AUTOMORPHISMS OF BUILDINGS CONSTRUCTED VIA COVERING SPACES

DISSERTATION

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ABSTRACT

Buildings constructed via covering spaces were first introduced by Davis in [9]. Such buildings make up an interesting class of examples and we will be interested in properties of such buildings involving their automorphism groups.

In Chapter 4 we show that the stabilizer of a chamber C in the group of structure preserving automorphisms of a regular right angled building, E, is an inverse limit of the structure preserving automorphism groups of larger and larger combinatorial balls centered at C. We show that any structure preserving automorphism of a suitably nice ball in E can be extended to a structure preserving automorphism on all of E. We use this to determine the stabilizer of a chamber in a graph product of buildings.

In Chapter 6 we show that any building $\widetilde{\Delta}$ arising from a building Δ by finitely many changes in the Coxeter matrix admits a strongly transitive automorphism group if Δ admits a chamber transitive automorphism group and a pre-Moufang set of root groups of an apartment. This immediately implies that in this case $\mathrm{Aut}(\widetilde{\Delta})$ admits a (B, N)-pair.

Finally in Chapter 7 we show that the notion of generalized graph product of a collection of groups $\{G_i\}_I$ from [13] with G_i acting on Δ_i corresponds to the group of automorphisms of $\prod_{\mathcal{G}} \Delta_i$ generated via lifts of the action of the product of the groups on $\prod_I \Delta_i$ up to a kernel of the G_i actions. We use these results to show that if each Δ_i is finite, with $G_i = \mathrm{Aut}(\Delta_i)$ then the generalized graph product of the G_i is residually finite.

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CHAPTER 1

INTRODUCTION

Buildings are combinatorial objects first introduced by Tits to generalize the study of reflection groups. In this paper we will study groups acting on buildings arising via universal covers.

Reflection groups are a major part of the study of Lie groups and Lie algebras. Coxeter classified reflection groups of both spherical and Euclidean spaces in [5]. Tits later introduced *Coxeter groups* [22] as a purely algebraic way of looking at actions on symmetric spaces in an effort to unify the study of reflection groups of all symmetric spaces, that is, products of spherical, Euclidean and hyperbolic spaces. We let (W, S) be a Coxeter system where W is a Coxeter group and S its generating set.

Tits then introduced the combinatorial notion of buildings as an analogue of symmetric spaces. Though these objects were introduced as combinatorial objects one can construct geometric realizations of these buildings, traditionally as simplicial complexes. Let Δ be a building of type (W, S). Then we can construct a realization of Δ, denoted $\mathcal{U}(\Delta, K)$ where K is a *mirrored space*, by gluing together copies of K along the mirrors. The study of group actions on these realizations have helped shape finite group theory and have been important in Lie theory. The definitions of Coxeter groups and buildings along with motivating examples will make up Chapter 2.

In Chapter 3 we discuss the notion of CAT(0) cube complexes introduced by

Gromov in his paper [10]. Such complexes generalize trees and products of trees. Results from that paper lead to the study of groups acting on CAT(0) cube complexes. Many classes of groups act on CAT(0) cube complexes, including Coxeter groups which were 'cubulated' by Niblo and Reeves in [18] using a construction of Sageev's from [20]. We are interested in spaces with the CAT(0) property, because the standard realization of any building, that is $\mathcal{U}(\Delta, K)$ where K is the *Davis chamber* of (W, S), is CAT(0). This was proved by Davis in [7]. Our interest in CAT(0) cube complexes specifically arises from the fact that a certain class of buildings, appropriately named *right angled buildings*, can be realized as CAT(0) cube complexes. Right angled buildings are of particular importance in geometric group theory.

In Chapter 4 we describe a construction of buildings and a special case of the construction both due to Davis in [9]. In the first construction Davis gave a general construction of a building $\widetilde{\Delta}$ arising from the universal cover of a nonstandard realization of some building Δ. Davis applied this construction to a collection of buildings $\{\Delta_i\}_I$ indexed by the vertex set of some graph $\mathcal{G}$. The graph product of the Δ_i is denoted by $\prod_{\mathcal{G}} \Delta_i$ and arises from the universal cover of a nonstandard realization of the product $\prod_{I} \Delta_i$. We give explicit definitions of the constructions in Sections 4.1 and 4.3. Davis's paper generated examples of buildings with interesting properties highlighting the complexity of buildings and the potential irregularity that may arise.

In Section 4.4 we give a combinatorial construction of a right angled building $\widetilde{E}$ which is isomorphic to a graph product of buildings, $\widetilde{\Delta}$, as sets. Call the isomorphism ϕ. We introduce a notion of a *structure preserving automorphism* of $\widetilde{E}$ which is an automorphism of $\widetilde{E}$ that induces a building automorphism of $\widetilde{\Delta}$ under ϕ.

Theorem 5.3.3 is a construction of the full stabilizer of a chamber in the group of all structure preserving automorphisms of $\widetilde{E}$. This is a generalization of the well known construction of the stabilizer in $\mathrm{Aut}(T)$, where T is an infinite bi-regular tree, of an edge in T as an inverse limit of iterated wreath products. We show that the this group is isomorphic to the stabilizer of a chamber in a graph product of buildings.

We are often concerned with properties of the full automorphism group of a building and what that says about the building. We will show that if the Coxeter matrix for $(\widetilde{W}, S)$ differs from the Coxeter matrix for (W, S) in finitely many places then if $\mathrm{Aut}(\Delta)$ is chamber transitive and admits a pre-Moufang set of root groups of an apartment, then the same is true for $\mathrm{Aut}(\widetilde{\Delta})$. In this case $\mathrm{Aut}(\widetilde{\Delta})$ is strongly transitive and admits a (B, N)-pair.

Finally we discuss lattices in the group of structure preserving automorphisms of a right angled building. If we consider the right angled building $\widetilde{E}$ as the graph product of rank-1 buildings $\{E_i\}_I$ and we have a collection of groups $\{G_i\}_I$ with G_i acting on E_i then $\prod G_i$ acts on $\prod E_i$. The group of lifts of the action of $\prod G_i$ is denoted $\prod_{\mathcal{G}} G_i$. Our first result on lattices is Lemma 7.1.2 which states that $\prod_{\mathcal{G}} G_i$ is equal to the *generalized graph product*, introduced in [13], modulo $\prod_I K_i$ where K_i is the kernel of the action of G_i on E_i. We use this lemma to show Theorem 7.2.2, that $\prod_{\mathcal{G}} G_i$ is residually finite.

CHAPTER 2

COXETER GROUPS AND BUILDINGS

2.1 Coxeter Groups

Finite reflection groups were first studied in connection with Lie groups and Lie algebras. These finite reflection groups act isometrically on the sphere $\mathbb{S}^n$, and thus are called *spherical* groups. These actions were instrumental in the classification of Lie groups and Lie algebras. One can also consider discrete isometric reflections of Euclidean space. Such reflection groups are called *Euclidean.* When the orbit space of such an action is compact, then the group is called *cocompact.*

Coxeter classified both spherical reflection groups and cocompact Euclidean reflection groups in [5]. The only other irreducible symmetric space is hyperbolic space, so any reflection group splits into factors of spherical groups, Euclidean groups and hyperbolic groups.

In [22] Tits introduced the following abstract notion of a reflection group, which is where we will begin.

A *Coxeter matrix over a set* $S = \{s_i\}_{i \in I}$ is an $S \times S$ symmetric matrix $M = (m(s,t))$ with each diagonal entry equal to 1 and each off-diagonal entry equal to either an integer ≥ 2 or the symbol ∞. The matrix M determines a presentation of a group W as follows: the set of generators is S and the relations have the form $(st)^{m(s,t)}$ where (s,t) ranges over all pairs in $S \times S$ such that $m(s,t) \neq \infty$. The pair

(W, S) is a *Coxeter system* and W is a *Coxeter group*. The *rank* of a Coxeter system is the cardinality of the generating set, $|S|$. When the generating set is understood we will refer to this as the rank of W.

We say that a *word* for w in (W, S) is a finite sequence $(s_1, s_2, \ldots, s_n)$ with s_i in S such that $s_1 \cdot s_2 \cdots s_n = w$. A word is said to be *reduced* if it is of minimal length. We denote $\ell(w)$ to be the length of any reduced word for w. While there need not be a unique reduced word for w, the length of all reduced words for w is the same. This follows from *Tits' solution to the word problem* [23] which states that the only two moves necessary to reduce a word are

1. replacing a subsequence of length $2k$ of the form $(s, t, s, t, \ldots, s, t)$ with a subsequence of length $2(n - k)$ of the form $(t, s, t, s, \ldots, t, s)$ when $m(s, t) = n$ in the Coxeter matrix, and

2. removing any subsequence (s, s).

Note that the definition above is independent of geometry. Tits developed Coxeter groups in order to develop a unifying theory to study Lie groups and Lie algebras. Thus Coxeter groups were introduced as a purely group theoretical tool to work with. We next define a special class of Coxeter group that will appear throughout the rest of this thesis.

A *right angled Coxeter group* is a Coxeter group defined by a Coxeter matrix with entries either 1, 2 or the symbol ∞. That is, the only relations are the generators being involutions and that certain pairs of generators commute. We can construct such a Coxeter group from a graph in the following manner. Given a simplicial graph $\mathcal{G}$ with vertex set S we define the *right angled Coxeter group associated to* $\mathcal{G}$ as the Coxeter group generated by S with entries in the Coxeter matrix given by $m(s, s) = 1$ as required, $m(s, t) = 2$ if $\{s, t\}$ is an edge in $\mathcal{G}$ and $m(s, t) = \infty$ otherwise. Any

Coxeter group which arises in this manner is a right angled Coxeter group and any right angled Coxeter group has a graph it is associated to, namely the graph with edge set S and edges $\{s, t\}$ when $m(s, t) = 2$.

In keeping with the terminology of reflection groups, we say that W is *spherical* if it is finite. We denote the subgroup of W generated by $T \subset S$ as W_T. We say that T is *spherical* if W_T is spherical. Spherical subsets will become important later when we begin constructing appropriate geometries from these groups. We begin by first defining abstract buildings.

2.2 Buildings

Just as reflection groups came about as an aid to study Lie groups and Lie algebras, a corresponding geometric notion, called buildings, was developed to generalize incidence geometries connected to classical algebraic groups over finite fields. Buildings were introduced by Tits as entirely combinatorial objects, just as Coxeter groups were entirely algebraic, but the geometric realizations of these buildings, traditionally as simplicial complexes, have helped shape finite group theory and have been important in Lie theory.

Definition 2.2.1. Suppose (W, S) is a Coxeter system. A *building of type* (W, S) is a pair (Δ, δ) consisting of a nonempty set Δ, whose elements are called *chambers* and a function $\delta : \Delta \times \Delta \to W$ so that the following conditions hold for all chambers $C, D \in \Delta$.

(WD1). $\delta(C, D) = 1$ if and only if $C = D$.

(WD2). If $\delta(C, D) = w$ and $C' \in \Delta$ satisfies $\delta(C', C) = s \in S$, then $\delta(C', D) = sw$ or w. If in addition $\ell(sw) = \ell(w) + 1$, then $\delta(C', D) = sw$.

(WD3). If $\delta(C, D) = w$, then for any $s \in S$ there is a chamber $C' \in \Delta$ such that $\delta(C', C) = s$ and $\delta(C', D) = sw$.

This is the more classical definition of building. A treatment of equivalent definitions can be found in [1].

We say two chambers C and D are *s-adjacent* if $\delta(C, D) = s$. We write $C \sim_s D$ if C is *s-equivalent* to D, that is if C is s-adjacent to D or $C = D$. For any chamber C and subset $T \subset S$ the *T-residue containing C or $\mathcal{R}_T(C)$* is the set of all chambers in Δ that can be connected to C by adjacencies whose types lie only in T. The chambers in such a path connecting C to C' is called a *gallery of type T*. If the set is understood, then we write $\mathcal{R}(C)$. If we are choosing a residue without specifying a chamber it contains, we will write $\mathcal{R}_T$ and if $T = \{s\}$ for some s, then we can write $\mathcal{R}_s(C)$, instead of $\mathcal{R}_T(C)$. We will often refer to such residues as the *s-panel containing C*.

