

THE ACTION DIMENSION OF ARTIN GROUPS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of
Philosophy in the Graduate School of The Ohio State University

By

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2016

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ABSTRACT

The *action dimension* of a discrete group G is the minimum dimension of a contractible manifold, which admits a proper G -action. In this dissertation, we study the action dimension of general Artin groups. The main result is that the action dimension of an Artin group with the nerve L of dimension n for $n \neq 2$ is less than or equal to $(2n + 1)$ if the Artin group satisfies the $K(\pi, 1)$ -Conjecture and the top cohomology group of L with \mathbb{Z} -coefficients is trivial. For $n = 2$, we need one more condition on L to get the same inequality; that is the fundamental group of L is generated by r elements where r is the rank of $H_1(L, \mathbb{Z})$. We prove our theorem by constructing an aspherical manifold of dimension $2n + 1$ which has the Artin group as its fundamental group.

AMS classification numbers. Primary: 20F36, 20F65 Secondary: 32S22

Keywords: Artin groups, action dimension, poset of groups, manifold, hyperplane arrangement.

To Bao Nguyen

ACKNOWLEDGMENTS

I would like to thank my advisor, Michael W. Davis, for his help, support and patience. Mike is very generous and flexible with his time. He agreed to meet and talked with me in the weekends during the period when I took care of my little daughter. Without his help, this work would not be fulfilled. Thank you very much, Mike.

I would like to thank my committee members, Jean Lafont and Thomas Kerler, for their time and suggestion. Special thank to Jean for providing me financial aid from his NSF grand.

I am also grateful to Ian Leary, who taught me algebraic topology and introduced me to Mike when he left OSU. I would like to express my gratitude to the faculty and staffs at the math department for everything they have done to help me in my studying and teaching.

Last but not least, thanks also go to my husband, Vinh Khuc, who never understands what I am doing but supports me nonetheless.

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Giang Le, The Relative Hyperbolicity of One-relator Relative Presentations, J. of group theory, 10.1515/JGT.2009.

Fields of Study

Major Field: Mathematics

Specialization: Geometric Group Theory

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Chapter 1

INTRODUCTION

1.1 Background

One of the main topics in geometric group theory is the study of (discrete) groups via their action on spaces. The action may reveal invariants of groups which we can use to distinguish them. For example, one standard invariant is the geometric dimension of a (torsion-free) group G , which is the minimum dimension of a contractible complex with a proper G -action. If we replace contractible complex by a contractible manifold, we get another invariant, called the $K(\pi, 1)$ -Conjecture action dimension. The action dimension was first defined by M. Bestvina, M. Kapovich and B. Keiner in their paper titled "*Van Kampen's embedding obstruction for discrete groups*" [BKK02]. In this paper, the authors introduced the so-called obstructor dimension of a discrete group. Roughly speaking, the obstructor dimension of a group relates to a maximum dimension of a complex which can be embedded into the boundary of the group (supposing such boundary exists). The main theorem of [BKK02] states that if a group has obstructor dimension m then the group cannot act properly discontinuously on a contractible manifold of dimension less than m . From this, the authors introduced the action dimension, which is an upper bound for the obstructor dimension. In the same paper, the authors also determined, as an example, the obstructor dimensions of the braid groups, which in turn can be used to determine their action dimensions. Another paper, in which the action dimension was studied, is "*The action dimension of right-angled Artin groups*" by G. Avramidi, M. Davis, B. Okun and K. Schreve [ADOS16]. In this paper, the authors studied right-angled Artin groups (RAAGs)

and determined the case when a RAAG has the maximum action dimension.

Braid groups or RAAGs are examples of Artin groups, which were introduced by J. Tits in 1965 as extensions of Coxeter groups. The name "Artin group" was suggested by E. Brieskorn and K. Saito [EB72] based on the fact that braid groups were first studied by Artin in 1925. Braid groups and RAAGs are well understood because of their additional geometric properties; however, Artin groups in general are poorly understood. Many classical problems in group theory for Artin groups are still open. For example, questions about torsion-freeness, the word problem, the conjugation problem, etc.. In this dissertation, we study the action dimension of a class of Artin groups which satisfy the $K(\pi, 1)$ -Conjecture.

