THE DEFORMATION THEORY OF DISCRETE REFLECTION GROUPS AND PROJECTIVE STRUCTURES

DISSERTATION

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ABSTRACT

We study deformations of discrete groups generated by linear reflections and associated geometric structures on orbifolds via cohomology of Coxeter groups with coefficients in the adjoint representation associated to a discrete representation. We completely describe a cochain complex that computes this cohomology for an arbitrary discrete reflection group and, as a consequence of this description, give a vanishing theorem for cohomology in dimensions greater than 2. As an application, we discuss some situations in which the cohomology vanishes in dimension 2 as well. In particular, we are able to give a proof of a recent result of Choi and Lee on deforming a certain class of hyperbolic orbifolds through non-hyperbolic projective structures in cohomological language and give some insight into how the result can be extended.

To Ellie, my wife, and my daughter Lily

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CHAPTER 1

INTRODUCTION

In recent years, the subject of discrete reflection groups has attracted interest for its connection with convex real projective structures [17], [5]. It is a result of Vinberg [18] that discrete reflection groups correspond in an essentially unique way to certain orbifolds with convex real projective structures. In the event that the corresponding orbifold is compact, this gives rise to "divisible convex sets" as studied by Benoist in his celebrated series of papers [1], [2], [3], [4]. Many of the known examples of such sets arise from discrete reflection groups.

On the other hand, a fundamental question in the geometry of Coxeter groups is: when a Coxeter group can be realized as a group generated by reflections in the facets of a compact, hyperbolic polytope? In two and three dimensions, this question is completely answered, in the latter case by Andreev's theorem. Andreev's theorem, combined with results of Moussong [16], can be rephrased as saying that any word hyperbolic Coxeter group with nerve homeomorphic to S^2 is isomorphic to a hyperbolic reflection group generated by reflections in the faces of some compact three dimensional hyperbolic polytope. Both conditions are essentially combinatorial and can be decided by inspection of Coxeter diagrams.

There is no such result in higher dimensions, though a theorem of Vinberg gives an upper bound on the dimension of such polytopes. [19]

One may hope to gain some insight into the higher dimensional situation by considering the same questions for convex real projective structures. Convex projective structures often occur as deformations of hyperbolic structures. Indeed, hyperbolic structures themselves are a special case of convex real projective structures. It is a result of Benoist that if a word hyperbolic group acts cocompactly on a convex open subset Ω of a projective space, then the boundary $\partial\Omega$ is at least twice differentiable and $\overline{\Omega}$ is strictly convex (= no line segments in $\partial\Omega$) and properly convex (= $\overline{\Omega}$ does not contain a line). The hyperbolic case occurs exactly when the boundary is smooth. It is an observation of Moussong that Vinberg's bound on the dimension of hyperbolic polytopes depends only on word hyperbolicity of the associated Coxeter group. Taken together, we believe this makes convex real projective structures a reasonable proxy for the more rigid class of hyperbolic structures.

Unfortunately, it turns out that Andreev's theorem has no direct analog in higher dimensions even for convex projective structures – there exist word hyperbolic Coxeter groups with nerve homeomorphic to S^n for some n that do not act cocompactly on any convex open subset of $\mathbb{R}P^{n+1}$. Examples of such groups go back to Moussong. [15][16]

Although a full account of when a word hyperbolic Coxeter group acts cocompactly on an open subset of a projective space does not appear to be any closer than a similar account of the hyperbolic situation at this point, results giving combinatorial conditions for existence of *deformations* of hyperbolic structures into real projective structures have appeared. Recent work of Choi and Lee [9], for example, gives a condition that is completely combinatorial in three dimensions providing a dimension count for the moduli space in a neighborhood of the hyperbolic structure. Such results give us some idea of the prevalence of such structures.

Another related direction is to attempt to extend Andreev's theorem to handle non-word hyperbolic Coxeter groups in three dimensions, where now we only ask for a cocompact action on a convex open subset of $\mathbb{R}P^3$ with no requirement on regularity or strict convexity of the boundary. This was in fact the original impetus for the theory developed here. We pose the question:

Question 1.0.1. *Is there combinatorial characterization of Coxeter groups with nerve isomorphic to S^2 that act cocompactly on an open convex subset of $\mathbb{R}P^3$?*

In both the construction of examples and in understanding convex projective structures in general, deformations play an important role. To date, most work on deformations of projective structures in connection with Coxeter groups have been based on Vinberg's classification of discrete reflection groups.

To a discrete reflection group, Vinberg associates an equivalence class of Cartan matrices, which, along with some additional data in the reducible and Euclidean cases, completely determine the underlying reflection group. Cartan matrices corresponding to abstractly isomorphic reflection groups have a similar form. The set of equivalence classes of such Cartan matrices has the structure of a real semialgebraic variety with points corresponding to specializations of certain polynomial matrices. This classification is very convenient for some purposes. For example, it is easy to read off from a Cartan matrix whether the associated reflection group preserves a quadratic form. The Cartan martrix encodes the dimension of the underlying representation in its rank in the irreducible, non-Euclidean case. For groups generated by reflections in the faces of a compact simplex, the picture is particularly nice, as the dimension of the associated irreducible representations does not depend on the choice of parameters chosen in the Cartan matrix. In general, though, the Cartan matrix is a somewhat unwieldy way to understand the space of projective structures, since here the issue of dimension is key and controlling the rank of a highly degenerate real polynomial matrices can be difficult.

Another possible approach is deformation theory. Here we consider a discrete reflection group as a representation of a Coxeter group into some Lie group, usually $GL_n(\mathbb{R})$ or $PO(n, 1)$, and consider cohomology with coefficients in the associated adjoint representation. The first cohomology parametrizes first order (infinitesmal) deformations. Roughly, we may think of this as the Zariski tangent space of an appropriate quotient of the representation variety. Again, roughly speaking, a deformation is a germ of an analytic path of representations. We build up this germ by a limiting process – at each stage, we extend to higher order, obtaining a power series in the limit. Each such extension has an associated obstruction, which lives in the second cohomology. Much of our discussion will revolve around situations where these obstructions vanish for structural reasons, especially when the second cohomology vanishes.

Our object here is to develop the deformation theory of discrete reflection groups by studying the group cohomology of Coxeter groups with coefficients. We use methods developed by Davis [10], [12], along with Vinberg's approach to discrete reflection groups. We describe a cochain complex that computes the cohomology in terms of data used to describe reflection groups in the Vinberg picture.

This yields a vanishing result for higher cohomology, which we exploit in applications. We are able to reinterpret recent results on deformations of hyperbolic reflection groups into non-hyperbolic projective structures within this framework and provide a different perspective.

CHAPTER 2

PRELIMINARIES: COXETER GROUPS AND REFLECTION GROUPS

A group is called a *Coxeter group* if it has a presentation with a generating set $S = \{s_1, ..., s_n\}$ such that $s_i^2 = 1$ for each i and for each i and j $(s_i s_j)^{m_{i,j}} = 1$ for some $m_{i,j} \in \{2, 3, ..., \infty\}$, where the case $m_{i,j} = \infty$ is interpreted as meaning there is no relation. Notice in particular that for $m_{i,j} = 2$, we have $s_i s_j = s_j s_i$. Coxeter groups are usually understood to come with a distinguished generating set as above. The exponents $m_{i,j}$ appearing in the definition can be arranged into a symmetric matrix $C = (m_{i,j})$ called a Coxeter matrix, where $m_{i,j}$ is understood to be 1 when $i = j$. We will often refer to elements of such a generating set without introducing indices as above. As a convenience, we use the notation $m_{s,t}$ to refer to the exponent appearing in the appropriate relation $(st)^{m_{s,t}} = 1$ for $s, t \in S$. A Coxeter group is called *reducible* if there is a decomposition $S = T_1 \sqcup T_2$ such that for each $s \in T_1$ and $t \in T_2$, $m_{s,t} = 2$. If there is no such decomposition, we call the group *irreducible*.

There is a useful graphical notation encoding presentations of Coxeter groups, so-called Coxeter diagrams. The Coxeter diagram of a Coxeter group W with generating set S is a graph, possibly with labelled edges, constructed as follows:

The vertex set is the generating set S. For each pair $s, t \in S$, there is an edge if $m_{s,t} > 2$, which is labelled $m_{s,t}$ if $m_{s,t} > 3$. Alternatively, especially in actual visual representations of these objects, an edge labelled 4 may be represented by a double edge and an edge labelled 6 may be represented by a triple edge, as with Dynkin diagrams. Some authors, notably Vinberg, use alternate stylized representations of edges labelled ∞ as well, often bold or dotted lines. We note that a Coxeter group is irreducible if and only if its Coxeter diagram is connected.

In the theory of infinite Coxeter groups, Coxeter groups generated by reflections in the faces of a compact, totally geodesic simplex embedded in a sphere, Euclidean space, or hyperbolic space with angles between faces corresponding to generators s and t given by $\frac{\pi}{m_{s,t}}$ play a special role. Such groups are called *simplex groups* or Lannèr groups. The spherical case corresponds to finite Coxeter groups, which are often called spherical for this reason. Similarly, we have Euclidean or affine simplex groups and hyperbolic simplex groups. Simplex groups of all three types are classified. The classification for the spherical case is both particularly simple and useful, so we record it below in diagrammatic form. See Humphreys for details on this classification as well as other simplex groups. [14]

We note that all these diagrams are trees. That is, they contain no circuits. The same holds for all but one infinite family of Euclidean simplex groups, the $\tilde{A}_n$ series. Many hyperbolic simplex groups have diagrams containing loops, including infinitely many hyperbolic triangle groups. Note also that for any $n \in \mathbb{N}$, there are irreducible finite Coxeter groups having n generators. The same holds for Euclidean simplex groups. In contrast, compact hyperbolic simplex groups exist only in dimensions up to four.