Example 2.2.2. Start with a Coxeter system (W, S) where S is just a singleton, $\{s\}$ and $W = \mathbb{Z}_2$. A building Δ of type (W, S) is just a set Δ with distance function $\delta(C, C') = s$ if $C \neq C'$ and $\delta(C, C) = 1$. We call such a building a *rank-1 building.*

In general the *rank of a building* Δ of type (W, S) is just the cardinality of S. The rank of a residue $\mathcal{R}_T$ is just the cardinality of T.

Example 2.2.3. Fix some Coxeter system (W, S). Then W is a building with W-distance $\delta(w, w') = w^{-1}w'$. Thus two chambers w and w' are s-adjacent if $ws = w'$. In such a building every panel has cardinality two. Any building in which every panel has cardinality exactly two can be identified with its Coxeter group in this way. Since every panel is of minimal size we call this building the *thin building of type* (W, S).

In most cases of interest to us at least some panels will have size larger than two. If every panel in a building has size at least three, we say the building is *thick.* We will often be concerned with thick buildings and this makes it important to consider panels and, more generally, residues. We begin with the following result.

Proposition 2.2.4. Any residue of type T is a building of type (W_T, T).

This follows from definitions. Further, for any $C \in \Delta$ and $T \subset S$, we have that C is contained in exactly one T-residue. If $\mathcal{R}$ is a T-residue, then $\mathcal{R}$ can also be viewed as a building of type W_T. In Example 2.2.3 the subset corresponding to the subgroup W_T is a T residue, and thus is a thin building of type (W_T, T). In this case the T residues correspond to the right cosets of the subgroup W_T in W.

Proposition 2.2.5. For any building Δ, any residue $\mathcal{R} \subset \Delta$ and any chamber $C_0 \in \Delta$ there exists a unique chamber $C \in \mathcal{R}$ such that $\ell(\delta(C, C_0))$ is minimal.

This proposition follows from Tits' solution to the word problem. The chamber C is called the *projection of* C_0 *onto* $\mathcal{R}$.

Definition 2.2.6. A *right angled building* is a building associated to some (W, S) where W is a right angled Coxeter group. We say a right angled building, is *regular* if for every s in S the cardinality of every s-panel is constant.

Example 2.2.7. An infinite tree $\mathcal{T}$ with no leaves can be viewed as a right angled building of type $\mathbb{Z}_2 \star \mathbb{Z}_2$ with chamber set the edge set of $\mathcal{T}$ and panels the set of edges in the star of a vertex, which we can identify with the vertices. The type of the panels alternates between vertices. If $\mathcal{T}$ is bi-regular then $\mathcal{T}$ is a regular right angled building.

Remark 2.2.8. It is known that a regular right angled building is completely determined by its type and the cardinality of each s-panel. This was written down by Haglund and Paulin in [12]. Thus in Example 2.2.7 combinatorially the trivalent tree is the unique building of type $\mathbb{Z}_2 \star \mathbb{Z}_2$ with all panels of size three.

The example of a tree gives a suggestion of the geometry which arises from this construction. We will now make rigorous this identification. Consider an arbitrary space X. A *mirror structure* over an arbitrary set S on X is a family of subspaces (X_s) indexed by S. The X_s are called *mirrors*. Given a mirror structure on X, a subspace $Y \subset X$ inherits a mirror structure by setting $Y_s := Y \cap X_s$. For each nonempty subset $T \subset S$, define subspaces X_T and X^T by

$$X_T := \bigcap_{s \in T} X_s \text{ and } X^T := \bigcup_{s \in T} X_s$$

Put $X_\emptyset := X$ and $X^\emptyset := \emptyset$. Given a subset c of X or a point $x \in X$ put

$$\begin{aligned} S(c) &= \{s \in S \mid c \subset X_s\} \\ S(x) &= \{s \in S \mid x \in X_s\}. \end{aligned}$$

A space with a mirror structure is a *mirror space*. Given a building Δ of type (W, S) and a mirrored space X over S define an equivalence relation $\sim$ on $\Delta \times X$ by $(C, x) \sim (D, y)$ if and only if $x = y$ and $\delta(C, D) \in S(x)$. We define the X-*realization* of Δ, denoted $\mathcal{U}(\Delta, X)$ as

$$\mathcal{U}(\Delta, X) := (\Delta \times X)/\sim.$$

In words, we identify s-mirrors of two chambers when those chambers are s-adjacent. We define the realization of a chamber $C \in \Delta$ inside a realization $\mathcal{U}(\Delta, X)$ to be the image of (C, X) and the realization of a residue as the union of the realizations of the chambers contained in that residue. Thus the realization of a T residue in $\mathcal{U}(\Delta, X)$ is the X realization of the residue as a building of type (W_T, T).

We say that two panels $\mathcal{R}_s$ and $\mathcal{R}_t$ in Δ are *adjacent* if they are contained in a spherical residue of type $\{s, t\}$ and the set of distances between chambers in $\mathcal{R}_s$ and $\mathcal{R}_t$ is a coset of $W_{\{s\}}$ in $W_{\{s,t\}}$. We extend these adjacencies to equivalence classes and define a *wall* in Δ as an equivalence class of panels. By the realization of a wall we mean the union of the mirrors corresponding to the panels in the wall. Let $\mathcal{D}$ be a convex subset of Δ. We define the *boundary of* $\mathcal{D}$ to be the set of all panels in Δ which have proper intersection with $\mathcal{D}$.

We now concern ourselves with what it means for a realization of a building to be 'nice.' There are several desirable properties we would like our realizations to have. Most notably, we would like the realization to be a simply connected simplicial complex. In order for the realization to be a simplicial complex we must begin with X being a simplicial complex, though in general X will not be a simplex. For the realization to be simply connected we must make sure all spherical residues are simply connected. To this end, denote the poset of spherical subsets of S partially ordered by inclusion by $\mathcal{S}(W, S)$, or $\mathcal{S}$ if the Coxeter system is understood. For any $T \subset S$ define $\mathcal{S}_{\geq T}$ to be the poset of spherical subsets of S which contain T.

Definition 2.2.9. Let $\mathcal{S}$ be as above and put $X = |\mathcal{S}|$, the geometric realization of the poset $\mathcal{S}$. (Recall that the geometric realization of a poset has simplices the chains in $\mathcal{S}$.) Define a mirror structure on X as follows: for each $s \in S$ put $X_s := |\mathcal{S}_{\geq\{s\}}|$ and for each $T \in \mathcal{S}$, let $X_T := |\mathcal{S}_{\geq T}|$. We say the complex X with this mirror structure is the *Davis chamber* of (W, S). In general we denote the Davis chamber K.

It follows from the Davis chamber being a realization of a poset that K is a flag complex. Often we determine the 1-skeleton and define K to be the flag complex with that 1-skeleton.

The *nerve* of (W, S), written $L(W, S)$, is the poset of nonempty elements in $\mathcal{S}$. It is an abstract simplicial complex. Thus for a Coxeter system (W, S) we have a simplicial complex $L(W, S)$ with vertex set S and whose simplicies are spherical subsets of S. The Davis chamber can also be defined as the cone on the barycentric subdivision of $L(W, S)$. These two definitions of the Davis chamber are equivalent. The empty set in Definition 2.2.9 corresponds to the cone point above.

It is proved in [7] that for any building Δ of type (W, S) with K its Davis chamber $\mathcal{U}(\Delta, K)$ is contractible. We say this is the *standard realization* of Δ.

Example 2.2.10. Let W be the dihedral group of order six, with generating set $S = \{s, t\}$. Let Δ be the thin building of type (W, S) as in Example 2.2.3. Then the left hexagon in Figure 2.1 is the standard realization of Δ while the right hexagon is the realization with just a cone on the generators. The mirrors in a single chamber in each are emphasized.

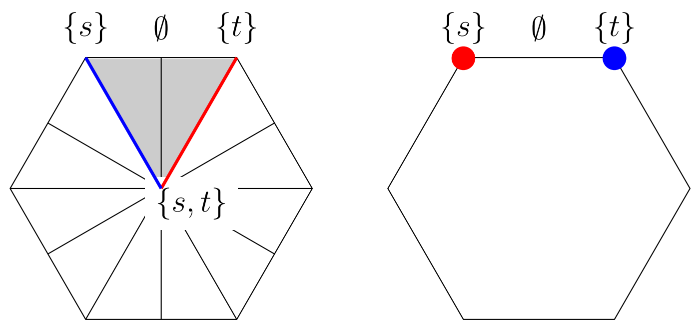

Figure 2.1: Two realizations of the thin building for the dihedral group of order six; the left with the Davis chamber, and the right with an edge.

We will take a side step to discuss how this geometry sheds light on the Coxeter group.

CHAPTER 3

CAT(0) CUBE COMPLEXES

In his paper [10] Gromov introduced CAT(0) cube complexes as a class of examples of complexes which generalized trees and products of trees. Theorems from that paper lead to the study of groups acting on CAT(0) cube complexes. Many classes of groups act on CAT(0) cube complexes, including Coxeter groups which were 'cubulated' using a construction of Sageev's in [20]. Our interest in CAT(0) cube complexes arises from the fact that right angled buildings can be realized as CAT(0) cube complexes. we will detail this realization in Section 3.3.

3.1 CAT(0)

Definition 3.1.1. Let M_ϵ be a simply connected 2-dimensional manifold of constant curvature ϵ. A geodesic metric space X is *CAT(ϵ)* if the following hold.

1. There exists a geodesic segment (of length less than $\dfrac{\pi}{\sqrt{\epsilon}}$ if $\epsilon > 0$) between any two points x and y.

2. Fix any geodesic triangle T_X in X with vertices x_1, x_2 and x_3, edges geodesic segments (with the sum of the lengths of the segments less than $\dfrac{2\pi}{\sqrt{\epsilon}}$ when $\epsilon > 0$) and a triangle T in M_ϵ with corresponding vertices x_1^*, x_2^* and x_3^* with distances between corresponding pairs of vertices the same. For any pair x, y in T_X we have that $d_X(x, y) \leq d(x*, y*)$.

In fact we need only require that the above inequality hold for the midpoints of the geodesic segments which make up the edges.

The following was known to Moussong, though not included in [17]. It was eventually written down by Davis in [7]

Theorem 3.1.2. The standard realization of any building is a complete CAT(0) space.

Hence it is natural for us to consider CAT(0) spaces and properties of such spaces. CAT(ϵ) spaces admit the following desirable properties which help motivate their study [2].

Theorem 3.1.3. If X is a geodesic space, we have the following:

1. If X is CAT(0) then X is contractible.

2. If X is CAT(1) any open ball of radius π contained in X is contractible.

Combining theorems 3.1.2 and 3.1.3 gives us that the standard realization of any building is contractible, which we stated in Section 2.2 and was first stated by Davis in [7]. The following result is due to Gromov [10]:

Theorem 3.1.4. If X is a compact geodesic space with curvature less than ϵ for some positive ϵ, then X is CAT(0) if and only if every closed geodesic has length less than $\dfrac{2\pi}{\sqrt{\epsilon}}$.

3.2 Cube Complexes

We now describe the construction of cube complexes. First let $Q = [0,1]^n$ be a cube in $\mathbb{R}^n$. We define the $n-1$ dimensional faces of Q to be the product of intervals except with 1 or 0 in the ith position for exactly one i. A face of Q is a non-empty intersection of $n-1$ dimensional faces.

We construct a cube complex by starting with a collection of cubes $\mathcal{C}$ and identifying faces of cubes of the same dimension via an isometry. We require that

1. distinct faces of Q are not identfied and

2. for any two cubes Q and Q' if two distinct pairs of i dimensional faces are identified, then they are contained in higher dimensional faces which are identified.

Next we define a metric on a connected cube complex X. For any two points x and y in X we construct a path $\{x_0, x_1, ..., x_k\}$ with $x_0 = x$ and $x_k = y$ and and x_i, x_{i+1} both in some cube Q_i for $0 \leq i < k$. Define $d(x, y) = \inf \sum d_{Q_i}(x_i, x_{i+1})$ where we take the infimum over all such paths. By *CAT(0) cube complex* we mean a geodesic cube complex with the above metric that is CAT(0).

In [10] Gromov gave the following combinatorial characterization of when cube complexes are CAT(0).

Theorem 3.2.1. A cube complex is CAT(0) if and only if it is simply connected and the links of 0-cells are flag complexes.

CAT(0) spaces differ from negatively curved spaces in that they can contain Euclidean codimension one spaces. Thus isometries which act on CAT(0) spaces need not be hyperbolic, specifically some isometries may be reflections. In [20] Sageev showed the following:

Theorem 3.2.2. Any hyperplane in a CAT(0) cube complex is itself a CAT(0) cube complex and separates the space into exactly two half spaces.