1.2 Main result

The main object of study in this dissertation is the class of Artin groups, which satisfy the $K(\pi, 1)$ -Conjecture. This conjecture states that a certain complex is a classifying space of the Artin group. We focus on this class of groups because when an Artin group satisfies the $K(\pi, 1)$ -Conjecture, it admits a finite classifying space [CD95a] and thus possesses many nice properties (e.g., it is torsion-free, its geometric dimension is known). The conjecture is proven true for many Artin groups and it is doubt that there exists an Artin group which does not satisfy the $K(\pi, 1)$ -Conjecture; but if there is, then it would be also difficult to understand it, let alone determine its action dimension.

To set up the main theorem, we need to introduce some notations. An Artin group is a group with a presentation by a system of generators $a_i, i \in I$, and relations

$$a_i a_j a_i \cdots = a_j a_i a_j \cdots; \quad i, j \in I$$

where the words on each side of these relations are sequences of m_{ij} ($= m_{ji}$) alternating letters a_i and a_j . We set $m_{ij} = \infty$ to mean that there is no relation between a_i and a_j , and $m_{ii} = 1$. The symmetric matrix $M = (m_{ij})_{i,j \in I}$ is a *Coxeter matrix* on I . Corresponding to a Coxeter matrix is a *Coxeter group*, which has the same definition as Artin group but with additional relations $a_i^2 = 1$, for all $i \in I$. Coxeter group and Artin group can also be

determined by a graph Γ , called the *Coxeter diagram*, which has I as its set of vertices and any two vertices i and j are connected if and only if $m_{ij} \geq 3$. If Γ is connected, we say that the Artin group (and the Coxeter group) is *irreducible*.

Related to an Artin group (or Coxeter group) is a simplicial complex, called the *nerve* L of the Artin group (or the Coxeter group). The definition of the nerve is following: the vertices of L are indexed by I , the index set of generators of the Artin group. Any collection of k vertices of L forms a $(k - 1)$ -simplex if and only if the Coxeter subgroup generated by those corresponding k generators is finite. We usually denote the Artin group and the Coxeter group corresponding to the nerve L by A_L and W_L , respectively. When the nerve L is a simplex σ , we call it a *spherical Artin group* A_σ and the Coxeter group in this case is finite.

Let L be a simplicial complex with edges labeled by integers greater than or equal to 2, and A_L is an Artin group associated with L . The main result of this dissertation is a manifold construction which proves the following theorem:

Theorem 1.1. *If A_L satisfies the $K(\pi, 1)$ -Conjecture and $H^n(L, \mathbb{Z}) = 0$ (and when $n = 2$, further suppose that $\pi_1(L)$ is generated by r elements where $r = rkH_1(L, \mathbb{Z})$), then $actdim(A_L) \leq 2n + 1$.*

Charney and Davis [CD95b] proved that when L is a flag complex, then A_L satisfies the $K(\pi, 1)$ -Conjecture. Thus, the main theorem implies the following corollary.

Corollary 1.2. *If L is an n -dimensional flag complex of dimension $n \neq 2$ and the top cohomology group $H^n(L, \mathbb{Z})$ is trivial, then the action dimension of the Artin group A_L is less than or equal to $2n + 1$.*

In Chapter 4, the action dimension of irreducible spherical Artin groups is studied. In this case, the action dimension of the irreducible Artin group A_σ is exactly $2n + 1$.

Corollary 1.3. *If the Artin group A_L in the main theorem has an irreducible spherical Artin group with the nerve of the same dimension as a subgroup, then the $actdim(A_L) = 2n + 1$.*

We prove the upper bound on the action dimension of the Artin groups by constructing an aspherical manifold which has dimension $2n + 1$. The idea of our construction is that we consider Artin group as the colimit group (and/or the fundamental group) of a poset of groups over the poset of simplices of L or a poset whose geometric realization is the cone on the barycentric subdivision of L . A poset of groups has realization by aspherical complexes, which is a complex which has fundamental group the fundamental group of the poset of groups. We will use manifolds instead of complexes in the realization to construct the manifold.