A subset $T \subset S$ determines a subgroup $\langle T \rangle \subset W$. Such subgroups are called

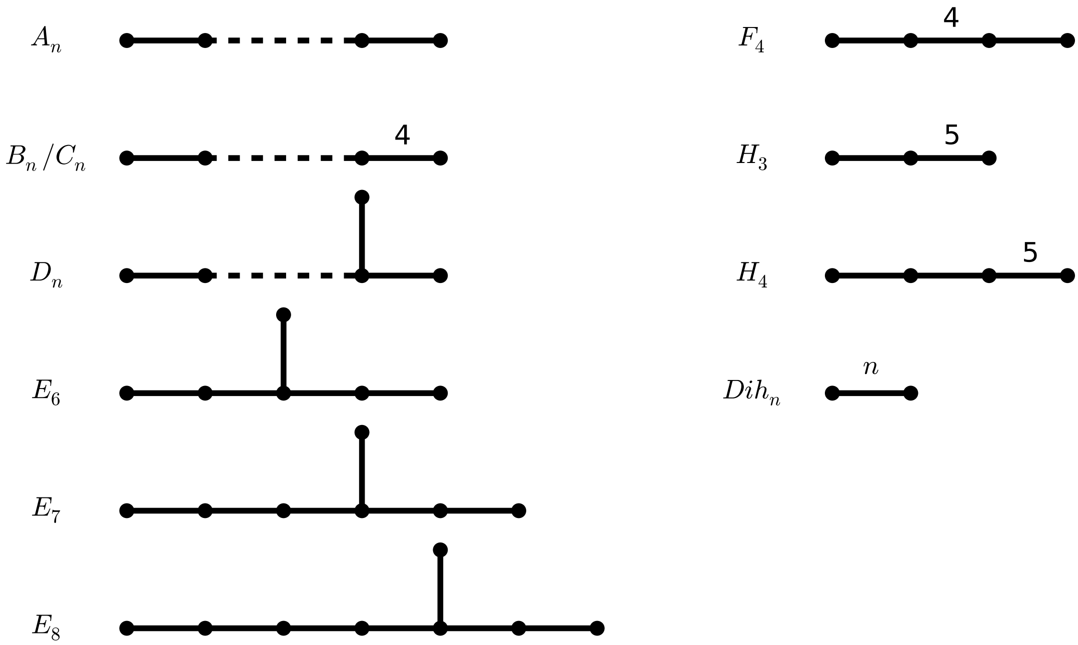

Figure 2.1: Diagrams for all finite Coxeter groups, including infinite families A_n, B_n/C_n, and D_n – here the dashed lines represent enough unlabeled edges to make the total number of nodes n.

special or *parabolic* subgroups. The subset T and the subgroup it generates are called *spherical* if $\langle T \rangle$ is finite. There is a simplicial complex $L(W, S)$ called the nerve of (W, S) that encodes important topological information about W. Its vertex set is the generating set S. Its faces are the simplices spanned by the spherical subsets of S. By results of Davis [10], [11], the nerve controls aspects of the topology of (W, S). For example, one can compute the cohomology of the Davis complex of (W, S) with compact supports in terms of certain subcomplexes of $L(W, S)$. We will not need the full force of these results, but they will provide combinatorial and topological restrictions on Coxeter groups giving rise to certain geometric structures.

2.1 Reflection groups

The approach to reflection groups we take here follows that of Vinberg [18]. We will consider reflections defined on real vector spaces and all vector spaces will be assumed to be defined over the field of real numbers. A *reflection* acting on a vector space V, is a linear endomorphism $s : V \to V$ with simple eigenvalues -1 and 1, with -1 occurring with multiplicity 1. Such a reflection is therefore determined by its eigenspaces. The positive eigenspace is called the *reflecting hyperplane* of the reflection. There is a convenient way to specify data defining a reflection. Given a reflection, we may choose a pair $(\alpha_s, h_s) \in V^* \times V$ such that $\alpha_s(h_s) = 2$, α_s vanishes along the reflecting hyperplane of s, and h_s is a (-1)-eigenvector of s. We call such a pair *defining data* of the reflection s. Then have

$$s : v \mapsto v - \alpha_s(v)h_s.$$

Equivalently, $s = \mathrm{I} - \alpha_s \otimes h_s$. This choice of α_s and h_s is not uniquely determined. We can scale α_s by some $a \in \mathbb{R}^+$ and h_s by $\frac{1}{a}$ without changing the reflection s. Nevertheless, such pairs will be our preferred description of reflections and for brevity we will tacitly assume that any reflection under consideration comes with a choice of such defining data except where this ambiguity is under discussion.

Now suppose we have a set S of reflections defined by pairs α_s, h_s for each $s \in S$ such that the intersection of half-planes $\{v \in V | \alpha_s(v) \leq 0\}$ defines a convex polyhedral cone K. Vinberg defines the group $\Gamma = \langle S \rangle$ to be a *discrete linear group generated by reflections* if for all $\gamma \in \Gamma - \{e\}, \gamma K^o \cap K^o = \emptyset$. We will call such groups *discrete reflection groups.* Vinberg goes onto prove that this definition is equivalent to the following conditions: for each pair of reflections $s, t \in S$ with $s \neq t$ we have $\alpha_s(h_t) \leq 0$ and either

- $\alpha_s(h_t) = 0$ and $\alpha_t(h_s) = 0$,
- $\alpha_s(h_t)\alpha_t(h_s) = 4\cos^2 \frac{\pi}{n_{s,t}}$ for some $n_{s,t} \in \{2, 3, ..., \infty\}$, or
- $\alpha_s(h_t)\alpha_t(h_s) \geq 4$.

The first condition implies that the (-1)-eigenvectors of each of s and t lie within the reflecting hyperplane of the other so that the two are simultaneously diagonalizable, hence $st = ts$. The last two conditions are best understood in terms of the trace of the product st in two dimensions. One can compute: $\mathrm{Tr}\, st = \mathrm{Tr}\,(\mathrm{I} - \alpha_s \otimes h_s) \cdot (\mathrm{I} - \alpha_t \otimes h_t) = \mathrm{Tr}\,\mathrm{I} - \alpha_s \otimes h_s - \alpha_t \otimes h_t + \alpha_s(h_t)\alpha_t \otimes h_s = 2 - \alpha_s(h_s) - \alpha_s(h_t) + \alpha_s(h_t)\alpha_t(h_s) = \alpha_s(h_t)\alpha_t(Hs) - 2$. The condition $\alpha_s(h_t)\alpha_t(Hs) = 4\cos^2 \frac{\pi}{n_{i,j}}$ is equivalent to $\alpha_s(h_t)\alpha_t(Hs) - 2 = 2\cos \frac{\pi}{n_{i,j}}$, so st is conjugate in $\mathrm{G}L_2(\mathbb{R})$ to a rotation through the angle $\frac{2\pi}{n_{i,j}}$. The condition $\alpha_s(h_t)\alpha_t(Hs) \geq 4$ is equivalent to $\mathrm{Tr}(st) \geq 2$. There are two cases then, reminscent of the classification of automorphisms of the hyperbolic plane. Namely, if $\mathrm{Tr}(st) = 2$, the action of st on the projective line $\mathbb{P}(\langle h_s, h_t \rangle)$ has a single fixed point x_0 and the action on $\mathbb{P}(\langle h_s, h_t \rangle) \setminus \{x_0\}$ is by translation under the obvious identification with $\mathbb{R}$. If $\mathrm{Tr}(st) > 2$, there are two fixed points for the action on the projective line and the action has sink-source dynamics. In the language of Coxeter groups, these three situations correspond to $m_{s,t} = 2$, $m_{s,t} = n_{s,t}$, and $m_{s,t} = \infty$ respectively. It follows that a discrete reflection group may be viewed as a representation of Coxeter group.

2.1.1 The main result on discrete reflection groups and convex real projective structures

The result is

Theorem 2.1.1. *(Vinberg) Suppose* Γ *is a discrete reflection group as above, generated by reflections in the faces of a polyhedral cone* $K \subset V$*. Then the union*

$$C = \bigcup_{\gamma \in \Gamma} \gamma K$$

is itself a convex cone.

Indeed, it follows from this result that discrete reflection groups are faithful representations of Coxeter groups.

This theorem provides the link between convex real projective structures and discrete reflection groups.

Definition 2.1.1. *A* real projective structure *on an orbifold* X *is an orbifold atlas with charts in a real projective space* $\mathbb{P}(V)$ *such that the transition maps between overlapping charts are given by projective transformations (= elements of* $PGL(V)$*). Two such structures are equivalent if the corresponding atlases have a common refinement together with projective isomorphisms of the charts compatible with the transition maps. To such a structure, there is a* holonomy representation *of the fundamental group of* X *defined by composing the sequence of transition maps encountered as one travels along a based loop. A real projective structure is called* convex *if it is induced by a projective isomorphism of the universal cover of* X *with a convex, open subset of* $\mathbb{P}(V)$*.*

A quadratic form Q of signature $(n, 1)$ defines a projective ellipsoid by the equation $Q(x) = 0$. The set of projective transformations that preserve this ellipsoid (and the disk it bounds) is precisely $PO(Q)$, so we see that hyperbolic structures are in fact a special case of convex projective structures.

If we projectivize the cone C of Theorem 2.1.1, we get a convex, open subset $\mathbb{P}(V)$ of the projective space $\mathbb{P}(V)$. The quotient of $\mathbb{P}(V)$ by Γ is an orbifold locally modeled on (a quotient of) the projective space $\mathbb{P}(V)$. The underlying topological space associated to this orbifold is canonically identified with the fundamental domain $\mathbb{P}(K)$. Such orbifolds are examples of so-called *Coxeter orbifolds*. Here we mean an orbifold whose underlying space is a polytope such that the isotropy groups are constant on the relative interiors of each face, generated by reflections in the facets that meet at that face.

2.2 Cartan matrices

For Γ a discrete reflection group with generating set $S = \{s_1, ..., s_n\}$ together with a choice of a pair (α_s, h_s) for each $s \in S$ as above, Vinberg defines the associated *Cartan matrix* to be $A = (\alpha_{s_i}(h_{s_j})$. We then have that $A_{i,i} = 2$ for each i and either $A_{i,j} = A_{j,i} = 0$, $A_{i,j}A_{j,i} = 4\cos^2 \frac{\pi}{n_{i,j}}$, or $A_{i,j}A_{j,i} \geq 4$ for each i and j. On the other hand, we call a matrix A a Cartan matrix if it satisfies these conditions.

We have observed that there is an ambiguity in defining a reflection s by a pair (α_s, h_s). We could just as well define s by the pair α'_s, h'_s where $\alpha'_s = z\alpha_s$ and $h'_s = \frac{1}{z}h_s$. Then for any $t \in S, t \neq s$, we have $\alpha'_s(h_t) = z\alpha_s(h_t)$ and $\alpha'_t(h_s) = \frac{1}{z}\alpha_t(h_s)$, so we have an ambiguity in the Cartan matrix as well. The effect on the Cartan matrix is easy to account for though, since all we have done is conjugate with a diagonal matrix with nonzero entries equal to 1, except for the one corresponding to the generator s, which is z. To eliminate the indeterminacy, we pass to equivalence classes of Cartan matrices, where two Cartan matrices are equivalent if they are conjugate to each other via a diagonal matrix with positive diagonal entries.

This notion of equivalence interacts in an interesting way with the topology of Coxeter diagrams. The nonzero, off-diagonal elements of the Cartan matrix of a reflection group Γ correspond naturally to edges of the Coxeter diagram of the associated Coxeter group. That is, for each pair $A_{i,j}$ and $A_{j,i}$ of nonzero entries of the Cartan matrix, there is an edge in the Coxeter diagram connecting the nodes s_i and s_j with weight determined by the product $A_{i,j}A_{j,i}$.

The following can be proved by induction on edges in a rooted spanning tree of a diagram:

Lemma 2.2.1. *Suppose Γ is an irreducible discrete reflection group with associated Coxeter system (W, S) and Coxeter diagram D with spanning tree T. Let A be a Cartan matrix associated to some choice of α_s and h_s for each $s \in S$. For each edge $\{s_i, s_j\}$ of T choose negative real numbers $B_{i,j}$ and $B_{j,i}$ with $B_{i,j}B_{j,i} = \alpha_{s_i}(h_{s_j})\alpha_{s_j}(h_{s_i})$. Then A is equivalent to a Cartan matrix A' with $A'_{i,j} = B_{i,j}$ and $A'_{j,i} = B_{j,i}$.*

Vinberg gives a classification of discrete reflection groups in terms of equivalence classes of Cartan matrices and some additional data on the relations between the α's and h's. The following result is enough to give some interesting examples of deformations of projective structures.