For a full treatment of hyperplanes in CAT(0) cube complexes see [19]. From these two theorems we start to appreciate how natural a generalization of trees CAT(0) cube complexes really are.

3.3 Right Angled Buildings Are CAT(0) Cube Complexes

Let Δ be a thin right angled building of type (W, S). Begin with the Cayley graph for W with generating set S and include an n-cube whenever its 1-skeleton appears in the graph. The standard realization of Δ is the barycentric subdivision of this cube complex. The simplices added in the subdivision comprise the mirrors of the standard realization. It is immediate from this construction that the realization satisfies Gromov's link condition, so that it is a CAT(0) cube complex.

For a general building Δ , construct the 1-skeleton of the complex by including a 0 cell for every chamber and connecting two 0-cells by a 1-cell if the corresponding chambers are adjacent in Δ. Then include cubes whenever the 1-skeleton of the cube exists. The standard realization of Δ is a subdivision of this cube complex. It is immediate that this cube complex satisfies Gromov's link condition so that it is also CAT(0).

Next, we construct an equivalence relation between i-dimensional faces in an arbitrary cube complex X. We start by making two i dimensional cubes $Q \sim Q'$ if they lie in an $i+1$ cube and do not intersect, and then extending. This equivalence relation has an analogue in right angled buildings in terms of the notion of parallel residues:

We will need some building notation.

$$\mathrm{Proj}_{\mathcal{R}}(\mathcal{R}') = \{C \in \mathcal{R} | C \text{ is the projection of some } C' \in \mathcal{R}' \text{ onto } \mathcal{R}\}.$$

Definition 3.3.1. We say that two residues $\mathcal{R}$ and $\mathcal{R}'$ are *parallel* if $\mathrm{Proj}_{\mathcal{R}}(\mathcal{R}') = \mathcal{R}$ and $\mathrm{Proj}_{\mathcal{R}'}(\mathcal{R}) = \mathcal{R}'$.

In [3] Caprace showed the following:

Lemma 3.3.2. Parallelism of residues in a building is an equivalence relation if and only if the building is right angled.

Specifically, the notion of parallel residues will not be transitive in general. His proof does not involve the geometry of right angled buildings as CAT(0) cube complexes, but with the visualization of parallelism in cube complexes one direction, at least, seems intuitive. This fact led to the following theorem:

Theorem 3.3.3. If Δ is a thick, regular right angled building of type (W, S), irreducible and nonspherical, then $\mathrm{Aut}(\Delta)$ is a simple group which acts strongly transitively on X.

These results of right angled buildings and the broader results of CAT(0) cube complexes motivate our study of graph products in Chapters 4 and 7.

CHAPTER 4

GRAPH PRODUCTS

In [6] Davis introduced a construction that involved removing relations in the presentation of a Coxeter group. Later he created an analogue construction for buildings in [9]. This construction offered concrete examples of buildings with complicated behavior. He constructed buildings whose maximal spherical residues contain an arbitrary set of spherical buildings. He also constructed examples of buildings such that if the building's spherical residues are associated to algebraic groups over finite fields the fields need not be the same.

We will give explicit descriptions of the Coxeter group and building constructions in Section 4.1. Next we will discus graph products of Coxeter groups and buildings as an application of that construction (also from [9]) in Sections 4.2 and 4.3. In Section 4.4 we will show that any graph product of buildings can be endowed with a new distance function making it a right angled building. This new building metric will yield a realization of graph products of buildings as CAT(0) cube complexes. We will then offer an immediate application of this fact by classifying when a graph product of buildings is Gromov hyperbolic as a generalization of Mousong's result in [17].

4.1 Buildings Via Covering Spaces

We start with a Coxeter system (W, S) with Coxeter matrix $(m(s,t))$ and a building Δ of type (W, S). Let $(m'(s,t))$ be a new $S \times S$ Coxeter matrix where $m'(s,t) = m(s,t)$ if $m'(s,t) \neq \infty$. Denote the Coxeter system with the new Coxeter matrix as (W', S).

Let K' be the Davis chamber for the Coxeter system (W', S). The non-standard realization of Δ with this Davis chamber, $\mathcal{U}(\Delta, K')$, is not simply connected unless $(m(s,t)) = (m'(s,t))$. In [9] Davis proved the following:

Theorem 4.1.1. The universal cover, $\widetilde{\mathcal{U}(\Delta, K')}$, of $\mathcal{U}(\Delta, K')$ is the standard realization of a building of type (W', S).

In light of this theorem we will usually denote W' as $\widetilde{W}$ and K' as $\widetilde{K}$. We let $\widetilde{\Delta}$ be the building with this realization. Explicitly we define $\widetilde{\Delta}$ to be the set of copies of $\widetilde{K}$ in the realization $\widetilde{\mathcal{U}(\Delta, \widetilde{K})}$. We define the s-adjacencies on $\widetilde{\Delta}$ to be those copies of $\widetilde{K}$ which share an s-mirror. We can extend this to a $\widetilde{W}$-metric. The result is a building of type $(\widetilde{W}, S)$.

Example 4.1.2. Let W be the dihedral group of order six, then the Coxeter matrix for W is:

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

and let Δ be a building of type (W, S) with each panel of size three. Let $\widetilde{W}$ be the Coxeter group with generating set S and Coxeter matrix:

$$\begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix}$$

In Figure 4.1 the realization on the left is $\mathcal{U}(\Delta, \widetilde{K})$ where $\widetilde{K}$ is the Davis chamber for the Coxeter system $(\widetilde{W}, S)$. The resulting graph is the incidence graph of the

Fano plane [8]. The tree on the right is the universal cover of the realization on the left which from Example 2.2.7 we know is the realization of a building of type the infinite dihedral group.

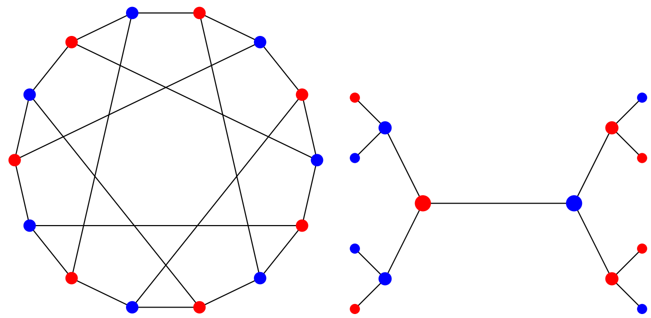

Figure 4.1: The incidence graph of the Fano plane and its universal cover, the trivalent tree.

4.2 Graph Products of Coxeter Groups

The classical ways to combine groups are with direct products and free products. Graph products are an intermediate where we allow some factors to commute, but not others. The first examples of graph products are right angled Coxeter groups, and right angled Artin groups. The automorphism groups of each of these have been well studied. Laurence, in his thesis [14], considers the case when the graph product is taken over groups which are all infinite cyclic or all finite with order p, some fixed prime. Laurence offers a finite generating set for the automorphism group of such graph products, and this work was later extended in [4] to any graph product of abelian groups.

We we will procede by giving a formal definition of graph products of Coxeter groups. This definition is equivalent to the quotient of the free product by relations where some sets of generators commute as in the more general case, which we will define in Chapter 7. However, we find it useful to define graph products in terms of a Coxeter matrix to stay in the context of the construction given in Section 4.1.

For the rest of this paper let $\mathcal{G}$ be a graph with vertex set $V(\mathcal{G}) = I$. Let $\{(W_i, S_i)\}_{i\in I}$ be a collection of Coxeter systems.

Let W be the product of the W_i, and S the disjoint union of the S_i so that (W, S) is a Coxeter system with Coxeter matrix:

$$m(s,t) = \begin{cases} m_i(s,t) & \text{if } i = j, \\ 2 & \text{otherwise.} \end{cases}$$

Then we apply the construction from Section 4.1 to this Coxeter system in the following way.

Definition 4.2.1. The *graph product of the W_i over $\mathcal{G}$* is the Coxeter group with generating set $S = \bigsqcup_{i\in I} S_i$ and Coxeter matrix $\widetilde{m}(s,t)$ with $s \in S_i$ and $t \in S_j$ where

$$\widetilde{m}(s,t) = \begin{cases} m_i(s,t) & \text{if } i = j, \\ 2 & \text{if } i \neq j \text{ and } \{i,j\} \in E(\mathcal{G}) \\ \infty & \text{otherwise.} \end{cases}$$

We denote this Coxeter group as $\prod_{\mathcal{G}} W_i$, or $\widetilde{W}$ when all data is understood.

In words, we include all of the relations from each of the W_i and then include the additional relations that the elements of S_i commute with those in S_j exactly when $\{i, j\}$ is an edge in the graph. Our new Coxeter matrix has the matricies m_i along the diagonal and off the diagonal there are blocks of either all 2 or all the symbol ∞.

Example 4.2.2. The graph product of copies of $\mathbb{Z}_2$, that is the graph product of Coxeter groups with just one generator, is a right angled Coxeter group. Further, any right angled Coxeter group can be written as such a graph product. Then the group $\prod_{\mathcal{G}} \mathbb{Z}_2$ is the right angled Coxeter group associated to $\mathcal{G}$ as constructed at the end of Section 2.1. In this thesis we will use nonstandard notation for the sake of simplicity and take the generating set to be exactly the vertex set of $\mathcal{G}$ rather than a set indexed by the vertices.

4.3 Graph Products of Buildings

We now apply the construction for buildings from Section 4.1 to define the graph product of buildings. This was also due to Davis in [9]. We first give an explicit construction the Davis chamber for the graph product of Coxeter groups.

Let K_i denote the Davis chamber for each (W_i, S_i) and K denote the Davis chamber for $W = \prod_I W_i$. Let the nerve $L(W_i, S_i)$ be denoted L_i so that K_i is the cone on the barycentric subdivision of L_i. We define $\widetilde{L}$ to be the nerve $L(\prod_{\mathcal{G}} W_i, \bigsqcup S)$. Then the 1-skeleton of $\widetilde{L}$ is formed from the disjoint union of the L_i by attaching 1-cells connecting vertices in L_i with vertices in L_j when $\{i, j\}$ is an edge in $\mathcal{G}$. Thus for every edge $\{i, j\}$ in $\mathcal{G}$ we have that $\widetilde{L}$ contains the complete bipartite graph between L_i and L_j. Because $\widetilde{L}$ is a flag complex it is completely determined by its one skeleton. Recall that the Davis chamber is the cone on the barycentric subdivision of $\widetilde{L}$, the nerve of the Coxeter system. The empty set in each K_i is the cone point of that K_i, and in $\widetilde{K}$, the Davis chamber for $(\widetilde{W}, S)$, each of these cone points is identified.

Note that $\widetilde{L}$ is a subcomplex of $\prod_I L_i$, which means that $\widetilde{K}$ is contained in the Davis chamber of (W, S). Thus we can also think of the construction as removing faces of K when edges in $\mathcal{G}$ are missing.

We now introduce new data: for each $i \in I$ let Δ_i be a building of type (W_i, S_i). First, $\Delta = \prod_I \Delta_i$ is a building of type $\prod_I W_i$. Since every S_i commutes with every S_j when $i \neq j$ the $\prod_I W_i$ distance between two chambers is just the product of the distances between corresponding components. Call this product of buildings Δ. We now apply the realization of Section 4.1 to Δ. Let $\widetilde{K}$ be the Davis chamber for $\prod_{\mathcal{G}} W_i$ constructed above. The realization $\mathcal{U}(\Delta, \widetilde{K})$ will not be simply connected (unless $\mathcal{G}$ is the complete graph on I), since $\widetilde{K}$ is not the Davis chamber for $\prod_I W_i$.

Thus we can apply Theorem 4.3.1 to graph products and gain the following:

Proposition 4.3.1. The universal cover of $\mathcal{U}(\Delta, \widetilde{K})$ is the standard realization of a building of type $\prod_{\mathcal{G}} W_i$.

Thus we define the graph product of buildings in the same way as in Section 4.1.

Definition 4.3.2. The *graph product of the* Δ_i *over* $\mathcal{G}$, denoted $\prod_{\mathcal{G}} \Delta_i$, is the set of copies of $\widetilde{K}$ in $\widetilde{\mathcal{U}(\Delta, \widetilde{K})}$, the universal cover of the $\widetilde{K}$ realization of Δ, with two chambers C and C' being s-adjacent when their intersection is a copy of $\widetilde{K}_s$. The $\widetilde{W}$-distance is defined by the s-mirrors that a path between two centers intersects.