There are two points in our proof of the main result. Firstly, we need to construct an aspherical manifold (so its universal cover is contractible). For this we will need the condition that the Artin group satisfies the $K(\pi, 1)$ -Conjecture. Charney and Davis [CD95a] proved that the $K(\pi, 1)$ -Conjecture is equivalent to a conjecture regarding the contractibility of the basic construction of a poset of groups for the Artin group. Haefliger [Hae92] related, under certain condition on the poset of groups, the contractibility of the basic construction to the sphericity of a realization by aspherical complexes of the poset of groups. We will use these two results to prove that our manifold is aspherical. Secondly, we want to minimize the dimension of our manifold. The best we can do is $2n + 1$ because the dimensions of the manifolds for local groups are already $2n + 1$. We can always be able to construction an aspherical manifold of dimension $2n + 2$ for the Artin group; however, if we want to decrease the dimension by one, we need to put condition on the nerve L . In particular, the constraint on the cohomology group of L implies that L can be embedded into a contractible complex of the same dimension. And this is a crucial point in our proof.

The dissertation is divided into the following chapters: Chapter 2 and Chapter 3 deal with poset of groups and poset of spaces, most of this follows the work in [BH99]. We also prove some technical lemmas which will be needed later. Chapter 4 provides basic information about Artin groups and different dimensions of a group (geometric and action dimensions). We study the geometric and action dimensions of irreducible spherical Artin groups here. Chapter 5 answers the question of when we can embed a simplicial complex

into a contractible simplicial complex of the same dimension. In the last chapter, Chapter 6, we describe our manifold construction and prove our results. We finish in Chapter 6 with a discussion about the idea and questions which we will pursue further in the future.

1.3 Notation

We set the following notation for this dissertation.

- D^i is an Euclidean disk of dimension i ,
- S^i is a sphere of dimension i ,
- L is a simplicial complex,
- L' is the barycentric subdivision of L ,
- \mathcal{P} is a poset,
- $\mathcal{P}(L)$ is the poset of nonempty simplices of L ,
- $\mathcal{S}(L)$ is a poset which is the union of $\mathcal{P}(L)$ with the empty simplex,
- $\mathcal{P}_{\geq\sigma}$ is the set of all $\tau \in \mathcal{P}$ such that $\tau \geq \sigma$,
- $\mathcal{P}_{\leq\sigma}$ is the set of all $\rho \in \mathcal{P}$ such that $\rho \leq \sigma$,
- D_σ is the *dual cone* for each $\sigma \in \mathcal{P}$, which is the geometric realization of $\mathcal{P}_{\geq\sigma}$,
- $\text{Lk}(\sigma, L)$ is the link of a simplex σ in a simplicial complex L ,
- $\mathbf{G}(\mathcal{P})$ is a poset of groups over a poset \mathcal{P} ,
- G_σ is a local group of $\mathbf{G}(\mathcal{P})$ for $\sigma \in \mathcal{P}$,
- $\text{Th}(L)$ is a thickening of L ,
- $T(\tau, Lk'_\sigma)$ is the thickening of τ in the thickening of Lk'_σ .

Chapter 2

POSETS OF GROUPS

In this chapter, we provide some basic information about posets of groups (or simple complexes of groups) and describe an important construction which is useful in building many examples of group actions on complexes. The idea is that if an action of a group G by isometries on a complex X has a strict fundamental domain Y (i.e. a subcomplex of X that meets each orbit in exactly one point), then one can recover X and the action of G directly from Y and the pattern of its isotropy subgroups.

2.1 Preliminaries

2.1.1 Abstract simplicial complex

Definition 2.1. An *abstract simplicial complex* L consists of a nonempty set of vertices V together with a collection \mathcal{P} of non-empty finite subsets of V , such that $\{v\} \in \mathcal{P}$ for all $v \in V$, and if $\sigma \in \mathcal{P}$ then every non-empty subset ρ of σ is also in \mathcal{P} .

