbf{C}^{L''}}$ is a nonstandard realization of $\mathcal{C}''$, i.e., $\widetilde{\mathbf{C}^{L''}} = \mathcal{U}(\mathcal{C}'', K(L''))$, where $K(L'')$ is the chamber dual to L'' and where $\mathcal{U}(\mathcal{C}'', K(L''))$ means the "basic construction" defined as in [2]. An apartment in this nonstandard realization is $\mathcal{U}(W'', K(L''))$. Since L'' is not a flag complex, the realization of the apartment, $\mathcal{U}(W'', K(L''))$, is not contractible (see [2, Thm.9.1.4, p.167]). Since $\mathcal{U}(\mathcal{C}'', K(L''))$ retracts onto the (nonstandard) realization of any apartment, we see that $\widetilde{\mathbf{C}^{L''}}$ cannot be contractible and hence, $\widetilde{\mathbf{C}^{L}}$ also cannot be contractible. □

Lemma 2. *Suppose* L *is a simplicial complex consisting of two vertices. Let* $I = \{1, 2\}$ *be its vertex set and let* $\mathbf{A} = \{(A(i), B(i))\}_{i\in I}$ *be two pairs of spaces such that* $A(i)$ *path connected, neither* $\widetilde{B}(1)$ *nor* $\widetilde{B}(2)$ *is acyclic (i.e.,*

$I_1 = \emptyset$), and condition (*) holds. Then $\mathbf{A}^L$ is aspherical if and only if, for $i = 1, 2$, $(A(i), B(i))$ is nearly aspherical.

Proof. For $i = 1, 2$, let E_i denote the set of path components $\widetilde{B}(i)$ and let $\mathbf{C}$ denote the family $\{(\mathrm{Cone}(E_i), E_i)\}_{i \in I}$. Then $\mathbf{C}^L$ is a complete bipartite graph Ω with vertex set $E_1 \coprod E_2$. Similarly, $\widetilde{\mathbf{A}}^L$ is a graph of spaces for the same graph Ω: the set of vertex spaces for one type is $\{\widetilde{B}(1)_e \times \widetilde{A}(2)\}_{e \in E_1}$ and for the other type it is $\{\widetilde{A}(1) \times \widetilde{B}(2)_f\}_{f \in E_2}$. Here $\widetilde{B}(1)_e$ (resp. $\widetilde{B}(2)_f$) means the path component corresponding to $e \in E_1$ (resp. $f \in E_2$). Since $\widetilde{A}(i)$ is simply connected, $\pi_1(\widetilde{\mathbf{A}}^L) \cong \pi_1(\Omega)$. (The reason is that the cover of $\widetilde{\mathbf{A}}^L$ corresponding to the universal cover of Ω is a tree of spaces with simply connected vertex spaces.)

Suppose $\mathbf{A}^L$ is aspherical. Since $A(i)$ is a retract of $\mathbf{A}^L$, it also is aspherical; so its universal cover $\widetilde{A}(i)$ is contractible. Since the universal cover of $\widetilde{\mathbf{A}}^L$ is contractible, the natural map $\widetilde{\mathbf{A}}^L \to \Omega$ is a homotopy equivalence. In particular, it induces an isomorphism on homology. On the other hand, when each $\widetilde{A}(i)$ is contractible, there is a formula for the homology of any polyhedral product (cf. [1, Thm.2.21]). In the case at hand, it gives that $\widetilde{\mathbf{A}}^L$ has the same homology as the suspension of a smash product, $S^0 * (\widetilde{B}(1) \wedge \widetilde{B}(2))$. So, $\widetilde{B}(1) \wedge \widetilde{B}(2)$ must have the same homology as the discrete space $E_1 \wedge E_2$. By condition (*) there is a field $\mathbb{F}$ so that $I_2 \cup I''$ is a partition of $\{1, 2\}$. With coefficients in $\mathbb{F}$, the homology of a smash product is concentrated in degree 0 if and only if both factors are concentrated in degree 0 or if one of them is acyclic over $\mathbb{F}$. If $I_2 = \{1, 2\}$, then both $\widetilde{B}(1)$ and $\widetilde{B}(2)$ are connected and have nontrivial homology in positive degrees with $\mathbb{F}$ coefficients, which is impossible. If I_2 is a singleton, then the other element of $\{1, 2\}$ is in I'' (since $I_1 = \emptyset$) and we again get a contradiction. So, $I_2 = \emptyset$, i.e., neither $\widetilde{B}(1)$ nor $\widetilde{B}(2)$ is connected. In the case at hand, this means that, for $i = 1, 2$, each component of $\widetilde{B}(i)$ is acyclic (over $\mathbb{Z}$). Hence $(\widetilde{A}(i), \widetilde{B}(i))$ is nearly aspherical.

Conversely, suppose each $(\widetilde{A}(i), \widetilde{B}(i))$ is nearly aspherical. There is a $\pi_1(\Omega)$-equivariant map of trees of spaces, from the universal cover of $\widetilde{\mathbf{A}}^L$ to $\widetilde{\mathbf{C}^L}$, which induces an isomorphism on homology. Hence the universal cover of $\widetilde{\mathbf{A}}^L$ is contractible. $\square$

Example 4. For $i = 1, 2$, suppose $\widetilde{A}(i)$ is contractible and that the reduced homology of $\widetilde{B}(i)$ is p_i-torsion for distinct primes p_1 and p_2. Then $\widetilde{B}(1) \wedge \widetilde{B}(2)$ is acyclic over $\mathbb{Z}$. Hence, for $L = \langle \{1, 2\} \rangle$ as in Lemma 2, $\widetilde{\mathbf{A}}^L$ is contractible.