When discussing graph products and all data is understood we will denote $\prod_I W_i$ as W and $\prod_{\mathcal{G}} W_i$ as $\widetilde{W}$. Also, Δ will denote $\prod_I \Delta_i$ and $\widetilde{\Delta}$ will denote $\prod_{\mathcal{G}} \Delta_i$.

Example 4.3.3. If each of the W_i are $\mathbb{Z}_2$ then $\widetilde{W}$ is a right angled Coxeter group and $\widetilde{\Delta}$ is a right angled building as in Definition 2.2.6. Further each i-panel has cardinality $|\Delta_i|$ making $\widetilde{\Delta}$ a regular right angled building. By Remark 2.2.8 any regular right angled building can be written as such a graph product. In this case each of the Δ_i is a rank one building, which we can think of as just a set as in Example 2.2.2.

Example 4.3.4. If $\mathcal{G}$ is the complete graph on I then $\widetilde{\Delta}$ equals Δ, the product of the Δ_i.

4.4 Graph Products of Buildings as Right Angled Buildings

In general, any building can be viewed as a building of type $\mathbb{Z}_2$ as in Example 2.2.2. Fix an arbitrary building Δ and let E be the set of chambers in Δ. Define the $\mathbb{Z}_2$-metric on E to be

$$\delta_E(C, C') = \begin{cases} s & \text{if } C \neq C' \\ 1 & \text{if } C = C'. \end{cases}$$

Where s is the non-identity element in $\mathbb{Z}_2$. Of course Δ and E are the same set with a different word metric, but we denote the buildings differently to avoid confusion.

Now, consider each Δ_i as a building of type $\mathbb{Z}_2$. We let E_i denote the set Δ_i with the $\mathbb{Z}_2$ metric. Because these are *the same sets*, just with a different metric, there is a "natural" identification between them. This identification induces a map between $\prod_I E_i$ and Δ. Let E denote $\prod_I E_i$ and $\widetilde{K}_E$ denote the Davis chamber for $\prod_{\mathcal{G}} \mathbb{Z}_2$. We know that $\widetilde{E} = \prod_{\mathcal{G}} E_i$ is a regular right angled building of type $\prod_{\mathcal{G}} \mathbb{Z}_2$.

Theorem 4.4.1. There exists a unique isomorphism between $\widetilde{E}$ and $\widetilde{\Delta}$ (up to a choice of basepoint) so that the diagram commutes:

$$\begin{array}{ccc} \widetilde{E} & \overset{\phi}{\longleftrightarrow} & \widetilde{\Delta} \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle p} \\ E & \underset{=}{\longleftrightarrow} & \Delta \end{array}$$

Where ϕ is the map induced from the isomorphism, and p_E is the projection map from $\widetilde{E}$ to E.

Proof. Let $\mathcal{S}_E = \mathcal{S}(\prod_{\mathcal{G}} \mathbb{Z}_2, I)$ and $\mathcal{S} = \mathcal{S}(W, S)$. We construct a map $f : \mathcal{S} \to \mathcal{S}_E$ in the following way. For any $T \subset S_i$ we let $f(T) = \{i\}$. If T is a spherical subset which intersects more than one S_i define $J = \{j | T \cap S_j \neq \emptyset\}$. Then the elements of J span a complete subgraph in $\mathcal{G}$. Thus $J \in \mathcal{S}_E$ and we let $f(T) = J$.

This induces a homotopy $f : \widetilde{K} \to \widetilde{K}_E$ by identifying the cone points and extending. Further, this gives a homotopy equivalence of $\mathcal{U}(\widetilde{K}, \Delta)$ with $\mathcal{U}(\widetilde{K}_E, E)$ by applying this map to every chamber. The definition of the identifications guarantees that this map is well defined. Thus $\mathcal{U}(\widetilde{K}, \Delta)$ and $\mathcal{U}(\widetilde{K}_E, E)$ have the same fundamental group, and combining this with $\Delta = E$ as sets we can apply the universal property of universal covers to see that there is a unique isomorphism between $\widetilde{\Delta}$ and $\widetilde{E}$ as sets (up to a change in basepoint,) as desired. □

Example 4.4.2. In Figure 4.2 we see the homotopy between the hexagon and the star on six vertices. All of the mirrors on the left will map to the central vertex on the right. This central vertex corresponds to the i-mirror in the standard realization of E_i.

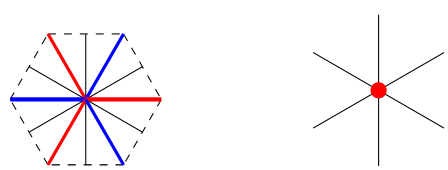

Figure 4.2: The standard realization of the Coxeter complex of the dihedral group of order six (left) and the standard realization of that set as a building of type $\mathbb{Z}_2$ (right.)

Remark 4.4.3. Note that two chambers in $\widetilde{E}$ are i-adjacent exactly when they lie in the same S_i residue in $\widetilde{\Delta}$. If we let $p : \widetilde{\Delta} \to \Delta$ be the map induced by the covering map from $\widetilde{\mathcal{U}(\Delta, \widetilde{K})} \to \mathcal{U}(\Delta, \widetilde{K})$ we see that S_i residues in $\widetilde{\Delta}$ are isomorphic to S_i

residues in Δ under p. This map extends to an isomorphism of $S_i \cup S_j$ residues exactly when $\{i, j\}$ is an edge in $\mathcal{G}$ and so on.

We will denote S_i residues in $\widetilde{\Delta}$ by $\mathcal{R}_i$. This is a slight abuse of notation since $\mathcal{R}_i$ also denotes an i-panel in $\widetilde{E}$, however such residues are identified in the map from Theorem 4.4.1, so that it will not cause ambiguity in practice. Further, for any $J \subset I$ we will say that any residue of type $T_J = \bigcup_{i \in J} S_i$ is a J-residue.

In Section 3.3 we stated the following lemma of Caprace:

Lemma. 3.3.2 Parallelism of residues in a building is an equivalence relation if and only if the building is right angled.

Applying this lemma to $\widetilde{E}$ we have the following result:

Lemma 4.4.4. Paralellism of J-residues in $\widetilde{\Delta}$ is an equivalence relation.

4.5 Hyperbolic Buildings

Gromov had a notion of hyperbolicity which loosely meant that any triangle in the space X is thin. Such a space is said to be *Gromov hyperbolic* [2].

Let (W, S) be a Coxeter system. Let Δ be the thin building of type (W, S) as in Example 2.2.3, and K be the cone on S with mirrors $K_s = s$ for each s in S. The *Cayley graph* of a Coxeter group W with generating set S is $\mathcal{U}(\Delta, K)$. One can check easily that this is a rephrasing of the standard definition of the Cayley graph in building terminology. We say that a group G is *hyperbolic* or *word hyperbolic* if its Cayley graph is Gromov hyperbolic.

In [17] Moussong proved the following result classifying word hyperbolic Coxeter groups:

Theorem 4.5.1. Let (W, S) be a Coxeter system. The following are equivalent:

1. W is word hyperbolic.

2. W has no subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

3. W does not contain a Euclidean sub-Coxeter system of rank greater than two and does not contain a pair of disjoint commuting Coxeter systems whose groups are both infinite.

Later Moussong observed that a building has negative curvature exactly when its Coxeter group is word hyperbolic and this was written down by Davis in [7]. The notion of hyperbolicity has offered a rich class of examples in geometric group theory.

Using these results and the isomorphism in Theorem 4.4.1 we can completely categorize the hyperbolic buildings arising from graph products as a generalization of Theorem 4.5.1. For this we first need the graph $\mathcal{G}$ to satisfy the *no squares condition*, that is that every square in $\mathcal{G}$ must have a diagonal inside.

Theorem 4.5.2. The standard realization of a building $\widetilde{\Delta}$ is hyperbolic if and only if the following conditions hold:

1. each Δ_i is spherical or infinite hyperbolic,

2. $\mathcal{G}$ satisfies the no squares condition

3. for all pairs of infinite hyperbolic buildings Δ_i and Δ_j we have that $\{i, j\}$ is not an edge in $\mathcal{G}$ and

4. if Δ_i is infinite hyperbolic then the star of i spans a complete subgraph in $\mathcal{G}$.

Proof. We need only show that if each of these is satisfied then $\widetilde{W}$ is hyperbolic. Seeing that each of these is necessary is immediate. By Moussong's Theorem we see that we need only guarantee that $\widetilde{W}$ does not contain a Euclidean sub-Coxeter system of rank greater than two, guaranteed by conditions 1 and 3 and that W

does not contain a pair of disjoint commuting Coxeter systems whose groups are both ininite. In order for a sub-Coxeter system to be infinite in $\widetilde{W}$ it could contain an infinite hyperbolic W_i or be the free product of two finite groups W_i and W_j. Conditions 2, 3, and 4 verify that no such direct product of two of these exist. $\square$

CHAPTER 5

AUTOMORPHISMS OF BUILDINGS

The study of automorphisms of spaces is at the heart of geometric group theory. We will give a brief description of the topology on our automorphism groups in Section 5.1, then define a subgroup of the automorphism group of a right angled building based on the construction from Section 4.4. Chapter 6 will be concerned with when buildings admit automorphism groups with special properties. Chapter 7 will give an isomorphism between the standard lattice in the automorphism group of a graph product of buildings and the generalized graph product of buildings.

5.1 The Compact-Open Topology

We will discuss the topology of the automorphism group of the building in order to motivate the study of the full automorphism group in Section 5.2 of transitive properties of this group in Chapter 6 and of lattices in Chapter 7.

First note that the standard realization of a building Δ is locally finite if and only if each panel in Δ is finite. For this section we will assume that each panel in Δ is finite. An automorphism g of Δ is said to be *type preserving* if for every s in S we have that g sends s-panels to other s-panels. If g is an automorphism of a realization of a building then g need not be type preserving, however we will only be considering actions on realizations of buildings that are type preserving, so in this paper an *automorphism of a building* Δ will always be a type preserving automorphism. The group of all automorphisms of Δ is denoted $\mathrm{Aut}(\Delta)$. We will refer to an automorphism of a building and its induced map on the realization of the building interchangeably.

We construct a topology on $\mathrm{Aut}(\Delta)$ as follows: Given any pair consisting of a finite set $F \subset \Delta$ and an automorphism $g \in \mathrm{Aut}(\Delta)$ we let $\mathcal{O}(g, F) = \{f \in \mathrm{Aut}(\Delta) : f|_F = g|F\}$. Then we take our topology to be the one for which $\{\mathcal{O}(F, g)\}$ is a subbasis. This coincides with the *compact-open topology* of $\mathrm{Aut}(\Delta)$ where we consider the induced action of $\mathrm{Aut}(\Delta)$ on the standard realization of Δ.

5.2 Structure Preserving Automorphisms

Given the natural identification between graph products of buildings and right angled buildings, it makes sense to consider when an action on $\widetilde{E}$ extends to an action on $\widetilde{\Delta}$ and vice versa. Certainly the full automorphism group of $\widetilde{E}$ will not act (as building automorphisms) on $\widetilde{\Delta}$ unless $\widetilde{\Delta}$ happens to be a graph product of rank-1 buildings. We will show, however, that any building automorphism of $\widetilde{\Delta}$ is a building automorphism of $\widetilde{E}$. We will give a complete characterization of $\mathrm{Aut}(\widetilde{\Delta})$ as a subgroup of $\mathrm{Aut}(\widetilde{E})$ and we will give a construction for the stabilizer of a chamber in each of these groups.

First, let $p : \widetilde{E} \to E$ be map induced by the covering map from $\widetilde{\mathcal{U}(E, K)} \to \mathcal{U}(E, K)$. Further, let p_i be p composed with the projection onto E_i in the product. Then p_i is a building isomorphism between any S_i residue in $\widetilde{E}$ and E_i.

Suppose that we are given as data a collection of groups $\{G_i\}_I$ with G_i acting on E_i for each $i \in I$. We let $\prod\limits_I G_i$ act on E component wise. Then we define our desired automorphisms of $\widetilde{E}$ as follows.

Definition 5.2.1. We say that an automorphism g of $\widetilde{E}$ is *structure preserving* if for each i-panel $\mathcal{R}_i$ there exists a $g_i \in G_i$ such that $p_i \circ g(C) = g_i \circ p_i(C)$ for all C in $\mathcal{R}_i$. We will denote the group of all structure preserving automorphisms of $\widetilde{E}$ by $\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})$.