The elements of \mathcal{P} are called the *simplices* of L . A simplex $\sigma \in \mathcal{P}$ is called an *n -simplex* if σ has cardinality $n + 1$; n is called the *dimension* of σ . The elements of σ are its *vertices* and the nonempty subsets of σ are its *faces*. The dimension, $\dim L$, of L is the supremum of the dimensions of its simplices.

Let L_1 and L_2 be abstract simplicial complexes with vertex sets V_1 and V_2 respectively. A *simplicial map* from L_1 to L_2 is a map $f : V_1 \rightarrow V_2$ that sends each simplex of L_1 to a simplex of L_2 .

Now suppose the vertex sets V_1 and V_2 are disjoint. The *simplicial join* of L_1 and L_2 , denoted by $L_1 * L_2$, is a simplicial complex with vertex set $V_1 \cup V_2$; a subset of $V_1 \cup V_2$ is a simplex of $L_1 * L_2$ if and only if it is a simplex of L_1 , a simplex of L_2 , or the union of a simplex of L_1 and a simplex of L_2 . Note that

$$\dim(L_1 * L_2) = \dim L_1 + \dim L_2 + 1.$$

If L is a simplicial complex with vertex set V , then the *simplicial cone* over L , denoted by $\text{Cone}(L)$, is the simplicial join of L and a complex with only one vertex v_0 . The vertex v_0 is called the *cone vertex (cone point)* of $\text{Cone}(L)$ and the dimension of $\text{Cone}(L)$ is $\dim L + 1$.

Given a simplex σ in an abstract simplicial complex L , the *link* of σ in L , denoted by $\text{Lk}(\sigma, L)$ or Lk_σ , is the subcomplex of L consisting of those simplices τ such that $\tau \cap \sigma = \emptyset$ and $\tau \cup \sigma$ is a simplex of L .

Note that the subcomplex of L whose simplices are the faces of simplices containing σ is isomorphic to the join $\sigma * \text{Lk}(\sigma, L)$.

Let L be an abstract simplicial complex with vertex set V . The *barycentric subdivision* L' of L is the abstract simplicial complex whose vertices are the simplices σ_i of L , and whose n -simplices are the sets $\{\sigma_0, \dots, \sigma_n\}$ where $\sigma_0 \subset \sigma_1 \subset \dots \subset \sigma_n$.

2.1.2 Geometric realization of an abstract simplicial complex

Let L be an abstract simplicial complex with vertex set V . Let W denote the real vector space with basis V . The *affine realization* $|\sigma|$ of a simplex σ of L is the affine simplex in W that has vertex set σ ; in other words, $|\sigma|$ consists of those vectors $x = \sum_{v \in \sigma} x_v v$ with $x_v \in [0, 1]$ and $\sum_{v \in \sigma} x_v = 1$. We give $|\sigma|$ the topology and affine structure induced from the finite dimensional subspace of W spanned by the vertices of σ .

Geometric realization. We define the *affine realization* (or *geometric realization*) of L to be the subset $|L|$ of W which is the union of the affine realizations of the simplices of L . We usually identify the abstract simplicial complex with its realization and write σ and L .

L to mean the realization $|\sigma|$ and $|L|$. The realization of σ is called *closed simplex* in L . The coordinates x_v of a point x of $L \subset W$ are called its *barycentric coordinates*.

Posets. A *poset* is a partially ordered set. Associated to any poset \mathcal{P} one has a simplicial complex whose set of vertices is \mathcal{P} and whose k -simplices are the strictly increasing sequences $\sigma_0 < \sigma_1 < \dots < \sigma_k$ of elements of \mathcal{P} . The geometric realization of this simplicial complex will be called the *geometric realization of the poset \mathcal{P}* , denoted $|\mathcal{P}|$.

Example 2.2. Let $\mathcal{P} = \mathcal{P}(L)$ the poset of a simplicial complex L . \mathcal{P} is partially ordered by inclusion. Then the simplicial complex associated to \mathcal{P} is the barycentric subdivision of L .