Theorem 2.2.1. *(Vinberg) There is a one-to-one correspondence between equivalence classes of Cartan matrices and isomorphism classes of irreducible discrete* *reflection groups. If the group is not Euclidean, then under this correspondence, the dimension of the representation is equal to the rank of the Cartan matrix.*

2.2.1 Othogonal reflection groups

Definition 2.2.1. *A reflection group is called* orthogonal *if it preserves a quadratic form Q.*

We will not distinguish between a quadratic form and its polarization, the symmetric bilinear form defined by

$$B(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y)).$$

We simply use the notation $Q(\cdot, \cdot)$ to refer to the associated bilinear form.

Given a reflection $s = \mathrm{I} - \alpha_s \otimes h_s$ that preserves a quadratic form Q, we may relate α_s and h_s. We have $Q(sv, sv) = Q(v, v)$ for all $v \in V$, so

$$\begin{aligned} Q(v - \alpha_s(v) \otimes h_s, & v - \alpha_s(v) \otimes h_s) \\ &= Q(v, v) - 2Q(v, \alpha_s(v)h_s) + Q(\alpha_s(v)h_s, \alpha_s(v)h_s) \\ &= Q(v, v), \end{aligned}$$

hence $2\alpha_s(v)Q(v, h_s) = \alpha_s(v)^2 Q(h_s, h_s)$. If may scale h_s so that $Q(h_s, h_s) = \pm 2$, then we get $\alpha_s = \pm Q(\cdot, h_s)$. The sign must be chosen so that $\alpha_s(h_s) = 2$.

Proposition 2.2.1. *(Vinberg) If Γ is a discrete reflection group with generating set S that preserves a quadratic form, then there is a choice of defining data $(\alpha_s, h_s), s \in S$ so that the associated Cartan matrix is symmetric.*

In particular, if Γ is generated by reflections in the faces a spherical, hyperbolic, or Euclidean polytope, then we can find a Cartan matrix for Γ that is symmetric.

2.3 The dual representation

A reflection s on V induces a reflection on the dual space V^*, which for simplicity we will also call s, as follows:

$$s : \lambda \mapsto (v \mapsto \lambda(sv)).$$

Unpacking this a bit, we get

$$(v \mapsto \lambda(sv) = (v \mapsto \lambda(v - \alpha_s(v)h_s)) = (v \mapsto \lambda(v) - \lambda(h_s)\alpha_s(v)) = \lambda - \lambda(h_s)\alpha_s.$$

So $s : \lambda \mapsto \lambda - \lambda(h_s)\alpha_s$. Notice that in the dual reflection, the roles of α_s and h_s are reversed, namely α_s is the (-1)-eigenvector and the evaluation map associated to h_s, thought of as a linear functional on V^*, defines the reflecting hyperplane.

To any representation of a group as automorphisms of a vector space V, there is an associated representation on the dual space V^*, called the contragredient or dual representation. The obvious action obtained by simply letting g act by $g : \lambda \mapsto (v \mapsto \lambda(gv))$ turns out to be a right action. To correct this and stay in the category of left modules, we compose with the inverse so that $g : \lambda \mapsto (v \mapsto \lambda(g^{-1}v)$. In the case of reflection groups, since the generators are involutions, the situation is simplifies. We can say that the dual representation is defined at the level of generators as the representation on V^* that sends each $s \in S$ to its dual reflection. Note, though, that the dual representation does not necessarily determine a discrete reflection group: the condition that the reflecting hyperplanes bound a convex polyhedral cone with nonempty interior may be violated. A simple example of this phenomenon is the two dimensional discrete reflection group isomorphic to the infinite dihedral group with generators s and t such that $\alpha_s(h_t)\alpha_t(h_s) = 4$. It turns out that h_s and h_t must be proportional in this case, so that in the dual representation, they determine the same reflecting hyperplanes.

2.4 The representation variety

Suppose we have a finitely presented group Γ with presentation $\langle x_1, ..., x_n | r_1, ..., r_m \rangle$, where the r_i are words in the generators $x_1, ..., x_n$, and an algebraic group G. The presentation defines a map

$$P : \underbrace{G \times ... \times G}_{n} \to \underbrace{G \times ... \times G}_{m}$$

sending a tuple $(g_1, ..., g_n)$ to a tuple $(h_1, ..., h_m)$ by substituting the letters x_i for g_i in the words r_i appearing in the presentation. This is an algebraic map, since it is obtained by taking a direct product of compositions of the multiplication and inverse maps from G. The representation variety $\mathcal{R}(\Gamma, G)$ is then defined to be $P^{-1}((1, ..., 1))$. Of course, this preimage is precisely the set of substitutions $x_i \mapsto g_i$ such that the defining relations of Γ are satisfied, in other words the points of $\mathcal{R}(\Gamma, G)$ are in one-to-one correspondence with homomorphisms $\Gamma \to G$. This construction therefore induces the structure of an algebraic variety on the set $\mathrm{Hom}(\Gamma, G)$. One can define a representation variety for finitely generated groups as well, but we will not need this here.

The representation variety comes with an action of G obtained by restricting the action of G on $\underbrace{G \times ... \times G}_{n}$ by conjugation on each factor – $\mathcal{R}(\Gamma, G)$ is preserved by this action, since the defining condition is preserved. We will say two representations are equivalent if they lie in the same orbit of this action. A representation is *locally rigid* if it lies in an open orbit of the action of G. Given a point

$\rho \in \mathcal{R}(\Gamma, G)$, we may consider the Zariski tangent space T_ρ at ρ. Inside T_ρ lies the tangent space to the orbit $G\rho$. The dimension of a complementary space is a measure of whether and in what different ways one can deform the representation ρ.

Discrete reflection groups correspond to a certain subset of the representation variety of the associated Coxeter group. The condition that the generators of the corresponding Coxeter group are sent to linear reflections is an open condition in the representation variety. Indeed, since the eigenvalues of an involution are all either 1 or -1, the set of possible traces of the images of the generating involutions of a Coxeter group forms a discrete set. Therefore, the value of the trace is locally constant. Vinberg's conditions characterizing which representations give rise to discrete reflection groups give this subset the structure of a semialgebraic subset. In particular, for a generic discrete reflection group, small deformations do not destroy discreteness.

CHAPTER 3

DEFORMATIONS AND GROUP COHOMOLOGY

The specific strain of deformation theory of interest to us involves the use of group cohomology to study deformations of discrete groups and geometric structures. It goes back to Calabi [7] and Weil [20]. A good account of the subject can be found in Goldman-Millson [13]. There, the authors describe Deligne's approach to deformation theory via differential graded Lie algebras and its application to representation varieties of fundamental groups of Kähler manifolds. We recall those parts of their treatment we will use in the sequel in the first section of this chapter. We will also need a rigidity result of Calabi for groups acting discretely and cocompactly on hyperbolic space. We will discuss the set up and give the needed statement in the second section.

3.1 Cocycles, brackets, and obstructions

There is a definition of the cohomology of a group Γ with coefficients in a Γ-module M in terms of M-valued functions on n-fold products of Γ. We define the cochain groups as follows:

$$C^n(\Gamma; M) = \mathrm{Fun}(\Gamma^n, M),$$

where $C^0(\Gamma; M)$ is understood to be M and Fun denotes set theoretic functions. The differential is defined by

$$d\varphi(g_1, ..., g_{n+1}) = g_1\varphi(g_1, ..., g_{n+1}) + \sum_{i=0}^{n} \varphi(g_1, ..., g_{i-1}, g_i g_{i+1}, g_{i+2}, ..., g_{n+1})$$
$$+ (-1)^{n+1}\varphi(g_1, ..., g_n)$$

for $\varphi \in C^n(\Gamma; M)$. In particular, the differential $d : C^1(\Gamma; M) \to C^2(\Gamma; M)$ is given by $d\varphi(g_1, g_2) = g_1\varphi(g_2) - \varphi(g_1 g_2) + \varphi(g_1)$. The group cohomology is defined to be the cohomology of this cochain complex.

If we have two modules M and N, there is a cup product map $\smile: C^p(\Gamma; M) \otimes C^q(\Gamma; N) \to C^{p+q}(\Gamma; M \otimes N)$ defined at the chain level by the formula

$$(\psi \smile \varphi)(g_1, ..., g_{p+q}) = \psi(g_1, ..., g_p) \otimes g_1...g_p\varphi(g_{p+1}, ..., g_{p+q}).$$

Now a map between modules induces a map on cochains. In particular, if we have a module with a map $\mu : M \otimes M \to M$, we get a map $\mu^* : C^*(\Gamma; M \otimes M) \to C^*(\Gamma; M)$. The main situation we have in mind is when $M = \mathfrak{g}$ is a the Lie algebra of a Lie group G with module structure induced by the adjoint action of $\Gamma \subset G$. Then we have a bracket operation $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. By combining the bracket operation with the cup product, we get a bracket operation $[\ ,\ ] : C^*(\Gamma; L) \otimes C^*(\Gamma; L) \to C^*(\Gamma; L)$. The most important case of this bracket operation for deformation theory is the pairing on 1-cochains, given by $[\psi, \varphi](g_1, g_2) = [\psi(g_1), \mathrm{A}d(g_1)(\varphi(g_2))]$.

Now suppose we have a homomorphism $\rho : \Gamma \to G(\mathbb{R})$ from a discrete group Γ to an algebraic group defined over $\mathbb{R}$. The general idea in deformation theory is to try to extend this homomorphism to the group $G(\mathbb{R}[t]/(t^n))$ for each n. Assuming this can be done, we can pass to the inverse limit to get a homomorphism into $G(\mathbb{R}[[t]])$, where $\mathbb{R}[[t]]$ denotes formal power series in t. In this way, in the limit, we arrive at a description of the analytic germ of the representation variety at ρ. A deformation of ρ, then, is essentially the germ of an analytic path in the representation variety through ρ.

Now suppose we take a set theoretic map $\varphi : \Gamma \to \mathfrak{g} \otimes \mathcal{I}$, where $\mathcal{I}$ is the ideal $(t) \subset \mathbb{R}[t]/(t^{n+1})$, and define $\rho_{\mathfrak{g}} \otimes \mathcal{I})$, that is a 1-cochain with values in $\mathfrak{g} \otimes \mathcal{I}\varphi : \Gamma \to G(\mathbb{R}[t]/(t^{n+1})$ by $\rho_\varphi : g \mapsto \exp(\varphi(g))\rho(g)$. Then the condition that ρ_φ is a homomorphism is

$$\exp(\varphi(g))\rho(g)\exp(\varphi(h))\rho(h) = \exp(\varphi(gh))\rho(gh)$$

for all $g, h \in \Gamma$. Cancelling $\rho(h)$ and rearranging, we get

$$\exp(\varphi(g))\rho(g)\exp(\varphi(h))\rho(g)^{-1} = \exp(\varphi(g))\rho(g)\exp(\mathrm{A}d_\rho(g)(\varphi(h)) = \exp(\varphi(gh)).$$

Now if $n = 1$, we may simply expand the exponentials. After some cancellations, we get

$$\varphi(g) + \mathrm{Ad}_\rho(g)\varphi(h) - \varphi(gh) = d\varphi(g, h) = 0.$$

So we see that first order deformations are in one-to-one correspondence with 1-cocycles.