Proof of Theorem 1. First suppose $\mathbf{A}^L$ is aspherical. As noted previously, this implies (i). Suppose $i \in I'$ is not conelike in L'. Then there is another vertex $j \in I'$ not connected by an edge to i. Since $\mathbf{A}^{\langle\{i,j\}\rangle}$ is a retract of $\mathbf{A}^L$, it is also aspherical. Applying Lemma 2, we see that (ii) holds. For each $i \in I_2$, $\widetilde{B}(i)$ is path connected and not acyclic, so, $(A(i), B(i))$ is not a nearly aspherical pair; hence, i must be conelike. By assumption, there is a field $\mathbb{F}$ so that for each $i \in I_2 \cup I''$, $\widetilde{B}(i)$ is not acyclic with coefficients in $\mathbb{F}$. (If $i \in I''$, then $\widetilde{B}(i)$ is not connected.) Suppose there is an "empty simplex" in $\langle I_2 \cup I'' \rangle$, i.e., a subset $J < I'$ so that $\langle J \rangle$ is isomorphic to the boundary of a k-simplex, $k \geq 2$. Since the reduced homology (with coefficients in $\mathbb{F}$) of smash product of $\{\widetilde{B}(i)\}_{i\in J}$ is nonzero, it follows from the formula in [1, Thm. 2.21] that this homology, shifted up in degree by k, appears as a direct summand of $H_*(\widetilde{\mathbf{A}}^{\langle I_2 \cup I''\rangle}; \mathbb{F})$. This shows, first of all, that J cannot be a subset of I_2. (Since $\mathbf{A}^{\langle I_2 \rangle}$ is aspherical, $\widetilde{\mathbf{A}}^{\langle I_2 \rangle}$ must be contractible.) Hence, $\langle I_2 \rangle$ is a simplex. Furthermore, J cannot decompose as $J_2 \cup J''$, where $J_2 = J \cap I_2$ and $J'' = J \cap I''$ are both nonempty. (This is because $\mathbf{A}^{\langle J \rangle}$ is aspherical and has the same fundamental group as $\mathbf{C}^{\langle J'' \rangle}$; hence, the homology of $\widetilde{\mathbf{A}}^{\langle J \rangle}$ must vanish in degrees $\geq \mathrm{Card}(J'')$.) Therefore, L' is the join $\langle I_2 \rangle * L''$, i.e., (iii) holds. A similar argument shows that $\langle I'' \rangle$ cannot contain an empty k-simplex for $k \geq 2$ with vertex set J'', for then the universal cover of $\widetilde{\mathbf{A}}^{\langle J' \rangle}$ has a nonzero homology in degree k (cf. the second paragraph of the proof of Theorem 2.22 in [4, p. 249]). Therefore, L'' and L' are both flag complexes, i.e., (iv) holds.

Conversely, suppose conditions (i), (ii), (iii), (iv) hold. Let J be the set of conelike vertices in L'. Since L' is flag, $\langle J \rangle$ is a simplex and $L' = \langle J \rangle * \langle I' - J \rangle$. Since each $\widetilde{A}(i)$ is contractible, so is $\widetilde{\mathbf{A}}^{\langle J \rangle} = \prod_{i \in J} \widetilde{A}(i)$. Therefore, $\widetilde{\mathbf{A}}^{\langle L' \rangle} = \widetilde{\mathbf{A}}^{\langle J \rangle} \times \widetilde{\mathbf{A}}^{\langle I' - J \rangle}$ is homotopy equivalent to $\widetilde{\mathbf{A}}^{\langle I' - J \rangle}$. Since the universal cover of $\mathbf{C}^{\langle I' - J \rangle}$ is contractible (being the standard realization of a RAB) and since for each $i \in I' - J$, the natural map $(\widetilde{A}(i), \widetilde{B}(i)) \to (\mathrm{Cone}(E_i), E_i)$ induces an isomorphism on homology, it follows that $\widetilde{\mathbf{A}}^{\langle I' - J \rangle}$ is contractible and therefore, so is $\widetilde{\mathbf{A}}^{\langle L' \rangle}$. For any set of vertices $H < I$, we have that the reduced homology of the smash product of $\{\widetilde{B}(i)\}_{i \in H}$ is trivial if $H \cap I_1 \neq \emptyset$. It follows from the formula in [1, Thm. 2.21] that $\widetilde{\mathbf{A}}^L \to \widetilde{\mathbf{A}}^{L'}$ induces an isomorphism on homology and hence is a homotopy equivalence. So, $\widetilde{\mathbf{A}}^L$ is aspherical and therefore, so is $\mathbf{A}^L$. $\square$

Acknowledgement We thank Boris Okun for his help with the proof of Lemma 2.

References

[1] A. Bahri, M. Bendersky, F.R. Cohen and S. Gitler, *The polyhedral product functor: a method of computation for moment-angle complexes, arrangements and related spaces*, Advances in Math. **225** (2010), 1634–1668.

[2] M.W. Davis, *The Geometry and Topology of Coxeter Groups*, London Math. Soc. Monograph Series **32**, Princeton University Press, Princeton, 2008.

[3] M.W. Davis, *Examples of buildings constructed via covering spaces*, Groups Geom. Dyn. **3** (2009), 279–298.

[4] M.W. Davis, *Right angularity, flag complexes, asphericity*, Geom. Dedicata **159** (2012), 239–262.

[5] P.H. Kropholler and A. Martino, *Graph-wreath products and finiteness conditions*, Preprint, arXiv:1407.0302v4, 12pp.

Michael W. Davis, Department of Mathematics, The Ohio State University, 231 W. 18th Ave., Columbus Ohio 43210, `davis.12@math.osu.edu`

Peter H. Kropholler, Mathematical Sciences, University of Southampton, Southampton SO17 1BJ `p.h.kropholler@soton.ac.uk`