We do not need to require the E_i to be rank 1 buildings, or $\widetilde{E}$ to be right angled, however in practice we will only use this definition in the right angled case.

5.3 Stabilizer of a Chamber

We next estabilish language to describe the full stabilizer of a chamber C_0 in $\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})$. This construction is a generalization of the construction of the stabilizer of an edge in a tree as an inverse limit of iterated wreath products as in [21]. For each $w \in \prod_{\mathcal{G}} \mathbb{Z}_2$ define $\mathrm{End}(w)$ to be the set of all $i \in I$ such that $\ell(wi) < \ell(w)$. Define D_n to be the set of all chambers C such that $\ell(\delta(C, C_0)) < n$.

If g is an automorphism of D_n such that for any S_i residue that intersects D_n nontrivially there exists $g_i \in G_i$ such that for each $C \in \Delta_i$ we have $p_i \circ g(C) = g_i \circ p_i(C)$ then g is a *structure preserving automorphism of* D_n. The group of all structure preserving automorphisms of D_n is denoted $\mathrm{Aut}_{\mathrm{SP}}(D_n)$. We will determine $\mathrm{Aut}_{\mathrm{SP}}(D_n)$ recursively, first noting that $\mathrm{Aut}_{\mathrm{SP}}(D_0) = id$. The restriction of $\mathrm{Aut}_{\mathrm{SP}}(D_{n+1})$ to D_n maps $\mathrm{Aut}_{\mathrm{SP}}(D_{n+1})$ to $\mathrm{Aut}_{\mathrm{SP}}(D_n)$. Let K_{n+1} to be the kernel of the restriction. We will show that the restriction is onto and describe the kernel.

We partition the set of chambers in $D = D_{n+1} \setminus D_n$ into two parts:

$$\begin{aligned} \mathcal{P}_1 &= \{C \in D |\ |\mathrm{End}(\delta(C, C_0)| = 1\} \\ \mathcal{P}_2 &= \{C \in D |\ |\mathrm{End}(\delta(C, C_0)| \geq 2\}. \end{aligned}$$

The sets $\mathcal{P}_1$ and $\mathcal{P}_2$ are disconnected. If any two chambers C_1 and C_2 in D are i-adjacent, then there is some unique C in the i-panel that is in D_n by Proposition 2.2.5.

Thus $\delta(C_1, C_0) = \delta(C, C_0) \cdot i = \delta(C_2, C_0)$. Hence $\mathrm{End}(\delta(C_1, C_0)) = \mathrm{End}(\delta(C_2, C_0))$ and C_1 and C_2 are in the same partition. It follows that there are no relations between chambers in $\mathcal{P}_1$ and $\mathcal{P}_2$.

Lemma 5.3.1. Let Φ denote the set $\{C \in D_n | C \in \mathcal{R}_i(C') \text{ for some } C' \in \mathcal{P}_1\}$. The group $\mathrm{Aut}_{\mathrm{SP}}(D_n \cup \mathcal{P}_1) = \mathrm{Aut}_{\mathrm{SP}}(D_n) \rtimes \prod_{\Phi} \mathrm{Stab}_{G_i}(p_i(C))$ up to isomorphism.

Proof. Fix any i panel $\mathcal{R}$ intersecting $\mathcal{P}_1$. Let $C \in \mathcal{R}$ be the projection of C_0 onto $\mathcal{R}$ as in Proposition 2.2.5. Then C is the only chamber in $\mathcal{R}$ contained in D_n. By the definition of $\mathrm{Aut}_{\mathrm{SP}}(D_n)$ for any element $g \in \mathrm{Aut}_{\mathrm{SP}}(D_n)$ we know there is at least one element $g_i \in G_i$ such that $p_i \circ g(C) = g_i \circ p_i(C)$ so that $g_i^{-1}\mathrm{Stab}_{G_i}(p(C))g_i$ is an extension of $\mathrm{Aut}_{\mathrm{SP}}(D_n)$ to that i panel. This g_i is not necessarily unique but a different choice induces an inner isomorphism. All such panels are disjoint so the actions on panels will commute, thus completing the claim. $\square$

Lemma 5.3.2. There is a unique extension of $\mathrm{Aut}_{\mathrm{SP}}(D_n)$ to $\mathrm{Aut}_{\mathrm{SP}}(D_n \cup \mathcal{P}_2)$.

Proof. Fix some $C_2 \in \mathcal{P}_2$ and let $\mathcal{R}$ denote the $\mathrm{End}(\delta(C_2, C_0))$ residue containing C_2. Let C be the projection of $\mathcal{R}$ onto C_0. Now, $\mathcal{R}$ is a right angled building of type $\prod_{\mathrm{End}(\delta(C_2,C_0))} \mathbb{Z}_2$. Thus it is isomorphic to its image under p in E. The action on $\mathcal{R} \cap D_n$ then corresponds to an element in $\prod G_i$ acting on $\prod E_i$, hence it can extend. Uniqueness follows in the same way since for any $g \in \mathrm{Aut}_{\mathrm{SP}}(D_n)$ we have that $g(C_2)$ must be i-adjacent to the projection of the i-panel onto C_0. Any two such adjacencies completely determine the chamber in the product. $\square$

Combining these we get the following result.

Theorem 5.3.3. $\mathrm{Stab}_{\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})}(C_0) = \varprojlim \mathrm{Aut}_{\mathrm{SP}}(D_n)$

Thus we have that any structure preserving automorphism of a finite ball contained in $\widetilde{E}$ can be extended to a structure preserving automorphism of the whole space. In Section 6.1 we will characterize when this stabilizer is part of a BN-pair which generates the whole $\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})$.

The previous theorem gives rise to the following results.

Corollary 5.3.4. If G_i acts freely on E_i for all $i \in I$ then $\mathrm{Stab}_{\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})}(C_0)$ is trivial.

Corollary 5.3.5. If $i \in I$ is such that $< s_i >$ is not contained in a finite factor of W and G_i has non-trivial stabilizer at some element of E_i, then $\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})$ is uncountable.

Corollary 5.3.6. In particular, if for some $i \in I < s_i >$ is not contained in a finite factor of W and E_i has at least three elements, then the full automorphism group of $\widetilde{E}$, where each G_i is the symmetric group on $|E_i|$, is uncountable.

5.4 Results for Graph Products

We now consider structure preserving automorphisms in terms of graph products of buildings. Fix $\mathcal{G}$ a graph with vertex set I, as before. Let $\{(W_i, S_i)\}_I$ be a collection of Coxeter systems with a collection of buildings $\{\Delta_i\}_I$ such that each Δ_i is a building of type (W_i, S_i). Let $\widetilde{\Delta}$ be the graph product of the Δ_i over $\mathcal{G}$. Further let E_i be the rank one building with chamber set Δ_i and $\widetilde{E}$ the graph product of the E_i over $\mathcal{G}$.

Theorem 5.4.1. For each $i \in I$ let $G_i = \mathrm{Aut}(\Delta_i)$. Then $\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E}) \cong \mathrm{Aut}(\widetilde{\Delta})$.

Proof. From Theorem 4.4.1 we have an isomorphism $\phi : \widetilde{\Delta} \to \widetilde{E}$ which preserves projections. The map ϕ induces an action of $\mathrm{Aut}(\widetilde{\Delta})$ on $\widetilde{E}$. The action will be type preserving and it is immediate that $\mathrm{Aut}(\widetilde{\Delta})$ injects into $\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})$.

We need to check that the action of $\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})$ on $\widetilde{\Delta}$ is type preservin. Fix some $g_{\widetilde{E}}$ in $\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})$. Let $g_{\widetilde{\Delta}}$ be the induced map on $\widetilde{\Delta}$ under ϕ. We verify that $g_{\widetilde{\Delta}}$ is a building automorphism. Fix two chambers C and C' in $\widetilde{\Delta}$ with $C \sim_s C'$ for some $s \in S_i$. We let $\phi(C) = C_E$ and $\phi(C') = C'_E$. Then we have that $C_E \sim_i C'_E$.

Since $g_{\widetilde{E}}$ is structure preserving there exists some $g_i \in G_i$ such that $p_i(g_{\widetilde{E}}(C_E)) = g_i(p_i(C_E))$ and $p_i(g_{\widetilde{E}}(C'_E)) = g_i(p_i(C'_E))$. Thus we have that $g_i(p_i(C)) \sim_s g_i(p_i(C'))$ which implies that $p_i(g_{\widetilde{E}}(C_E)) \sim_s p_i(g_{\widetilde{E}}(C'_E))$ in Δ_i. Hence $\phi(g_{\widetilde{E}}(C_E)) \sim_s \phi(g_{\widetilde{E}}(C'_E))$ so that $g_{\widetilde{\Delta}}(C) \sim_s g_{\widetilde{\Delta}}(C')$ as desired. □

Thus we can consider the automorphism group of any graph product of buildings as a subgroup of the automorphism group of the underlying right angled building. We can apply the previous results of right angled buildings to $\mathrm{Aut}(\widetilde{\Delta})$ to get the following:

Corollary 5.4.2. If there exists a thick Δ_i with W_i not contained in a finite factor of W, then $\mathrm{Aut}(\widetilde{\Delta})$ is uncountable.

Corollary 5.4.3. Any automorphism $g \in \mathrm{Aut}(X)$ where X is a combinatorial ball in $\widetilde{\Delta}$ centered at a chamber can be extended to an automorphism of $\widetilde{\Delta}$.

CHAPTER 6

STRONG TRANSITIVITY

We now consider *all* buildings arising from universal covers as defined in Section 4.1. We let $\widetilde{\Delta}$ be constructed from Δ via universal covers where we change finitely many entries in the Coxeter matrix for the system (W, S) to infinity. We study when $\widetilde{\Delta}$ inherits structure from Δ, specifically transitivity properties of the automorphism group. In Section 6.1 we define apartment systems and root groups in general along with W-transitivity and strong transitivity. In Section 6.2 we introduce a property of a set of root groups of an apartment called pre-Moufang which will help determine the regularity of the building. In Section 6.3 we give a construction of apartments and subgroups of root groups in $\widetilde{\Delta}$ arising from apartments and root groups in Δ. Finally, the main result from this chapter is that if we begin with a building that admits a pre-Moufang set of root groups of an apartment and apply the construction from Section 4.1 by switching finitely many entries in the Coxeter matrix to ∞ then our new building will also admit a pre-Moufang set of root groups of an apartment. This will immediately imply that our new building is strongly transitive and admits a (B, N)-pair.

Unless otherwise stated, in this chapter (W, S) and Δ will denote an arbitrary Coxeter system and building of type (W, S) respectively. We will let $\widetilde{W}$ and $\widetilde{\Delta}$ denote a Coxeter group and building arising from the covering space construction on (W, S) and Δ as in Section 4.1.

We will begin with definitions which hold for any building.

6.1 Apartment Systems and Root Groups

We start by defining some general notions for buildings. A full treatment of this theory along with all relevant proofs can be found in [1].

Definition 6.1.1. Recall that we say a building Δ is *thick* if every panel has size greater than two and that a building is *thin* if each panel has size exactly two. We refer to the thin building of type (W, S) that is identified with W, as in Example 2.2.3, as the *Coxeter complex*. If Δ is of type (W, S) then a sub-chamber complex isometric to the Coxeter complex of (W, S) under the W-metric is called an *apartment in* Δ.

A realization of an apartment A in the K realization of Δ is the union of the images of all (C, K) in $\mathcal{U}(\Delta, K)$ where C ranges over all chambers in A. Thus the realization of an apartment A in $\mathcal{U}(\Delta, K)$ is isometric to $\mathcal{U}(W, K)$, the K realization of the Coxeter complex, and this bijection sends copies of K to other copies of K.

Every building of type (W, S) contains at least one apartment. If we fix an apartment A in a building Δ and some automorphism $g \in \mathrm{Aut}(\Delta)$, then $g(A)$ is also an apartment in Δ.

Next, we recall from Theorem 2.2.5 the notion of projection. The projection of a chamber C onto a residue $\mathcal{R}$ is the unique chamber in $\mathcal{R}$ closest to C. Further recall from Section 3.1 the notation:

$$\mathrm{Proj}_{\mathcal{R}}(\mathcal{R}') = \{C \in \mathcal{R} | C \text{ is the projection of some } C' \in \mathcal{R}' \text{ onto } \mathcal{R}\}.$$

We say that two residues $\mathcal{R}$ and $\mathcal{R}'$ are parallel if $\mathrm{Proj}_{\mathcal{R}}(\mathcal{R}') = \mathcal{R}$ and $\mathrm{Proj}_{\mathcal{R}'}(\mathcal{R}) = \mathcal{R}'$.