For $n = 2$, we proceed using the Campbell-Baker-Hausdorff expansion of the logarithm map applied to a product of exponentials,

$$\log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X, Y] + ...,$$

where the remaining terms involve higher order brackets. We then get

$$\varphi(g) + \mathrm{A}d_\rho(g)\varphi(h) - \varphi(gh) + \frac{1}{2}[\varphi(g), \mathrm{A}d_\rho(g)\varphi(h)] = 0,$$

where the higher order terms of the Campbell-Baker-Hausdorff vanish because they contain powers of t greater than 2. This equation may be rewritten to read

$$d\varphi + \frac{1}{2}[\varphi, \varphi] = 0.$$

This is the *Maurer-Cartan equation.* Writing $\varphi = t\varphi_1 + t^2\varphi_2$, and equating coefficients, we get

$$d\varphi_1 = 0$$

and

$$d\varphi_2 + \frac{1}{2}[\varphi_1, \varphi_1] = 0.$$

In particular, the cohomology class $-\frac{1}{2}[\varphi_1, \varphi_1] \in \mathrm{H}^2(\Gamma; \mathfrak{g})$ is zero. This class is called the *obstruction* to extending the first order deformation φ_1.

By a similar procedure, obstructions to further extensions may be defined. We will confine ourselves to the second order case here, however.

3.2 Groups that preserve a quadratic form and rigidity

For us, G will usually be $GL_{n+1}(\mathbb{R})$ for some n, but the discrete group Γ that we want to deform will often preserve a nondegenerate quadratic form Q. The most interesting case is when Q has signature $(n, 1)$, so that Γ acts on hyperbolic space. We can say more about the cohomology with coefficients in $\mathfrak{gl}_n(\mathbb{R})$ by breaking decomposing $\mathfrak{gl}_n(\mathbb{R})$ as a module over the orthogonal group $SO(Q)$. This yields two pieces, one of which is the Lie algebra of the orthogonal group $\mathfrak{so}(Q)$. In the cocompact hyperbolic case, the cohomology with coefficients in the adjoint representation on $so(Q)$ is the subject of classical rigidity theorems of Calabi and Weil.

In this section, we review this decomposition and the relevant rigidty results.

3.2.1 The decomposition over $SO(Q)$

As before, we will not distinguish between the quadratic form Q and its polarization, a symmetric, bilinear form. Such a bilinear form defines a canonical isomorphism $q : V \to V^*, q : v \mapsto Q(\cdot, v)$. We note that in an orthogonal reflection group, we have $\alpha_s = q(h_s)$ for each s. Q gives rise to the notion of the adjoint X^* of an element $X \in \mathrm{End}(V)$

We briefly review the adjoint construction in terms of the identifications $\mathfrak{gl}(V) = \mathrm{End}(V) = V^* \otimes V$. The adjoint is obtained by switching the tensor factors appearing in an expression of $X \in V^* \otimes V$ using the identification of V with V^*. More precisely, the adjoint is obtained by pulling back the usual transpose map $X^T : V^* \to V^*$, $X^* : \lambda \mapsto (v \mapsto \lambda(Xv))$ via q. If we have a pure tensor $X = Q(\cdot, x) \otimes y \in V^* \otimes V$ and $v \in V$, $X^\dagger v$ is obtained as follows: First, push v forward to V^* to obtain $Q(\cdot, v)$ and apply X^*: $X^*(Q(\cdot, v) : u \mapsto Q(Xu, v)$, where $Q(Xu, v) = Q(Q(u, x)y, v) = Q(y, v)Q(u, x)$, so $X^T(q(v)) = Q(y, v)Q(\cdot, x)$. Now applying q^{-1}, we get $X^*v = Q(v, y)x$, so $X^* = Q(\cdot, y) \otimes x$. The adjoint operation is linear, so the general case can be treated by writing X as a sum of pure tensors and applying the procedure above term by term.

It is clear from the preceding discussion that the adjoint defines an involution on $\mathrm{End}(V)$, which in turn gives rise to positive and negative eigenspaces. We will call elements of the positive eigenspace *selfadjoint* and elements of the negative eigenspace *antiselfadjoint*. The isometry group of Q, $\mathrm{SO}(Q)$, has Lie algebra $\mathfrak{so}(Q)$ identified with the space of antiselfadjoint elements of $\mathrm{End}(V)$. $\mathcal{S}(Q)$ will denote the space of selfadjoint elements. This space, too, is preserved by the action of $\mathrm{S}O(Q)$, so we have $V^* \otimes V = \mathfrak{so}(Q) \oplus \mathcal{S}(Q)$ as a Γ-module. Of course, the cohomology $\mathrm{H}^*(\Gamma; \mathfrak{gl}(V))$ inherits this decomposition.

3.2.2 Calabi-Weil infinitesimal rigidity

Suppose Γ is a uniform lattice acting on hyperbolic n-space, that is Γ acts discretely with compact quotient. It is natural to ask whether we can find deformations of Γ as a subgroup of $\mathrm{S}O(n, 1)$. If $n > 2$, it turns out the answer is no:

Theorem 3.2.1. *(Calabi [7], Weil[20]) With* Γ *as above,* $\mathrm{H}^1(\Gamma; \mathfrak{so}(n, 1)) = 0$.

The original result of Calabi appears in more geometric language. The group cohomology formulation first appeared in Weil's papers, which generalized the result to other symmetric spaces.

The utility of this result for our purposes is that when we have a hyperbolic reflection group, the vanishing of the first cohomology will be enough to completely determine the cohomology.

CHAPTER 4
COHOMOLOGY OF COXETER GROUPS

There is a standard way to compute cohomology of Coxeter groups, namely via equivariant cohomology of the Davis complex [10], [12]. By making explicit local calculations using a well chosen cellulation or some special feature of our coefficient module, we may gain some insight into the structure of the cohomology or even calculate it outright.

As a matter of notation, we will switch freely between modules over a group Γ and coefficient systems on spaces acted upon by Γ. To clarify what is meant here, given any Γ-module M, there is an associated coefficient system on $\mathrm{E}\Gamma$, a space on which Γ acts freely – this space is unique up to equivariant homotopy. Then, given any other space X with an action of Γ, there is a map, again uniquely determied up to equivariant homotopy, $X \to \mathrm{E}\Gamma$. By the coefficient system M on X, then, we mean the coefficient system obtained by pulling back from $\mathrm{E}\Gamma$ along this map.

4.1 Equivariant cohomology

The usual way to define equivariant cohomology is via the Borel construction. We start with a space X with an action of a group Γ. If the action is free, then equivariant cohomology reduces to ordinary cohomology of the quotient X/Γ. If not, then we replace X with $X \times \mathrm{E}\Gamma$, where $\mathrm{E}\Gamma$ is a space on which Γ acts freely and define the equivariant cohomology $\mathrm{H}^*_\Gamma(X; M)$ to be $\mathrm{H}^*((X \times \mathrm{E}\Gamma)/\Gamma; M)$. From this definition, it is easy to see that if X is acyclic, e.g. if X is contractible, then $\mathrm{H}^*_\Gamma(X; M) \cong \mathrm{H}^*(\Gamma; M)$.

We will look at equivariant cohomology slightly differently. Naively, the equivariant cohomology $\mathrm{H}^*_\Gamma(X; M)$ of a space X with an action of a group Γ is what one should get by taking the invariants of the action of G on the singular chain complex $C^*(X; M)$ and computing cohomology. This is not at all how it works in general, but if X is a CW-complex, G acts cellularly with finite cell stabilizers, and M is a vector space over a field of characteristic zero, this description is in fact correct. The Davis complex allows us to exploit this situation and since this will be the only case we will deal with, we will simply make the above statement precise and leave it at that. We follow the exposition given in Brown [6] so we may quickly arrive at a nice description of the differential.

Proposition 4.1.1. *If X is a CW-complex with a cellular action of a group Γ with finite cell stabilizers and M a coefficient module defined over a field k of characteristic zero, then $\mathrm{H}^*_G(X; M)$ is computed by a cochain complex $C^*(X; M)$ with*

$$C^p_\Gamma(X; M) = \bigoplus_{c \in X^{(p)}/\Gamma} (M \otimes \mathcal{O}_c)^{\Gamma_c},$$

where $\mathcal{O}_c$ is the one-dimensional representation of Γ_c by the action on $\mathrm{H}_{|c|}(c, \partial c; k)$.

Proof. Given a CW-complex X with a cellular action of Γ, there is a spectral sequence with E_1-term given by

$$E_1^{p,q} = \bigoplus_{c \in X^{(p)}/\Gamma} \mathrm{H}^q(\Gamma_c; M \otimes_k \mathcal{O}_c),$$

where $\mathcal{O}_c$ is the orientation character of Γ_c given by the action on $\mathrm{H}_{|c|}(c, \partial c; \mathbb{Z})$, which converges to $\mathrm{H}^*_\Gamma(X; M)$. See [6], Chapter VII for details.

Now, since M is a vector space over a field of characteristic zero, for each cell c, the cohomology $\mathrm{H}^q(\Gamma_c; M \otimes_k \mathcal{O}_c)$ vanishes for $q > 0$. Therefore, the spectral sequence is nonzero only along the bottom row, hence the only nontrivial differential is d_1 and we have a complex as claimed. □

4.1.1 The differential

Brown gives a description of the differential d_1 for the first page of the spectral sequence used in the proof of Proposition 4.1.1. The ingredients of the differential are the usual differential for cellular cochains on X, the transfer maps between spaces of invariants for stabilizers of incident cells, and a choice of orbit representatives for the cells of X together with maps translating a given cell to its chosen representative. Again, the cases we will deal with lead to simplifications.

First, we will be dealing with complexes whose cells are polytopes and whose attaching maps have local degree ± 1. The cellular differential is then determined entirely by incidence and choice orientation. Given two cells $a \in X^{(p)}$ and $b \in X^{(p-1)}$ and a choice o of orientation for the cells of X, we write $o(a, b)$ for this sign.

The complexes we will work with also come with a natural choice of fundamental domain for the action of Γ. This leads to a natural choice of cell orbit representatives, namely the element of each orbit that intersects nontrivially with the fundamental domain. This frees us from the complication of translating cells not in our set of orbit representatives back into that set and the associated twisting of coefficients, except in the case that we have some cell c that is incident to more than one cell in a given orbit. This situation is not so bad though, because those incident cells must be related by elements of the cell stabilizer Γ_c. This is where the transfer comes into play.

With regard to cell stabilizers and transfer maps, there are two situations of interest to us. Suppose we have two cells $a \in X^{(p)}$ and $b \in X^{(p-1)}$ with $b \subset \partial a$. Let $\Gamma_{a,b} = \Gamma_a \cap \Gamma_b$. In general, there are of course various possible intersections, but in the cases of interest to us, we will have either $\Gamma_{a,b} = \Gamma_a$ or $\Gamma_{a,b} = \Gamma_b$. When $\Gamma_{a,b} = \Gamma_a$, a is incident with only one cell in the orbit of b and we are mercifully spared the complication of a transfer map. If on the other hand, we have $\Gamma_{a,b} = \Gamma_b$, there is a translate of b in ∂a for each coset of Γ_b in Γ_a and there is a transfer map in the differential. We summarize these situations in the following propositions:

Proposition 4.1.2. *With X, Γ, and M as in Proposition 4.1.1, suppose that for each $a \in X^{(p)}$ and $b \in X^{(p-1)}$ with b incident to a, $\Gamma_{a,b} = \Gamma_a$. Then the component of the differential in the complex from 4.1.1 mapping M^{Γ_b} to M^{Γ_a} is given by*

$$m \mapsto o(a,b) i_{a,b}(m) \tag{4.1}$$

where $i_{a,b}$ is the inclusion $M^{\Gamma_b} \hookrightarrow M^{\Gamma_a}$.

Proposition 4.1.3. *With X, Γ, and M as above, suppose that for each $a \in X^{(p)}$ and $b \in X^{(p-1)}$ with b incident to a, $\Gamma_{a,b} = \Gamma_b$. Then component of the differential in the complex of Proposition 4.1.1 mapping M^{Γ_b} to M^{Γ_a} is*

$$m \mapsto o(a,b) t_{\Gamma_b}^{\Gamma_a}(m), \tag{4.2}$$

where $t_{\Gamma_b}^{\Gamma_a}$ denotes the transfer map from invariants of the action of Γ_b to invariants of the action of Γ_a.

Both propositions are immediate consequences of the discussion in Brown, Chapter VII, section 8.

4.2 The Davis Complex

With this proposition in hand, we now construct a suitable complex for a Coxeter group to act on. This is the Davis complex. Let us review the construction.

Let (W, S) be a Coxeter system and let $\mathcal{S}$ be the simplex with facets in one to one correspondence with the generating set S. Then the vertices of the barycentric subdivision of $\mathcal{S}$ are in one to one correspondence with proper subsets $T \subsetneq S$. Let $K(W, S)$ be the full subcomplex of the barycentric subdivision spanned by the vertices corresponding to spherical subsets of S – note the intersection of $K(W, S)$ with $\partial\mathcal{S}$ is the barycentric subdivision of the nerve $L(W, S)$. To each spherical subset $T \subset S$ we associate a subcomplex K_T of $K(W, S)$, which is the full subcomplex spanned by the vertices that correspond to subsets $T' \subset S$ with $T \subset T'$. The *Davis complex* is then

$$D(W, S) := W \times K(W, S)/ \sim,$$

where $(w, x) \sim (w', y)$ if $x = y \in K_T$ and $w\langle T\rangle = w'\langle T\rangle$.

Proposition 4.2.1. *(Davis [11]) The Davis complex is complex is contractible and the obvious action of W on $D(W, S)$ has finite stabilizers.*

4.3 Coxeter cells

The Davis complex comes to us as a simplicial complex. We want a somewhat coarser cell structure that will interact nicely with the coefficient modules we consider. For this we will use so-called Coxeter cells. With notation as in the previous subsection, we first associate to each spherical subset $T \subset S$ a cube $c_T \subset K(W, S)$ by taking the full subcomplex spanned by the vertices that correspond to subsets of T (including the empty set). The Coxeter cell C_T associated to T is the union

$$\bigcup_{w \in \langle T \rangle} wc_t \subset D(W, S).$$

Clearly, for any $T' \subset T$, $C_{T'} \subset \partial C_T$. In fact, the translates $wC_{T'}$ where w and T' range over all $w \in \langle T \rangle$ and all strict subsets $T' \subset T$ gives a cell structure on ∂C_T.

Proposition 4.3.1. *The translates of the Coxeter cells C_T for the various spherical $T \subset S$ give an equivariant cell structure for the action of W on $D(W, S)$.*

Proof. See [11], section 7.3. □

The cell stabilizer of C_T is $\langle T \rangle$. Each $s \in T$ acts by an orientation reversing involution on C_T, so we have

Proposition 4.3.2. *The representation of $\langle T \rangle$ on $\mathrm{H}^{|T|}(C_T, \partial C_T)$ is the sign representation $\sigma_{\langle T \rangle}$.*

There is another way to construct Coxeter cells. If we have a realization of W as a discrete reflection group generated by reflections in the faces of a convex polyhedral cone P, we may take a point x in the interior of P and look at its orbit under the action of a finite special subgroup $\langle T \rangle$, $T \subset S$. The convex hull of such an orbit is a simple polytope of dimension $|T|$. The faces of this polytope are in one-to-one correspondence with cosets of finite special subgroups $\langle T' \rangle$ with $T' \subset T$. This polytope is combinatorially equivalent to C_T.

By theorem 2.1.1 the translates of P under the action of W is a convex cone K. Using this realization of the Coxeter cells, we can construct a natural embedding of the Davis complex in K from an orbit of a point x as above. For each spherical $T \subset S$, we embed a Coxeter cell as above by taking the orbit under $\langle T \rangle$ of the point

x. The translates of these embedded cells under the linear action of W determines an equivariant embedding of the Davis complex.

4.4 A dual perspective

From the preceding discussion, we have a way to embed the Davis complex in a convex cone $C = \bigcup_{w \in W} wP$. Let $i : D(W,S) \hookrightarrow C$ be the inclusion and $\dim C = n$. Then we may take a tubular neighborhood $\hat{\eta}$ of $i(K(W,S))$ and let $\eta = \overline{\hat{\eta}} \cap P$. Then $\bigcup_{w \in W} w\eta$ is a manifold possibly with boundary (it may be that the intersection of P with the union of Coxeter cells is just P) equivariantly homotopy equivalent to $D(W,S)$. We call this manifold $D_K(W,S)$. We can give $D_K(W,S)$ a cell structure in which the interior of η is a cell and $\eta \cap F$ is a cell for each face F of P. Such a cell structure induces an equivariant cell structure on $D_K(W,S)$.

The various cell structures on $\partial D_K(W,S)$ are possible according to the above description (again, unless $\eta = P$). The difficulties associated with these boundary cells can be avoided by considering the Poincaré dual picture. One then gets the usual description of the cup product as dual to the intersection pairing. Passing to cellular chains and then to invariants, we obtain a description of the cohomology identical to the one given above, where the sign representation arises from the twist by the orientation bundle introduced by Lefschetz duality.

This perspective is mentioned as an aside, as it was the original approach taken to find the results below and because it provides some useful geometric perspective on the cup product.

4.5 The adjoint representation and invariants

Deformations of representations of discrete reflection groups correspond to elements of $H^*(\Gamma; \mathfrak{gl}(V))$, where Γ is a discrete reflection group acting on V. It is convenient, in view of our description of the dual action of Γ on V^*, to identify $\mathfrak{gl}(V)$ with $\mathrm{End}(V) = V^* \otimes V$. We will also be interested in deforming orthogonal representations through other orthogonal representations, that is representations that preserve a quadratic form Q. Such representations can be viewed as representations into the Lie group $PO(Q)$, with Lie algebra $\mathfrak{so}(Q)$. The deformation theory is then controlled by $H^*(\Gamma; \mathfrak{so}(Q))$.

First we will find the invariants of the isotropy groups acting on $\mathfrak{gl}(V)$, then intersect with $\mathfrak{so}(Q)$ to get a handle on the orthogonal situation.

4.6 Invariants

In the previous section, we found that the relevant representations have the form $V^* \otimes V \otimes \sigma_{\Gamma_c}$, where Γ_c is a finite Coxeter group and V is a reflection representation. The invariants of such a representation are just the intersection of the (-1)-eigenspaces of the generating reflections of Γ_c.

It turns out there is a fairly simple description of the invariants:

Theorem 4.6.1. *Let Γ be a finite reflection group with generating set S acting on a real vector space V of dimension $n+1$ and let $U = V^* \otimes V \otimes \sigma_\Gamma$. Then*

- *if S consists of a single generator s, then $U^\Gamma = \langle \alpha_s \rangle \otimes \ker \alpha_s + \ker h_s \otimes h_s$. In particular, $\dim U^\Gamma = 2n$.*
- *if Γ has two generators so that $S = \{s, t\}$, there are two cases:*

1. *if* $m_{s,t} = 2$*, we have* $U^\Gamma = \langle \alpha_s \otimes h_t, \alpha_t \otimes h_s \rangle$ *so* $\dim U^\Gamma = 2$

2. *if* $m_{s,t} > 2$*, the space of invariants is one dimensional spanned by the element*

$$X_{s,t} = \alpha_s(h_t)\alpha_t \otimes h_s - \alpha_t(h_s)\alpha_s \otimes h_t.$$

- *if* $|S| > 2$*, then* $U^\Gamma = 0$.

Remark: In the statement of the theorem we have identified U with $V^* \otimes V$ in the obvious way. We will make further identifications of this kind without comment.

Proof. First, it is clear that the eigenspaces of a single reflection s acting on U are given by taking tensor products of eigenspaces of the action of s on V and V^*. Let V_+ (resp. V_+^*) denote the +1-eigenspace of s acting on V (resp. V^*) and define V_- and V_-^* to be the corresponding (-1)-eigenspaces. Then we have that the +1-eigenspace for the action of s on U is $U_+ = V_+^* \otimes V_+ + V_-^* \otimes V_-$ and the (-1)-eigenspace is $U_- = V_-^* \otimes V_+ + V_+^* \otimes V_-$. Of course, $V_+ = \ker \alpha_s$, $V_+^* = \ker h_s$, $V_- = \langle h_s \rangle$, and $V_-^* = \langle \alpha_s \rangle$, so the result follows.

For the two generator cases, we carry out a similar analysis based on the decomposition of the tensor product. If $st = ts$, we have that V decomposes as a Γ-module as $V = \langle h_s \rangle + \langle h_t \rangle + \ker \alpha_s \cap \ker \alpha_t$, where the action on the intersection of the kernels is trivial and t acts trivially on $\langle h_s \rangle$ while s acts trivially on $\langle h_t \rangle$. The decomposition for V^* is entirely analogous, given by $V^* = \langle \alpha_s \rangle + \langle \alpha_t \rangle + \ker h_s \cap \ker h_t$. We then get the following decomposition for the tensor product

$$\begin{aligned} U = & \langle \alpha_s \otimes h_s \rangle + \langle \alpha_t \otimes h_s \rangle \\ & + \langle \alpha_s \rangle \otimes (\ker \alpha_t \cap \ker \alpha_s) + \langle \alpha_t \rangle \otimes (\ker \alpha_t \cap \ker \alpha_s) \\ & + (\ker h_s \cap \ker h_t) \otimes h_s + (\ker h_s \cap \ker h_t) \otimes h_t \\ & + (\ker h_s \cap ker h_t) \otimes \ker(\alpha_s \cap \ker \alpha_t). \end{aligned}$$

Of these, only the first two terms carry the sign representation $\sigma_{\langle s,t \rangle}$, since on each other term, at least one of the generators acts trivially.