Definition 6.1.2. Two panels $\mathcal{R}_s$ and $\mathcal{R}_t$ in an apartment A are *adjacent* if they are parallel and are contained in some rank two spherical residue.

The type of two opposite panels may be the same, so that we cannot say that if $\mathcal{R}_s$ and $\mathcal{R}_t$ are opposite then they are contained in an $\{s, t\}$ residue. If two panels $\mathcal{R}$ and $\mathcal{R}'$ are adjacent, then their realizations intersect. Suppose $\{s, t\}$ is the type of the rank two spherical subgroup containing $\mathcal{R}$ and $\mathcal{R}'$. In the standard realization of Δ the intersection of the corresponding mirrors is the geometric realization of the subposet of spherical subsets of S containing $\{s, t\}$. This subposet is contained in every chamber in each panel.

Definition 6.1.3. A *wall* m *in* A is a collection of panels intersecting A any two of which can be connected to one another via a sequence of such adjacencies in A. When we say a chamber C is in m we mean that it is contained in a panel contained in m.

The more standard definition of a wall is the the set of panels in an apartment A stabilized by a reflection in $\mathrm{Aut}(A)$. The realization, then, is the fixed point set of the induced automorphism of the realization of A. We require a more combinatorial definition, which is why we add the additional language of a chamber in m, since this will come up regularly. Note that the realization of a wall is always connected.

A mirror K_s in the realization of a building Δ corresponds to a panel $\mathcal{R}$ if K_s is the intersection of the realization of every chamber in $\mathcal{R}$. The type of the mirror will be the type of the panel. The realization of a wall m in $\mathcal{U}(\Delta, K)$ is the union of all mirrors corresponding to a panel in m. In the standard realization the realization of a wall will be codimension one in a neighborhood of the wall. Thus we think of m as a hyperplane in the building.

Proposition 6.1.4. Any wall m of A separates A into two components. Likewise, the realization of m separates the realization of A into two components.

Note that in general m need not separate the building Δ.

Definition 6.1.5. Fix an apartment A and a wall m in A. We define a *root* α *of* A *bounded by* m (or just a *root* when the other data is understood) to be a component of A separated by m unioned with those chambers in m which lie adjacent to the component. Those chambers in A not in α are denoted $-\alpha$ and make up the root *opposite* α so that for any root α of A we have that $A = \alpha \cup -\alpha$.

The root opposite α depends on the choice of apartment containing α. If two different apartments, A and A', both contain α then the root opposite α in A is not the root opposite α in A'. We denote the set of all roots in an apartment A as $\Phi(A)$.

Example 6.1.6. Let Δ be the building of type the dihedral group of order six with each panel of size three. In the standard realization every circuit of length six would span a solid hexagon, and that hexagon would be the realization of the apartment. Any two parallel mirrors in the hexagon would be the realization of a wall in the apartment and either component of that apartment separated by the wall would be a root in the apartment. Figure 6.1 is a nonstandard K'-realization of Δ with an apartment in bold and a root highlighted. Here K' is the Davis chamber for the infinite dihedral group as in Example 4.1.2.

Example 6.1.7. Recall from Example 2.2.7 that any regular tree can be viewed as a building of type $\mathbb{Z}_2 \star \mathbb{Z}_2$. Any infinite line in the tree is a realization of an apartment and any ray in that line a root of that apartment.

Definition 6.1.8. For any root α of an apartment A in Δ we let G_α denote the set of automorphisms of Δ which fix chamberwise every panel intersecting $\alpha \setminus \partial\alpha$. We

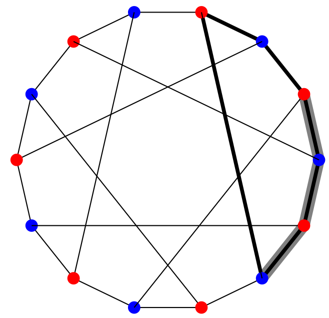

Figure 6.1: The incidence graph of the Fano plane with an apartment and root emphasized

call G_α the *root group* for α. For any fixed apartment A we call $\{G_\alpha\}_{\Phi(A)}$ the *set of root groups of* A.

The group G_α fixes all of α chamberwise, not just $\alpha \setminus \partial(\alpha)$, because the chambers on the boundary of α that are contained in α are also contained in other panels not on the boundary of α. One can check that $gG_\alpha g^{-1} = G_{g\alpha}$ for all $g \in \mathrm{Aut}(\Delta)$.

We see that G_α acts on the chambers in the wall m_α bounding α. Also, if $\mathcal{A}$ is an apartment system for Δ then G_α acts on $\mathcal{A}(\alpha)$, where $\mathcal{A}(\alpha)$ is the set of apartments containing α.

Next we define our first type of transitivity.

Definition 6.1.9. We say the action of a group G on a building Δ is W*-transitive* if for all $w \in W$ we have that G is transitive on all pairs (C, D) where $\delta(C, D) = w$.

An equivalent statement of this definition is that G acts transitively on Δ and that for all w the stabilizer of a chamber C_0 is transitive on all chambers of distance w from C_0. This does not guarantee that G_α acts transitively on $\mathcal{R}(C) \setminus \{C\}$ where $C \in \alpha$ and $\mathcal{R}(C)$ is in the wall bounding α.

Definition 6.1.10. Let $\mathcal{A}$ be a collection of apartments in a building Δ satisfying the following:

1. For any two chambers C and D in Δ there exists some $A \in \mathcal{A}$ containing them.

2. If two apartments A and A' both contain C and D then there exists an isomorphism from $A \to A'$ which fixes C and D.

Then $\mathcal{A}$ is called an *apartment system.*

In [1] there is an equivalent definition of building which begins with an apartment system rather than a type (W, S).

Definition 6.1.11. We say that an automorphism group $\mathrm{Aut}(\Delta)$ is *strongly transitive with respect to an apartment system* $\mathcal{A}$ if it is transitive on all pairs (C, A) consisting of an apartment A in $\mathcal{A}$ of Δ and a chamber C in A. We say that $\mathrm{Aut}(\Delta)$ is strongly transitive if it is transitive with respect to any apartment system of $\mathrm{Aut}(\Delta)$.

It follows that if $\mathrm{Aut}(\Delta)$ is strongly transitive with respect to some apartment system $\mathcal{A}$ then for every α a root of an apartment A in the apartment system we have that G_α is transitive on $\mathcal{A}(\alpha)$ the set of apartments in $\mathcal{A}$ which contain α. However, in general, A may be the only such apartment in $\mathcal{A}$ which contains α so this does not tell us anything about the action of G_α near m, the wall bounding α.

It is immediate that if an automorphism group is strongly transitive then it is W-transitive, however the reverse is also true which we can see when we choose the right apartment system.

Proposition 6.1.12. For any apartment A of Δ if $\mathrm{Aut}(\Delta)$ is W-transitive then $\{g(A)\}_{g \in \mathrm{Aut}(\Delta)}$ is an apartment system.

Definition 6.1.13. We say that a pair of subgroups B and N of a group G is a *(B, N)-pair* if $G =< B, N >$, $T := B \cap N$ is normal in N and the quotient $W := N/T$ has a generating set S such that:

1. for $s \in S$ and $w \in W$, we have that $sBw \subseteq BswB \cup BwB$ and
2. for $s \in S$, $sBs^{-1} \nleq B$.

We say that *$Aut(\Delta)$ admits a (B, N) pair* if for some apartment A and chamber C in A we have that $B := \text{Stab}_{\text{Aut}(\Delta)}(C)$ and $N := \text{Stab}_{\text{Aut}(\Delta)}(A)$ make up a (B, N)-pair where (W, S) is the type of Δ.

We can begin with a group which admits a (B, N)-pair and construct a building and Coxeter system from them. A great treatment of this can be found in Section 6.2.6 of [1].

Proposition 6.1.14. If a building Δ is such that $\text{Aut}(\Delta)$ is strongly transitive, then $\text{Aut}(\Delta)$ admits a (B, N)-pair.

6.2 The Pre-Moufang Property

For this section we assume that $\text{Aut}(\Delta)$ is chamber transitive. Fix α a root in some apartment A of Δ. For any chamber C in α on the boundary of α where $\mathcal{R}(C)$ is a panel in the wall m bounding α we have that G_α acts on the set $\mathcal{R}(C) \setminus \{C\}$.

Definition 6.2.1. Fix an apartment A in Δ. We say that the set of root groups of an apartment $\{G_\alpha\}_{\alpha \in \Phi(A)}$ is *pre-Moufang* if for each $\alpha \in \Phi(A)$ the above action of G_α is transitive for some C in $\alpha \cap \partial(\alpha)$.

Note that a different choice of C in α along the boundary would give an equivalent action of G_α.

If Δ is spherical we have that Δ admits a pre-Moufang set of root groups of an apartment if and only if Δ has the Moufang property as in [1]. Specifically, when Δ is spherical then G_α is transitive on $\mathcal{R}(C) \setminus \{C\}$ if and only if G_α is transitive on apartments containing α.

Theorem 6.2.2. If there exists some apartment A of Δ such that $\mathrm{Aut}(\Delta)$ admits a pre-Moufang set of root groups of A then $\mathrm{Aut}(\Delta)$ is W-transitive.

Proof. Fix A so that $\{G_\alpha\}_{\alpha\in\Phi(A)}$ is pre-Moufang. Fix a pair of chambers (C_0, C) in A. Let $w = \delta(C_0, C)$. Fix C' some chamber in Δ of distance w from C_0. We will construct an automorphism g that fixes C_0 and maps C to C'.

First we show that transitivity of $\mathrm{Aut}(\Delta)$ makes this case sufficient to prove the statement. For two more general pairs (C_0, D) and (C_0, D') such that $\delta(C_0, D) = \delta(C_0, D') = w$ with C_0 in A but D and D' not necessarily in A we use transitivity to find two automorphisms h and h' in $\mathrm{Aut}(\Delta)$ such that $h(C_0) = h'(C_0) = C_0$ and $h(D) = h'(D') = D_0$, the unique chamber in A of distance w from C_0. Then $h^{-1} \cdot h'(C_0) = C_0$ and $h'^{-1} \cdot h(D) = h'^{-1}(D_0) = D'$ as desired. For two arbitrary pairs of some fixed distance (C, D) and (C', D') we use transitivity to find some h_1 which maps C to some C_0 in A and h_2 which maps C_0 to C' so that if g maps $(C_0, h_1(D))$ to $(C_0, h_2^{-1}(h_1(D)))$ then $h_2 \cdot g \cdot h_1(C) = C'$ and $h_2 \cdot g \cdot h_1(D) = h_2 \cdot h_2^{-1}(D) = D$.

Now we shall construct g. Let γ be a minimal gallery from C_0 to C and $\mathbf{w}$ the word associated to $\gamma = (C_0, \ldots, C_k)$. Because γ is minimal it lies entirely in A. There exists $\gamma' = (C'_0, \ldots, C'_k)$ a gallery from C_0 to C' with the same associated word $\mathbf{w}$. Let $\mathbf{w} = (s_1 \ldots, s_k)$.

Let m_1 be the wall containing the s_1 panel containing C_0. Let α_1 be the root of A bounded by m_1 containing C. We will construct a sequence of automorphisms $(g_l)_{l=0}^{k}$ in G_{α_1} such that g_l maps the first $l+1$ chambers in γ to the first $l+1$ chambers in γ'. Let g_0 be the identity. Assume that $g_l \in G_{\alpha_1}$ maps the first $l+1$

chambers in γ to the first $l+1$ in γ'. Let m_{l+1} be the wall containing the s_{l+1} panel of C_l. Let α_{l+1} be the root of A bounded by m_{l+1} containing C_0. By assumption there exists $\bar{g}_{l+1} \in G_{\alpha_{l+1}}$ such that $\bar{g}_{l+1}(C_{l+1}) = g_l^{-1}(C'_{l+1})$. Then we have that $g_l\bar{g}_{l+1}(C_{l+1}) = g_l g_l^{-1}(C'_{l+1}) = C'_{l+1}$. So that $g_{l+1} := g_l\bar{g}_{l+1} \in G_{\alpha_1}$ maps the first $l+2$ chambers in γ to the first $l+2$ chambers in γ' as desired.