The non-right angled case is similar but more complicated. Now V decomposes as $V_2 + \ker \alpha_s \cap \ker \alpha_t$ and $V^* = V_2^* \ker \alpha_s \cap \ker \alpha_t$, where $V_2 = \langle h_s, h_t \rangle, V_2^* = \langle \alpha_s, \alpha_t \rangle$ are two dimensional irreducible submodules of V and V^* respectively. Taking the tensor product, we get

$$\begin{aligned} U = & V_2^* \otimes V_2 \\ & + V_2^* \otimes \ker \alpha_s \cap \ker \alpha_t + \ker h_s \cap \ker h_t \otimes V_2 \\ & + (\ker h_s \cap \ker h_t) \otimes (\ker \alpha_s \cap \ker \alpha_t). \end{aligned}$$

Clearly, the only part of this decomposition in which a sign representation might occur is in the first term, since the remaining parts either consist of copies of V_2 or V_2^* or are acted on trivially. So the question comes down to finding the common (-1)-eigenspace for the actions of s and t on $\langle \alpha_s, \alpha_t \rangle \otimes \langle h_s, h_t \rangle$. From one generator case, we know the (-1)-eigenspace of s is $\alpha_s \otimes \ker \alpha_s + \ker h_s \otimes h_s$, which intersects $\langle \alpha_s, \alpha_t \rangle \otimes \langle h_s, h_t \rangle$ in a two dimensional space. To see the common (-1)-eigenspace is not all of this two dimensional space, we can take an element, say $X = \alpha_s \otimes (h_t - \frac{\alpha_s(h_t)}{2} h_s)$ (note that $h_t - \frac{1}{2}\alpha_s(h_t)h_s \in \ker \alpha_s$) and compute tX – it is easy to see the result is not $-X$. On the other hand,

$$tX_{s,t} = \alpha_s(h_t)(-\alpha_t) \otimes (h_s - \alpha_t(h_s)h_t) - \alpha_t(h_s)(\alpha_s - \alpha_s(h_t)\alpha_t) \otimes (-h_t)$$
$$= -X_{s,t}.$$

The calculation for s is similar. The result is that the common (-1)-eigenspace is one dimensional, spanned by $X_{s,t}$ as claimed.

Finally, for three or more generators, from the classification of finite Coxeter groups (see Chapter 2), it must be that some two of the generators commute. We may then pick some three generators, say r, s, t with $st = ts$. Then $U^{\langle s,t \rangle} = \alpha_s \otimes h_t + \alpha_t \otimes h_s$. Now, $r\alpha_s \otimes h_t = (\alpha_s - \alpha_s(h_r)\alpha_r) \otimes (h_t - \alpha_r(h_t)h_r)$ and $r\alpha_t \otimes h_s = (\alpha_t - \alpha_t(h_r)\alpha_r) \otimes (h_s - \alpha_r(h_s)h_r)$. There are two cases to consider, either r commutes with both s and t, in which case $\alpha_r(h_s) = \alpha_r(h_t) = 0$ and $\alpha_s(h_r) = \alpha_t(h_r) = 0$, or r does not commute with one or more of the other two, say s. In the first case, it is easy to see that r leaves $\alpha_s \otimes h_t$ and $\alpha_t \otimes h_s$ invariant, so its (-1)-eigenspace intersects trivially with the common (-1)-eigenspace of s and t. In the second case, $\alpha_r(h_s)$ and $\alpha_s(h_r)$ are nonzero, so from the above calculations, $\alpha_s \otimes h_t$ and $\alpha_t \otimes h_s$ are not eigenvectors of r at all. Therefore $U^\Gamma = 0$.

□

4.7 Transfer maps

To understand the differentials, we need a description of the transfer maps between the spaces of invariants appearing in the cochain spaces. The standard description of transfer between spaces of invariants of represenation spaces of finite groups involves a sum over coset representatives. For $H \subset G$, V a representation of G, the transfer map $t_H^G : V^H \to V^G$ is given by

$$t_H^G : v \mapsto \sum_{xH \in G/H} xv.$$

It is convenient to consider the transfer as a restriction of an averaging operator, i.e. the operator $A_G : v \mapsto \sum_{g \in G} gv$. If we have a subgroup H as above, and $v \in V^H$, then we have $A_G(v) = \sum_{g \in G} gv = |H| \sum_{gH \in G/H} gv = |H| t_H^G(v)$.

Lemma 4.7.1. *If G is a finite group acting irreducibly on a vector space V, then the averaging operator A_G on V is either multiplication by $|G|$ or the zero map according to whether the action on V trivial or not.*

Proof. The image of A_G is invariant. Indeed, for $v \in V, g \in G$, $gA_G(v) = \Sigma_{h \in G} ghv = \Sigma_{h \in G} hv$. Since the action was assumed irreducible, it follows the image of A_G is either zero or all of V, in which case V is the 1-dimensional trivial module. It is easy to see in the case of a trivial action that A_G acts by multiplication as claimed. □

It follows that in the nonirreducible case, the kernel of A_G is exactly the maximal nontrival submodule and that the trivial part is acted upon by scalar multiplication. Returning to the situation above with $H \subset G$ and G acting on V, we have that the transfer map is simply the projection with image consisting of the invariants of G and kernel equal to the span of nontrivial G-submodules, followed by multiplication by $|H|$.

For the cohomological set up we are considering, in view of Theorem 4.6.1, we need only consider transfer from invariants of one generator groups to two generator groups and from the group with one element to the group with two elements generated by a reflection. This latter case is a simple calculation.

Proposition 4.7.1. *Let X be a pure element of $V^* \otimes V$, i.e. $X = \beta \otimes u$ for some β and u and let s be a reflection on V. Then $t^{\langle s \rangle}_{\{\mathrm{id}\}}(X) = \beta(h_s)\alpha_s \otimes u + \alpha_s(u)\beta \otimes h_s - \beta(h_s)\alpha_s(u)\alpha_s \otimes h_s$.*

Proof.

$$\begin{aligned} t^{\langle s \rangle}_{\{\mathrm{id}\}}(X) &= X - sX = \beta \otimes u - s\beta \otimes su \\ &= \beta \otimes u - (\beta - \beta(h_s)\alpha_s) \otimes (u - \alpha_s(u)h_s) \\ &= \beta(h_s)\alpha_s \otimes u + \alpha_s(u)\beta \otimes h_s - \beta(h_s)\alpha_s(u)\alpha_s \otimes h_s. \end{aligned}$$

□

We note that

$$\beta(h_s)\alpha_s \otimes u + \alpha_s(u)\beta \otimes h_s - \beta(h_s)\alpha_s(u)\alpha_s \otimes h_s = \\ \beta(h_s)\alpha_s \otimes (u - \frac{1}{2}\alpha_s(u)h_s) + \alpha_s(u)(\beta - \frac{1}{2}\beta(h_s)\alpha_s) \otimes h_s,$$

so X lands in the right subspace.

With transfer from a one generator group to a two generator group, there are two cases to consider: Right angled and not right angled. The right angled case is easily handled by the coset sum definition:

Proposition 4.7.2. *Let $X = \beta \otimes h_s \otimes + \alpha_s \otimes u$ be an element of $(V^* \otimes V \otimes \sigma_{\langle s \rangle})^{\langle s \rangle}$. Then the transfer $t^{\langle s,t \rangle}_{\langle s \rangle} : (V^* \otimes V \otimes \sigma_{\langle s \rangle})^{\langle s \rangle} \to (V^* \otimes V \otimes \sigma_{\langle s,t \rangle})^{\langle s,t \rangle}$ maps*

$$X \mapsto \beta(h_t)\alpha_t \otimes h_s + \alpha_t(u)\alpha_s \otimes h_t.$$

Proof. We pick coset representatives 1 and t. Let $X = \beta \otimes h_s + \alpha_s \otimes u \in (V^* \otimes V \otimes \sigma_{\langle s \rangle})^{\langle s \rangle}$ so that $\beta(h_s) = \alpha_s(u) = 0$. Then $t_{\langle H \rangle}^{\langle G \rangle}(X) = X - tX = (\beta \otimes h_s + \alpha_s \otimes u) - (\beta - \beta(h_t)\alpha_t) \otimes h_s + \alpha_s \otimes (u - \alpha_t(u)h_t) = \beta(h_t)\alpha_t \otimes h_s + \alpha_t(u)\alpha_s \otimes h_t$, where the second equality uses the fact that t fixes α_s and h_s. □

In the non-right angled case, we use the projection description. In this case, we know the averaging operator $A_{\langle s,t \rangle}$ and hence the transfer map $t_{\langle s \rangle}^{\langle s,t \rangle}$ has image spanned by $X_{s,t}$, so $A_{\langle s,t \rangle} = \Lambda_{s,t} \otimes X_{s,t}$ for some $\Lambda_{s,t} \in (V^* \otimes V)^*$ with kernel equal to the maximal nontrivial submodule for the action of $\langle s, t \rangle$. Any such functional can be realized by a map of the form $X \mapsto \mathrm{Tr}(XY)$ for some element Y. This allows us to write down an explicit formula for $t_{\langle s \rangle}^{\langle s,t \rangle}$.

Proposition 4.7.3. *For $m_{s,t} > 2$, the transfer $t_{\langle s \rangle}^{\langle s,t \rangle} : (V^* \otimes V \otimes \sigma_{\langle s \rangle})^{\langle s \rangle} \to (V^* \otimes V \otimes \sigma_{\langle s \rangle})^{\langle s,t \rangle}$ is given by*

$$t_{\langle s \rangle}^{\langle s,t \rangle} : \beta \otimes h_s + \alpha_s \otimes u \mapsto \frac{\alpha_s(h_t)\beta(h_t) - \alpha_t(h_s)\alpha_t(u)}{\alpha_s(h_t)\alpha_t(h_s)(\alpha_s(h_t)\alpha_t(h_s) - 4)} X_{s,t}.$$

Proof. We begin by identifying the functional $\Lambda_{s,t}$ mentioned above in connection with the averaging operator. The obvious first attempt would be $\widetilde{\Lambda}_{s,t} = (X \mapsto \mathrm{Tr}(X \cdot X_{s,t}))$. We have $w\widetilde{\Lambda}_{s,t}(X) = \mathrm{Tr}((w^{-1}Xw) \cdot X_{s,t}) = \mathrm{Tr}(X \cdot (wX_{s,t}w^{-1})) = \sigma_{s,t}(w)\,\mathrm{Tr}(X \cdot X_{s,t})$, so $\Lambda_{s,t}$ is invariant under the twisted adjoint action, so its kernel is a submodule. On the other hand,

$$\begin{aligned}\mathrm{Tr}\, X_{s,t}^2 &= \mathrm{Tr}(\alpha_s(h_t)^2\alpha_t(h_s)\alpha_t \otimes h_s - \alpha_s(h_t)\alpha_t(h_s)\alpha_t(h_t)\alpha_s \otimes h_s \\ &\quad - \alpha_s(h_t)\alpha_t(h_s)\alpha_s(h_s) + \alpha_t(h_s)^2\alpha_s(h_t)\alpha_s \otimes h_t) \\ &= 2\alpha_s(h_t)\alpha_t(h_s)(\alpha_s(h_t)\alpha_t(h_s) - 4) \neq 0,\end{aligned}$$

so $\Lambda_{s,t}$ is nonvanishing on $\langle X_{s,t} \rangle$.