Hence $\mathrm{Aut}(\Delta)$ is W-transitive. □

Theorem 6.2.3. If there is some apartment A in Δ such that $\mathrm{Aut}(\Delta)$ contains a pre-Moufang set of root groups of A then $\mathrm{Aut}(\Delta)$ is strongly transitive.

Proof. This follows from the previous theorem and Proposition 6.1.12. □

This gives the immediate corollary:

Corollary 6.2.4. If there is some apartment A in Δ such that $\mathrm{Aut}(\Delta)$ contains a pre-Moufang set of root groups of A then Δ admits a (B, N)-pair.

6.3 Lifts

Let $\widetilde{\Delta}$ be a building constructed from Δ via covering spaces as in Section 4.1. We wish to construct lifts of apartments, walls, and roots which make sense in the context of $\widetilde{\Delta}$. We first define what we mean by lift of a set of chambers.

Definition 6.3.1. We say the *lift of a set of chambers* $\mathcal{D}$ in Δ is the inverse image of $\mathcal{D}$ under the projection map from $\widetilde{\Delta}$ to Δ.

Now, in general, the lift of chambers in an apartment, wall, or root of Δ will not be an apartment, wall or root of $\widetilde{\Delta}$. However, we offer a procedure to obtain apartments, walls and roots of $\widetilde{\Delta}$ from those in Δ. First, begin with a Coxeter complex A and apply the universal cover construction to get a new building $\widetilde{A}$. Each panel in $\widetilde{A}$ will have the same cardinality as its projection in A, namely two, so that $\widetilde{A}$ is not only a building of type $(\widetilde{W}, S)$ but the Coxeter complex of type $(\widetilde{W}, S)$.

We need a notion of connectedness in a building. We say a subset of Δ, $\mathcal{D}$, is *connected* if for any two chambers C and C' in $\mathcal{D}$ there exists a gallery connecting C to C' which lies entirely in $\mathcal{D}$. We now consider the lift of chambers in an apartment in Δ. The lift of these chambers may not be connected.

Lemma 6.3.2. If A is an apartment in Δ and A' is a connected component of the lift of A in $\widetilde{\Delta}$ then A' is an apartment in $\widetilde{\Delta}$.

Proof. We know that for any C in A' and s in S that $\mathcal{R}_s(C) \cap A'$ contains two chambers, because A' is a lift of A which has this local property. It remains only to show that A' is a building of type $(\widetilde{W}, S)$. Equip A' with the induced $\widetilde{W}$ metric from $\widetilde{\Delta}$. Then (WD1) and (WD2) from Definition 2.2.1 hold because $A' \subset \widetilde{\Delta}$, a building of type $(\widetilde{W}, S)$. Thus we need only show that for any pair of chambers C and D in A' and generator s we have that if $\delta(C, D) = w$ then there is a chamber $C' \in A'$ such that $\delta(C', C) = s$ and $\delta(C', D) = sw$. Because every panel has just two chambers there is a unique choice for C'. Let γ and γ' be two minimal gallerys in A' with γ from C to D and γ' from C' to D. Let $(s_1, \ldots, s_n)$ be the type of γ and $(s'_1, \ldots, s'_m)$ be the type of γ'. Then, because each panel is of minimal size, C is the unique chamber of distance $s'_1 \cdots s'_m s_n^{-1} \cdots s_1^{-1}$ from C' so that

$$\delta(C', C) = s = s'_1 \cdots s'_m s_n^{-1} \cdots s_1^{-1} = \delta(C', D)\delta(D, C) = \delta(C', D)w^{-1}$$

hence $s = \delta(C', D)w^{-1}$ and $sw = \delta(C', D)$ as desired. Thus A' is an apartment in $\widetilde{\Delta}$. □

We say that A' is a *lift of* A and denote it $\widetilde{A}$. Next, we consider walls in such apartments.

Lemma 6.3.3. Let m be a wall in A and $\widetilde{A}$ a lift of A. Let $\widetilde{m}$ be a set of panels whose corresponding mirrors in the realization make up a connected component of the lift of the realization of m which intersects $\widetilde{A}$. Then $\widetilde{m}$ is a wall in $\widetilde{A}$.

Proof. Fix m' a component of the lift of m. Suppose m' contains a single panel $\mathcal{R}'$. Then for any panel $\mathcal{R}$ in m adjacent to $p(\mathcal{R}')$, the projection of $\mathcal{R}'$ to Δ, we have that none of the mirrors correspond to the lift of $\mathcal{R}$. Thus p is not an isomorphism from any rank two spherical residue containing $\mathcal{R}'$ to its image, a contradiction. Hence $m' = \{\mathcal{R}'\}$ is a wall of $\widetilde{A}$.

Suppose m' contains at least two panels $\mathcal{R}_s$ and $\mathcal{R}_t$. Without loss of generality assume the realizations of $\mathcal{R}_s$ and $\mathcal{R}_t$ intersect so that they are contained in some rank two spherical residue. They cannot intersect at a chamber because their images under p do not, so that their intersection must be contained in mirrors. Thus the intersection of their realizations is a realization of the poset of spherical subsets of S which contain both s and t. Let T be some spherical subset of S such that $\mathcal{R}_s$ and $\mathcal{R}_t$ are both contained in the T-residue $\mathcal{R}_T$. Because T is spherical $\mathcal{R}_T$ is isomorphic to its image under the projection map p. Thus $\text{Proj}_{\mathcal{R}_s}(\mathcal{R}_t) = \text{Proj}_{\mathcal{R}_t}(\mathcal{R}_s)$. Hence each component in the lift of m is a wall in $\widetilde{A}$. □

We call m' a lift of m to $\widetilde{\Delta}$ and denote it $\widetilde{m}$. It is important to consider connected components of the lift in the realizations of the walls rather than the set of chambers lying adjacent to the wall. We need the panels to be adjacent in order to build a wall, not just a gallery connecting chambers adjacent to the wall.

We next consider roots, though these will not be as straightforward. To see this, consider α and β, roots of apartments in a spherical building. We have that $\alpha \subseteq \beta$ implies that $\alpha = \beta$. However, in the case of a tree we have many properly nested roots. Thus we can see that the lift of a root in Δ is, in general, not a root in $\widetilde{\Delta}$. However, just as a chamber determines the apartment in the lift, if we choose that chamber to be adjacent to the wall which defines the boundary of the root, then we can choose which root we want (we have to decide between a root and its opposite) by picking the one that contains the lift of our chamber on its boundary. Note that both roots will contain lifts of our chamber, but only one will contain the chamber adjacent to our choice of wall.

In Figure 6.1 the boundary of the highlighted root is two distinct vertices, which is not connected. This is because the figure does not show the standard realization of the building. In the standard realization the interiors of all hexagons are included, so the mirrors in the realization of the wall would be the diameter of a hexagon. When we take the non-standard realization we remove the interiors leaving two vertices. In the universal cover, the trivalent tree, a wall is just a vertex.

Example 6.3.4. Figure 6.2 is the universal cover of the complex in Example 6.1 along with the lift of the apartment and the lift of the roots. Further from this apartment lie other components of the lift of the apartment. Note in Figure 6.2 that the lift of the root is not connected and that taking a component would not be sufficient in this case. The selected wall and chamber are indicated. We will choose the ray bounded by our choice of wall component containing our choice of chamber.

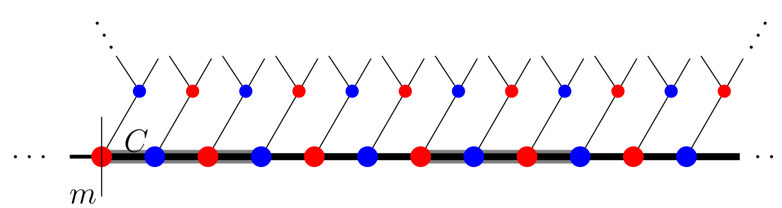

Figure 6.2: The lift of the complex in figure 6.1 with the lift of the apartment and root indicated.

Fix a gallery γ in an apartment A and a wall m in A. We say that γ *intersects* m if γ contains a subgallery (C, C') where $\{C, C'\}$ is contained in m.

6.4 Lifts of Group Actions

First note the following claim:

Proposition 6.4.1. If a building Δ is such that $\mathrm{Aut}(\Delta)$ is chamber transitive, then any $\widetilde{\Delta}$ constructed from Δ via covering spaces is such that $\mathrm{Aut}(\widetilde{\Delta})$ is chamber transitive.

Thus, for the rest of this section we will assume that all buildings are such that their automorphism group acts chamber transitively.

Suppose that the Coxeter matrix for (W, S) differs from the Coxeter matrix for $(\widetilde{W}, S)$ in finitely many places. Our goal in this final section is to show that if Δ admits a pre-Moufang set of root groups of an apartment then so does $\mathrm{Aut}(\widetilde{\Delta})$. From this we will have that $\mathrm{Aut}(\widetilde{\Delta})$ is strongly transitive and admits a (B, N)-pair.

We construct a sequence of buildings $\{\Delta_l\}_{l=1}^{n}$ with each Δ_l of type (W_l, S), and with $\Delta_0 = \Delta$ and $\Delta_n = \widetilde{\Delta}$, so that the Coxeter matrix for (W_l, S) differs from the Coxeter matrix for (W_{l+1}, S) in exactly two places. Thus the presentation for (W_{l+1}, S) is that of (W_l, S) with one relation removed.

We will show that if $\mathrm{Aut}(\Delta_l)$ admits a pre-Moufang set of root groups of an apartment then the same is true of $\mathrm{Aut}(\Delta_{l+1})$. By iterating this process we will have shown that if $\mathrm{Aut}(\Delta)$ is admits a pre-Moufang set of root groups of an apartment then the same is true of $\mathrm{Aut}(\widetilde{\Delta})$, as desired. Assume now that the presentation for (W_{l+1}, S) differs from (W_l, S) by removing the relation between s_1 and s_2. Let $T = S \setminus \{s_1, s_2\}$.

The plan is to generate a subgroup $\widetilde{G}_{\widetilde{\alpha}}$ of $G_{\widetilde{\alpha}}$ arising from the root group G_α of Δ_l

where $\widetilde{\alpha}$ is a lift of α. We construct $\widetilde{G}_{\widetilde{\alpha}}$ so that its action near the wall $\widetilde{m}$ bounding $\widetilde{\alpha}$ will agree with the action of G_α under the projection map.

Fix an apartment A of Δ_l so that the set of root groups of an apartment $\{G_\alpha\}_{\Phi(A)}$ is pre-Moufang. We construct modified lifts of elements of G_α, which will generate a subgroup $\widetilde{G}_{\widetilde{\alpha}}$ of $\mathrm{Aut}(\Delta_{l+1})$, where $\widetilde{\alpha}$ is bounded by a lift of the wall bounding α. These $\widetilde{G}_{\widetilde{\alpha}}$ will be contained in the root groups $G_{\widetilde{\alpha}}$. We will show that for any $\widetilde{\alpha}$ and C in $\widetilde{\alpha}$ with $\mathcal{R}_s(C)$ contained in the wall bounding $\widetilde{\alpha}$ the action of $\widetilde{G}_\alpha$ on $\mathcal{R}_s(C) \setminus \{C\}$ is transitive. Thus the action of $G_{\widetilde{\alpha}}$ on $\mathcal{R}_s(C) \setminus \{C\}$ is transitive. Hence we will have shown$\{G_{\widetilde{\alpha}}\}$ is a pre-Moufang set of root groups of an apartment as desired.

Let K' be a 1-simplex. Equip K' with a mirror structure as follows. Let K_{s_1} be one vertex, K_{s_2} be the other and K_s be K' whenever $s \neq s_i$ for $i \in \{1, 2\}$.

Remark 6.4.2. The realization $\mathcal{U}(\Delta_{l+1}, K')$ is a tree. Two chambers C and C' have the same realization in $\mathcal{U}(\Delta_{l+1}, K')$ exactly when they lie in the same T-residue.

When Δ_{l+1} is a Coxeter complex we can see this using the fact that any Coxeter group decomposes as an amalgamated free product of $< s_1, T >$ and $< s_2, T >$ over T, so that this realization is a $|W_{l,T}|$ valent tree.

We wish to consider our actions in terms of this realization. For this we construct the following two sets:

$$\mathcal{R}_{\widetilde{\alpha}} = \bigcup_{C \in \widetilde{\alpha}} \mathcal{R}_T(C) \qquad\qquad \mathcal{R}_{\widetilde{m}} = \bigcup_{C \in \widetilde{m}} \mathcal{R}_T(C)$$

Fix some $g \in G_\alpha$ and let $\bar{g}$ denote the lift of g to Δ_{l+1} which fixes some chamber in $\widetilde{m} \cap \widetilde{\alpha}$. Then $\bar{g}$ fixes every such chamber. We define $\widetilde{g}$, the *modified lift of* g, by its action on Δ_{l+1}.

$$\widetilde{g}(C) = \begin{cases} C & \text{if there exists a gallery from } C \text{ to } \mathcal{R}_{\widetilde{\alpha}} \text{ which does not intersect } \mathcal{R}_{\widetilde{m}} \\ \bar{g}(C) & \text{otherwise.} \end{cases}$$

Claim 6.4.3. The modified lift $\widetilde{g}$ is a building automorphism.

Proof. We must show that $\mathcal{R}_{\widetilde{\alpha}}$ separates Δ_{l+1} into components, one of which contains $\widetilde{\alpha} \setminus \mathcal{R}_{\widetilde{\alpha}}$ and does not intersect $-\widetilde{\alpha}$ so that $\widetilde{g}$ is type preserving.

In this context when we say a gallery $\gamma = (C_1, \ldots, C_n)$ is minimal we mean that if C_i and C_j are in the same T-residue so is C_k for all $i < k < j$. Fix C and γ such that γ is a minimal gallery from C to some C' in $\mathcal{R}_{\widetilde{\alpha}}$ which does not intersect $\mathcal{R}_{\widetilde{m}}$. Let γ' be another minimal gallery from C to C'.

Now, consider the realizations of γ and γ' in $\mathcal{U}(\Delta_{l+1}, K')$. These are two minimal paths from the realization of C to the realization of C' and since $\mathcal{U}(\Delta_{l+1}, K')$ is a tree by Remark 6.4.2 the paths must be the same. Thus, γ and γ' intersect the same T-residues in the same order. Hence γ' does not intersect $\mathcal{R}_{\widetilde{m}}$. Thus $\widetilde{g}$ is a building automorphism. □

Claim 6.4.4. The modified lift $\widetilde{g}$ is contained in $G_{\widetilde{\alpha}}$.

Proof. From the definition of $\widetilde{g}$ we have that $\widetilde{g}|_{\widetilde{\alpha} \setminus \mathcal{R}_{\widetilde{m}}}$ is the identity since a chamber C in $\widetilde{\alpha} \setminus \mathcal{R}_{\widetilde{m}}$ is in $\mathcal{R}_{\widetilde{\alpha}}$ so that (C) is a gallery from C to $\widetilde{\alpha}$ which does not intersect $\mathcal{R}_{\widetilde{m}}$. Thus we need only check that $\widetilde{g}$ fixes $\mathcal{R}_{\widetilde{m}} \cap \widetilde{\alpha}$. Let C be a chamber in $\mathcal{R}_{\widetilde{m}} \cap \widetilde{\alpha}$. We know that $p(C) \in A$ and that $C \in \mathcal{R}_T(C')$ for some $C' \in \widetilde{m} \cap \widetilde{\alpha}$. The panel $\mathcal{R}_T(C')$ is isomorphic to its image under p and $\widetilde{m} \cap \mathcal{R}_T(C')$ is a wall in $\mathcal{R}_T(C')$ bounding $\widetilde{\alpha} \cap \mathcal{R}_T(C')$, a root. Since p induces an isomorphism between $\mathcal{R}_T(C')$ and its image, $\mathcal{R}_T(p(C'))$, it induces an isomorphism between $\widetilde{\alpha} \cap \mathcal{R}_T(C')$ and $\alpha \cap \mathcal{R}_T(p(C'))$. But $C \in \widetilde{\alpha} \cap \mathcal{R}_T(C')$, so that $p(C) \in \alpha$. Hence $\widetilde{g}(C) = \bar{g}(C) = C$ as desired. □

Theorem 6.4.5. If A is an apartment in Δ_l such that $\{G_\alpha\}_{\Phi(A)}$ is a pre-Moufang set of root groups of A then $\widetilde{A}$, any lift of A, is such that $\{G_{\widetilde{\alpha}}\}_{\phi(\widetilde{A})}$ is a pre-Moufang set of root groups of $\widetilde{A}$.

Proof. Fix a triple (A, α, C) such that α is a root in A with C a chamber in α and on the boundary of α and $\mathcal{R}_s(C)$ contained in the wall of A bounding α. Then for any lift $(\widetilde{A}, \widetilde{\alpha}, \widetilde{C})$ of this triple in Δ_{l+1}, where $\widetilde{\alpha}$ is in $\widetilde{A}$ and $\widetilde{C}$ is in $\widetilde{\alpha}$ and on the boundary of $\widetilde{\alpha}$, we have that the restriction of the projection map p to $\mathcal{R}_s(\widetilde{C})$ induces an isomorphism from $\widetilde{G}_{\widetilde{\alpha}}$ to G_α. Thus, since G_α is transitive on $\mathcal{R}_s(C) \setminus C$ we have that $\widetilde{G}_{\widetilde{\alpha}}$ is also transitive on $\mathcal{R}_s(C) \setminus C$. Since $\widetilde{G}_{\widetilde{\alpha}} \leq G_{\widetilde{\alpha}}$ the same is true of $G_{\widetilde{\alpha}}$. The choice of triple was arbitrary so that $\{G_{\widetilde{\alpha}}\}_{\phi(\widetilde{A})}$ is a pre-Moufang set of root groups of an apartment, as desired. □

We iterate this process and get the following results for any building $\widetilde{\Delta}$ constructed from Δ by removing a finite number of relations from the Coxeter group presentation. This construction combined with the results in Section 6.2 give us the following theorems:

Theorem 6.4.6. If $\mathrm{Aut}(\Delta)$ admits a pre-Moufang set of root groups of an apartment, then so does $\mathrm{Aut}(\widetilde{\Delta})$.

Theorem 6.4.7. If $\mathrm{Aut}(\Delta)$ admits a pre-Moufang set of root groups of an apartment, then $\mathrm{Aut}(\widetilde{\Delta})$ is strongly transitive.

Theorem 6.4.8. If $\mathrm{Aut}(\Delta)$ admits a pre-Moufang set of root groups of an apartment, then $\mathrm{Aut}(\widetilde{\Delta})$ admits a (B, N)-pair.

CHAPTER 7

LATTICES

As we discussed in Section 5.1 if Δ is locally finite then $\mathrm{Aut}(\Delta)$ is a locally compact group and the chamber and panel stabilizers are open and compact. We say that a subgroup G of $\mathrm{Aut}(\Delta)$ is discrete if for some $C \in \Delta$ we have that $\mathrm{Stab}_G(C)$ is finite. A *lattice* in an automorphism group of a space with a measure is a discrete subgroup whose quotient has finite volume. A lattice is uniform if its quotient is compact.

7.1 Generalized Graph Product of Groups

As before, let $\{E_i\}_{i\in I}$ be a collection of rank one buildings and $\{G_i\}_{i\in I}$ be a collection of groups with G_i acting on E_i. We are now going to require that these actions are *transitive*. Define $E = \prod_I E_i$ and $\widetilde{E} = \prod_{\mathcal{G}} E_i$ as before and let W be the right angled Coxeter group assoiated to $\mathcal{G}$ with generating set $I = V(\mathcal{G})$.

We will now define the *generalized graph product* of groups acting on sets. This definition is due to [13].

Fix some $C = \prod_I C_i \in E$. Let B_i be the stabilizer of C_i in G_i and $B = \prod_I B_i$. Let H_i be the product of the B_k except with G_i in the i-th position rather than B_i and for all $\{i, j\}$ edges in $\mathcal{G}$ define $H_{i,j}$ to be the product of the B_k except with G_i in the i-th position and G_j in the j-th position. We have obvious inclusion maps $f_i : B \hookrightarrow H_i$, and $f_{i,j} : H_i \hookrightarrow H_{i,j}$ when $\{i, j\}$ is an edge in $\mathcal{G}$.

Definition 7.1.1. We define the *generalized graph product* of the G_i over $\mathcal{G}$ to be the direct limit of B, the H_i, and $H_{i,j}$ with inclusion maps f_i and $f_{i,j}$.

Call this generalized graph product $\overline{G}$. We then have the universal property that if any group H has maps $h_i : H_i \to H$ and $h_{i,j} : H_{i,j} \to H$ where the h_i and $h_{i,j}$ respect the inclusion maps f_i and $f_{i,j}$ then there exists a unique homomorphism $h : \overline{G} \to H$ which respects the inclusions as well.

Note that because the G_i act transitively, a different choice of C would give stabilizers conjugate to these B_i so that this generalized graph product is unique up to an inner automorphism of $\prod_I G_i$.

So far we have not assumed the G_i act effectively on the E_i. Let N_i be the kernel of the action of G_i on E_i and $N = \prod_I N_i$. Then $N \trianglelefteq B$. Further let $\widetilde{G}$ be the group of all lifts of the action of $\prod_I G_i$ on E to $\widetilde{E}$. Then $\widetilde{G}$ neccessarily acts effectively on $\widetilde{E}$.

Lemma 7.1.2. If $\overline{G}$ and $\widetilde{G}$ are as above, then $\widetilde{G} = \overline{G}/N$.

Proof. Fix $\widetilde{C_0}$ a chamber in the lift of C_0 and let $\widetilde{B}$ be the lift of B which stabilizes $\widetilde{C_0}$. Let $\widetilde{H_i}$ be the lift of H_i stabilizing the i-panel containing $\widetilde{C_0}$ and for each edge $\{i,j\}$ in the graph $\mathcal{G}$ let $\widetilde{H_{i,j}}$ be the lift of $H_{i,j}$ stabilizing the $\{i,j\}$-residue containing $\widetilde{C_0}$. This induces maps from B, H_i and $H_{\{i,j\}}$ to $\widetilde{G}$ the group of lifts. It follows from the construction that these maps respect inclusion. This gives us a unique map $h : \overline{G} \to \widetilde{G}$ that agrees with the embeddings.

The $\widetilde{H_i}$ generate all of $\widetilde{G}$ so that h must be onto. The map h induces an action of $\overline{G}$ on $\widetilde{E}$. Because $\widetilde{G}$ is defined as the group of lifts, $\widetilde{G}$ must act effectively so that the kernel of h is the kernel of the action. We have an action of each H_i on $\widetilde{E}$ by lifts of the action of H_i on E. The kernel of the action of H_i on E is N the product of the N_i, thus $N < \overline{G}$ is the kernel of h. This gives us that $\widetilde{G} = \overline{G}/N$ as desired. □

It immediately follows that $\overline{G}/N \subset \mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})$. Note that when the G_i act effectively, N is trivial so these groups are the same. In most cases of interest these notions coincide.

7.2 Residual Finiteness

When the G_i are finite we have that $\widetilde{G}$ is a lattice in $\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})$. We refer to $\widetilde{G}$ as the *standard lattice* of $\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})$. Margulis proved that a lattice Γ of a group G is arithmetic in some group G if and only if the commensurator of Γ in G is dense in G [16]. Liu proved that the commensurator of the standard uniform lattice of a tree in the full automorphism group is dense in that group [15]. Haglund generalized this to the following statement in [11].

Theorem 7.2.1. If we let $E_i = G_i$ then the commensurator of $\widetilde{G}$ in $\mathrm{Aut}(\widetilde{E})$ is dense in that group.

In this paper, Haglund also proved that the standard lattice is residually finite. If we let $\widetilde{G}^0$ be the group of lifts of the identity, we have that $\widetilde{G}^0$ is finite index in $\widetilde{G}$. Thus, $\widetilde{G}^0$ is also residually finite.

Let the E_i again be finite rank one buildings, and require only that the G_i act transitively on E_i. We apply Haglund's result to the lifts of the identity, $\widetilde{G}^0$ in $\mathrm{Aut}_{\mathrm{SP}}(\widetilde{E})$, and see that the action of $\widetilde{G}^0$ on $\widetilde{E}$ is residually finite. Now, if the G_i are all finite, then $\widetilde{G}^0$ is finite index in $\widetilde{G}$. Thus the action of $\widetilde{G}$ on $\widetilde{E}$ is residually finite. Hence we have the following result:

Theorem 7.2.2. Let each Δ_i be finite, and $G_i = \mathrm{Aut}(\Delta_i)$ then the standard lattice in $\widetilde{G}$ is residually finite.

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