It follows that the kernel of $\widetilde{\Lambda}_{s,t}$ is the span of nontrivial submodules for the twisted adjoint action.

We can now recover the desired functional: $\Lambda_{s,t} = \frac{m_{s,t}}{\alpha_s(h_t)\alpha_t(h_s)(\alpha_s(h_t)\alpha_t(h_s)-4)}\widetilde{\Lambda}_{s,t}$, where we have renormalized so $\Lambda(X_{s,t}) = 2m_{s,t} = |\Gamma|$. We can then write down a nice formula for the transfer. Take $X \in (V^* \otimes V \otimes \sigma_s)^{\langle s\rangle}$. Then X has the form $\beta \otimes h_s + \alpha_s \otimes u$ with $\beta(h_s) = \alpha_s(u) = 0$, so

$$\begin{aligned}
\widetilde{\Lambda}_{s,t}(X) &= \mathrm{Tr}(X \cdot X_{s,t}) = \mathrm{Tr}((\beta \otimes h_s + \alpha_s \otimes u) \cdot X_{s,t}) \\
&= \mathrm{Tr}(-\alpha_t(h_s)\beta(h_t)\alpha_s \otimes h_s + \alpha_s(h_t)\alpha_s(h_s)\alpha_t \otimes u - \alpha_t(h_t)\alpha_s(h_t)\alpha_s \otimes u) \\
&= 2(\alpha_s(h_t)\alpha_t(u) - \alpha_t(h_s)\beta(h_t)).
\end{aligned}$$

Since $A_{\langle s,t\rangle} : X \mapsto \Lambda(X)X_{s,t}$, we have $t^{\langle s,t\rangle}_{\langle s\rangle} : X \mapsto \frac{1}{m_{s,t}}\Lambda(X)X_{s,t}$ and the result follows.

□

4.8 The orthogonal case

Here we record the results for the modules $\mathfrak{so}(Q)$ and $\mathcal{S}(Q)$ (= selfadjoint elements) in the orthogonal case:

Theorem 4.8.1. *Let Γ be a finite, orthogonal discrete reflection group acting on V preserving a nondegenerate, symmetric bilinear form Q. Then*

- *if Γ is generated by a single reflection, $\Gamma = \langle s\rangle$,*

$$(\mathfrak{so}(Q) \otimes \sigma_\Gamma)^\Gamma = \{Q(\cdot, h_s) \otimes u - Q(\cdot, u) \otimes h_s | u \in V, Q(u, h_s) = 0\}$$

  and similarly,

$$(\mathcal{S}(Q) \otimes \sigma_\Gamma)^\Gamma = \{Q(\cdot, h_s) \otimes u + Q(\cdot, u) \otimes h_s | u \in V, Q(u, h_s) = 0\}.$$

- *if* Γ *has two generators,* $\Gamma = \langle s, t \rangle$,
  - *if* $m_{s,t} = 2$, *we have*

$$(\mathfrak{so}(Q) \otimes \sigma_\Gamma)^\Gamma = \langle Q(\cdot, h_s) \otimes h_t - Q(\cdot, h_t) \otimes h_s \rangle$$

    and

$$(\mathcal{S}(Q) \otimes \sigma_\Gamma)^\Gamma = \langle Q(\cdot, h_s) \otimes h_t + Q(\cdot, h_t) \otimes h_s \rangle.$$

  - *if* $m_{s,t} > 2$, *we have*

$$(\mathfrak{so}(Q) \otimes \sigma_\Gamma)^\Gamma = \langle Q(h_s, h_t)(Q(\cdot, h_s) \otimes h_t - Q(\cdot, h_t) \otimes h_s) \rangle$$

    and

$$(\mathcal{S}(Q) \otimes \sigma_\Gamma)^\Gamma = 0.$$

- *if* Γ *has more than two generators,*

$$(\mathfrak{so}(Q) \otimes \sigma_\Gamma)^\Gamma = (\mathcal{S}(Q) \otimes \sigma_\Gamma)^\Gamma = 0.$$

Proof. These are immediate from Theorem 4.6.1, in view of the fact that for an orthogonal reflection s, $\alpha_s = Q(\cdot, h_s)$ and the definition of selfadjointness. □

CHAPTER 5

THE COMPLEXES $C^*(\Gamma, M)$

We are now ready write down complexes computing the various flavors of group cohomology of interest. As usual, Γ will be a discrete reflection group with generating reflections S acting on V and, if it is orthogonal, preserving a form Q. M is a coefficient module equal to either $V^* \otimes V$ or, if Γ is orthogonal, one of $so(Q)$ or $\mathcal{S}(Q)$. For the purpose of choosing orientations, we choose an ordering, $<$, of S – obviously, a different choice of order makes no essential difference.

With these notations, we define $C^*(\Gamma; M)$ to have possibly nonzero cochains in dimensions 0, 1, and 2 given by:

$$C^0(\Gamma; M) = M,$$

$$C^1(\Gamma; M) = \bigoplus_{s \in S} (M \otimes \sigma_{\langle s \rangle})^{\langle s \rangle},$$

$$C^2(\Gamma; M) = \bigoplus_{s,t \in S, s<t, m_{s,t}<\infty} (M \otimes \sigma_{\langle s,t \rangle})^{\langle s,t \rangle}$$

and differentials

$$d^0 : X \mapsto \oplus_{s \in S} t^{\langle s \rangle}_{\langle \mathrm{id} \rangle}(X)$$

and

$$d^1 : \bigoplus_{s \in S} X_s \mapsto \bigoplus_{s,t \in S, s<t, m_{s,t}<\infty} t^{\langle s,t \rangle}_{\langle t \rangle}(X_t) - t^{\langle s,t \rangle}_{\langle s \rangle}(X_s).$$

Theorem 5.0.2. *With notation as above the cohomology of* $C^*(\Gamma; M)$ *is* $\mathrm{H}^*(\Gamma; M)$.

Proof. The theorem is immediate from the propositions of the previous section. We should say a word about d^1, however. We orient the two skeleton of the Davis complex $D(W, S)$ so that the orientation of a 2-cell $C_{\{s,t\}}$, $s < t$ agrees with the orientation of $C_{\{t\}}$ and hence disagrees with the orientation of $C_{\{s\}}$. This accounts for the minus sign in the formula for d^1. □

Corollary 5.0.1. *For* M *as above,* $\mathrm{H}^*(W; M) = 0$ *for* $* > 2$.

5.1 Euler characterisitics

In view of the dimension counts of the previous section, we can calculate Euler characteristics. We summarize the results.

Proposition 5.1.1. *If* $\dim V = n + 1$, $f = |S|$, e_2 *is the number of pairs* $s, t \in S$ *with* $st = ts$, *and* e_+ *is the number of edges in the Coxeter diagram with finite weight for* Γ *the Euler characteristics* $\chi(C^*(\Gamma; M))$ *are as follows:*

- *for* $M = V^* \otimes V$, *we have* $\chi(C^*(\Gamma; M)) = 2e_2 + e_+ - 2fn + (n + 1)^2$
- *for* $M = \mathfrak{so}(Q)$, *we have* $\chi(C^*(\Gamma; M)) = e_2 + e_+ - fn + \frac{n(n+1)}{2}$
- *for* $M = \mathcal{S}(W)$, *we have* $\chi(C^*(\Gamma; M)) = e_2 - fn + \frac{n(n+3)}{2}$.

We note that these Euler characteristics give the naive expected dimensions for the corresponding (subsets of) the representation varieties. That is, these are the same numbers we get by counting equations and degrees of freedom in the Vinberg equations.

If Γ is the holonomy of a hyperbolic structure on a compact Coxeter orbifold, $\chi(C^*(\Gamma; \mathfrak{so}(Q)))$ has an interesting combinatorial interpretation. If (W, S) is the associated Coxeter system with nerve $L = L(W, S)$, f_i denotes the i^{th} face number of L, $\chi(C^*(\Gamma; \mathfrak{so}(Q))) = f_0 - nf_1 + \binom{n+1}{2} = g_2(\mathcal{N})$, an invariant of simplicial spheres that appears in the combinatorics literature. It turns out that for simplicial 2-spheres, hence for compact 3-dimensional hyperbolic Coxeter orbifolds, this number vanishes by an Euler characteristic calculation making use of the fact that all faces are triangles (or dually, all vertices have valence three – that is, the polytope is simple). In higher dimensions, there is a classification of simplicial spheres with $g_2 = 0$ or 1. It turns out that g_2 is stable under "stacking," that is taking a connected sum along a top dimensional face with the boundary of a simplex. This has the effect of truncating a vertex in the dual cell structure on the sphere. Simplicial spheres of dimension greater than three with $g_2 = 0$ are those that can be obtained by a sequence of such stackings, beginning with the boundary of a simplex. Those with $g_2 = 1$ can be obtained by the same process applied either to a join of the boundaries of two simplices or a join of a polygon – i.e. a "triangulated" 1-sphere – and the boundary of a simplex.

In this case we also have,

Proposition 5.1.2. $\dim H^2(\Gamma; \mathfrak{so}(Q)) = \chi(C^*(\Gamma; \mathfrak{so}(Q)))$*, for* $n > 2$.

Proof. Γ has trivial centralizer in $\mathrm{SO}(Q)$, so $H^0(\Gamma; \mathfrak{so}(Q)) = 0$ and by Calabi-Weil rigidity, $H^1(\Gamma; \mathfrak{so}(Q)) = 0$ if $n > 2$. $\square$

5.2 A long exact sequence

We would like to have a way to build the cohomology of reflection groups up from the cohomology of their subgroups. Suppose we have reflections $s, t \in S$ with $s < t$ in an ordering as chosen above and we associate subsets $T_s = \{r \in S | r \leq s\}$ and $T_t = \{r \in S | r \leq t\}$. Then the inclusion $\langle T_s \rangle \hookrightarrow \langle T_t \rangle$ induces a map $C^*(\langle T_t \rangle) \to C^*(\langle T_s \rangle)$. It is clear from inspection that this map is surjective. We define $C^*(s,t;M)$ to be the kernel and $H^*(s,t;M)$ to be the cohomology of this kernel. Explicitly,

$$C^0(s,t;M) = 0,$$

$$C^1(s,t;M) = \bigoplus_{r \in T_t, s < r \leq t} (M \otimes \sigma_{\langle r \rangle})^{\langle r \rangle},$$

and

$$C^2(s,t;M) = \bigoplus_{r,r' \in T_t, r<r', s<r'} (M \otimes \sigma_{\langle r,r' \rangle})^{\langle r,r' \rangle},$$

with the induced differential.

It follows from the definition that there is a long exact sequence:

$$\begin{aligned} 0 \to \mathrm{H}^0(\langle T_t \rangle; M) \to \mathrm{H}^0(\langle T_s \rangle; M) &\to \mathrm{H}^1(s,t;M) \to \mathrm{H}^1(\langle T_t \rangle; M) \to \mathrm{H}^1(\langle T_s \rangle; M) \\ &\to \mathrm{H}^2(s,t;M) \to \mathrm{H}^2(\langle T_t \rangle; M) \to \mathrm{H}^2(\langle T_s \rangle; M) \to 0. \end{aligned}$$

(The initial $H^0(s,t;M)$ term is omitted, since $C^*(s,t;M) = 0$.) A consequence of this long exact sequence is

Proposition 5.2.1. *If $H^2(s,t;M) = 0$, then $H^2(\langle T_s \rangle; M) \cong H^2(\langle T_t \rangle; M)$.*

CHAPTER 6

APPLICATIONS

The best situation in deformation theory is when $\mathrm{H}^2 = 0$. Then we have local smoothness and the dimension of the deformation space is the dimension of H^1. This motivates the following

Definition 6.0.1. *We call a discrete reflection group* Γ *on* V orderly *if*

$$H^2(\Gamma; V^* \otimes V) = 0.$$

Our first application is to the case that Γ is isomorphic to the fundamental group of a compact, two dimensional Coxeter orbifold.

Theorem 6.0.1. *If* Γ *is a discrete reflection group acting on* V *abstractly isomorphic to a polygon group such that for each pair of generators* s, t *with* $\alpha_s(h_s)\alpha_t(h_t) \geq 4$, *we have* $\alpha_s(h_s)\alpha_t(h_t) > 4$, *then* Γ *is orderly.*

Proof. Let $S = \{s_1, ..., s_f\}$ be the generating set of Γ ordered so as to be compatible with a cyclic ordering of the underlying polygon. We will proceed by induction using [prop] and adding one generator at a time according to the above ordering. Obviously, $H^2(\langle s_1 \rangle; V^* \otimes V) = 0$. For $1 < i < f$, $H^2(s_{i-1}, s_i; V^* \otimes V) = 0$, since the differential is just $t^{\langle s_{i-1}, s_i \rangle}_{\langle s_i \rangle} : (V^* \otimes V)^{\langle s_i \rangle} \to (V^* \otimes V)^{\langle s,t \rangle}$, which is surjective, so we have $H^2(\langle s_1, ..., s_{f-1} \rangle; V^* \otimes V) = 0$.

The last piece to consider is $H^2(s_{f-1}, s_f; V^* \otimes V)$.

$$C^2(s_{f-1}, s_f; V^* \otimes V) = (V^* \otimes V \otimes \sigma_{\langle s_1, s_f\rangle})^{\langle s_1, s_f\rangle} \oplus (V^* \otimes V \otimes \sigma_{\langle s_{f-1}, s_f\rangle})^{\langle s_{f-1}, s_f\rangle},$$

so the question of surjectivity comes down to the linear independence of certain linear functionals on $(V^* \otimes V \otimes \sigma_{\langle s_f\rangle})^{\langle s_f\rangle}$ that appear in the formula for the differentials. As usual, for any $X \in (V^* \otimes V \otimes \sigma_{\langle s_f\rangle})^{\langle s_f\rangle}$ we write $X = \beta \otimes h_{s_f} + \alpha_{s_f} \otimes u$ with $\alpha_{s_f}(u) = \beta(h_{s_f}) = 0$. The functionals of interest are:

$$\Lambda_1 : X \mapsto \beta(h_{s_1}),$$

$$\Lambda_2 : X \mapsto \alpha_{s_1}(u),$$

$$\Lambda_3 : X \mapsto \beta(h_{s_{f-1}}),$$

and

$$\Lambda_4 : X \mapsto \alpha_{s_{f-1}}(u).$$

These are linearly independent if and only if h_{s_1} and $h_{s_{f-1}}$ are linearly independent and α_{s_1} and $\alpha_{s_{f_1}}$ are linearly independent. But this is equivalent to the statement that a certain minor of the Cartan matrix,

$$\begin{pmatrix} 2 & \alpha_{s_1}(h_{s_{f-1}}) \\ \alpha_{s_{f-1}}(h_{s_1}) & 2 \end{pmatrix}$$

has nonzero determinant – i.e. that $4 - \alpha_s(h_s)\alpha_t(h_t) \neq 0$. If $m_{s,t} < \infty$, this is automatic. Otherwise, it is true by hypothesis.

$\square$

Remark: The hypothesis above is satisfied by holonomy groups of hyperbolic and convex projective structures on polygonal reflection orbifolds. In fact, the hypothesis on the cyclic products is only needed at the last step of the argument, so all that is really required is that there is a subset $r, s, t \subset S$ with $m_{r,s} < \infty$, $m_{r,t} < \infty$, and $\alpha_s(h_s)\alpha_t(h_t) > 4$ to make the above argument work, after a suitable choice of ordering of S. The above argument also works for noncompact polygon groups – indeed, one may drop the hypothesis on cyclic products altogether, since one can choose the ordering so that at the last stage, $m_{s_f,s_1} = \infty$, so one need add at most one 2-cell.

6.1 Weakly orderable Coxeter polytopes

In recent work, Choi and Lee [9] introduced a class of hyperbolic Coxeter orbifolds with nice deformation properties in the neighborhood of the hyperbolic structure.

Definition 6.1.1. *Suppose X is a compact hyperbolic Coxeter orbifold of dimension n with underlying polytope P and face numbers $f_0, ..., f_{n-1} = f$. X is called* weakly orderable *if:*

- $f_{n-2} - nf_{n-1} + \binom{n}{2} = 0$,
- *there exists an ordering of $F_1, ..., F_f$ of the facets so that for each i, there at most n facets F_j with $j < i$ such that F_i and F_j meet at a right angle, and*
- *whenever $F_{i_1}, ..., F_{i_m}$ are m such faces, those faces must be in general position in the sense that the linear functionals $\alpha_{i_1}, ..., \alpha_{i_m}$ defining support hyperplanes at those faces are linearly independent.*

In their paper, Choi and Lee note that this last condition is automatic in dimension three, as proven in a previous paper with Hodgson [8]. As we mentioned above, the condition that $f_{n-2} - nf_{n-1} + \binom{n}{2} = 0$ imposes a severe restriction on the combinatorial type of the underlying polytope in higher dimensions.

Question 6.1.1. *Can this restriction be used to remove the general position hypothesis, i.e. is this condition automatic in higher dimensions as well?*

Choi and Lee prove the following result:

Theorem 6.1.1. *If X is a weakly orderable Coxeter orbifold of dimension n with e_2 pairs of faces meeting at right angles and e_+ pairs of faces meeting at acute angles, then*

- *the deformation space of projective structures is smooth in a neighborhood of the hyperbolic structure and*
- *the dimension of the moduli space is $2nf - 2e_2 - e_+ - n^2 - 2n$.*

Their proof is rather homological in nature in that it uses infinitesimal rigidity in combination with computations with the Jacobian of a system of equations derived from the Cartan matrix associated to X. We give a more directly homological statement and proof:

Theorem 6.1.2. *If Γ is the holonomy group of a weakly orderable Coxeter orbifold X, then Γ is orderly, i.e.* $\mathrm{H}^2(\Gamma; V^* \otimes V) = 0$.

Proof. First, by hypothesis, Γ preserves a form Q of signature $(n, 1)$. We note that the combination of face numbers in the definition of "weakly orderable" is the Euler characteristic of $C^*(\Gamma; \mathfrak{so}(Q))$. As noted in the previous section, it follows that $\mathrm{H}^2(\Gamma; \mathfrak{so}(Q)) = 0$. Therefore, $\mathrm{H}^*(\Gamma; V^* \otimes V) \cong \mathrm{H}^*(\Gamma; \mathcal{S}(Q))$. Our task then is to show that $\mathrm{H}^2(\Gamma;\mathcal{S}(Q)) = 0$. As before, we proceed by induction on faces using Proposition 5.2.1.

For each i, let s_i be the reflection the face F_i and let $S = \{s_1, ..., s_f\}$ be ordered in the obvious way. Let $S_i = \{s_j \in S | j < i, m_{i,j} = 2\}$. Then the base case of $H^2(\langle s_1\rangle;\mathcal{S}(Q))$ vanishes since $C^2(\langle s_1\rangle;\mathcal{S}(Q)) = 0$. For the induction step, we have $d^1 : C^1(s_{i-1}, s_i;\mathcal{S}(Q)) \to C^2(s_{i-1}, s_i;\mathcal{S}(Q))$ given by

$$d^1 : X \mapsto \oplus_{t\in S_i} t^{s_i,t}_{\langle s_i\rangle}.$$

Writing $X = Q(\cdot, u) \otimes h_{s_i} + Q(\cdot, h_{s_i}) \otimes u$ where $Q(u, h_{s_i}) = 0$, we have $t^{\langle s_i,t\rangle}_{\langle s_i\rangle}(X) = Q(h_t, u)Q(\cdot, h_t)\otimes h_{s_i} + Q(u, h_t)Q(\cdot, h_{s_i})\otimes h_t = Q(u, h_t)(Q(\cdot, h_t)\otimes h_{s_i} + Q(\cdot, h_{s_i})\otimes h_t)$ for each $t \in S_i$. So surjectivity comes down to whether the linear functionals $Q(\cdot, h_t), t \in S_i$ defined on $\ker Q(\cdot, h_{s_i})$ appearing in the transfer maps are linearly independent. This is so by hypothesis. It follows that $\mathrm{H}^2(s_{i-1}, s_i;\mathcal{S}(Q)) = 0$ for each $i > 1$, so by induction $\mathrm{H}^2(\Gamma;\mathcal{S}(Q)) = 0$.

□

Choi and Lee provide examples that show the condition on face numbers is necessary to guarantee smoothness and the dimension statement. In fact, the homological statement is never true without it.

We can say a little more about the situation when only the face number condition is satisfied:

Proposition 6.1.1. *If X is a hyperbolic Coxeter orbifold of dimension n with holonomy group Γ and underlying polytope P having face numbers $f_0, f_1, ..., f_{n-1}$ satisfying $f_{n-2} - nf_1 + \binom{n}{2} = 0$, then the obstruction to extending any first order deformation $[\varphi] \in \mathrm{H}^1(\Gamma;\mathfrak{gl}_{n+1}(\mathbb{R}))$ to a second order deformation vanishes.*

Proof. The obstruction is $-\frac{1}{2}[[\varphi], [\varphi]]$. For any two elements $g, h \in \Gamma$, we have $[\varphi, \varphi](g, h) = [\varphi(g), \mathrm{Ad}(g)\varphi(h)]$. Now since $\mathrm{H}^1(\Gamma; \mathfrak{so}(n, 1)) = 0$, we may choose a cocycle representative φ with values in $\mathcal{S}(Q)$ for some quadratic form Q of signature $(n, 1)$. It follows that $[\varphi(g), \mathrm{Ad}(g)\varphi(h)] \in \mathfrak{so}(n, 1)$ for all g, h. Therefore, $[[\varphi], [\varphi]] \in \mathrm{H}^2(\Gamma; \mathfrak{so}(n, 1)) = 0$. □

Question 6.1.2. *Do the higher obstructions also vanish? In other words, does the smoothness part of the Choi-Lee result depend only on the face number condition?*

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