Let $\mathcal{S} = \mathcal{S}(L) = \mathcal{P}(L) \cup \{\emptyset\}$. The simplicial complex associated to \mathcal{S} is the cone on the barycentric subdivision of L .

2.1.3 Dual cones

Let \mathcal{P} be a poset. The *dual cone* of $\sigma \in \mathcal{P}$, denoted by D_σ , is the geometric realization of the set $\mathcal{P}_{\geq \sigma}$ of all elements $\tau \in \mathcal{P}$ such that $\tau \geq \sigma$.

If \mathcal{P} is the poset of non-empty simplexes of a simplicial complex L , then the dual cone of a simplex $\sigma \in L$ is the cone on the barycentric subdivision of the link of σ in L , Lk'_σ . It is also a subcomplex of L' , the barycentric subdivision of L . Some examples of the dual cones of simplex $\sigma \in L$ are:

1. If σ is top-dimensional, then D_σ is a point.
2. If the codimension of σ is 1, then D_σ is a cone on a discrete set.
3. If the codimension of σ is 2, then the link of σ is a graph and D_σ is the cone on the barycentric subdivision of this graph.
4. if L is a PL-triangulation of a manifold then D_σ is the dual cell of σ .

2.1.4 Stratified spaces

This section is devoted to stratified space, a useful generalization of simplicial complex. The discussion follows from [BH99], chapter II.12.

Definition 2.3. (Stratified sets and spaces). A *stratified set* $(X, \{X^\sigma\}_{\sigma \in \mathcal{P}})$ consists of a set X and a collection of subsets X^σ called *strata*, indexed by a set \mathcal{P} , such that

1. X is a union of strata,
2. if $X^\sigma = X^\tau$ then $\sigma = \tau$,
3. if an intersection $X^\sigma \cap X^\tau$ of two strata is non-empty, then it is a union of strata,
4. for each $x \in X$ there is a unique $\sigma(x) \in \mathcal{P}$ such that the intersection of the strata containing x is $X^{\sigma(x)}$.

The inclusion of strata gives a partial ordering on the set \mathcal{P} , namely $\tau \leq \sigma$ if and only if $X^\tau \subseteq X^\sigma$. We shall often refer to $(X, \{X^\sigma\}_{\sigma \in \mathcal{P}})$ as "a stratified space X with strata indexed by the poset \mathcal{P} " or "a stratified space over \mathcal{P} ".

Stratified simplicial complexes: If a stratified set X is the geometric realization of an abstract simplicial complex and each stratum X^σ is a simplicial subcomplex, then X is called a *stratified simplicial complex*.

Example 2.4. The geometric realization $|\mathcal{P}|$ of a poset \mathcal{P} has a natural stratification indexed by \mathcal{P} : the stratum $|\mathcal{P}_{\leq \sigma}|$ indexed by $\sigma \in \mathcal{P}$ is the union of the k -simplices $\sigma_0 < \sigma_1 < \dots < \sigma_k$ with $\sigma_k \leq \sigma$.

Strata-preserving maps and actions. Given two stratified sets $(X, \{X^\sigma\}_{\sigma \in \mathcal{P}})$ and $(Y, \{Y^\sigma\}_{\sigma \in \mathcal{Q}})$, a map $f : X \rightarrow Y$ is called *strata-preserving* if it maps each stratum X^σ of X bijectively onto some stratum of Y . The map $f : \mathcal{P} \rightarrow \mathcal{Q}$ obtained by defining $f(\sigma)$ to be the index of the stratum $f(X^\sigma)$ is a morphism of posets (a morphism of posets is an order-preserving map). If X and Y are stratified spaces of the same type, then f is called a *strata-preserving morphism* if its restriction to each stratum preserves the stratum's structure

(i.e. if each stratum is a topological space the restriction must be a homeomorphism, if each stratum a simplicial complex, it must be a simplicial isomorphism). If there is a strata-preserving inverse to f , then f is called an *isomorphism of stratified spaces*.

The action of a group G on a stratified space $(X, \{X^\sigma\}_{\sigma \in \mathcal{P}})$ is said to be *strata-preserving* if for each $g \in G$ the map $x \rightarrow g.x$ is a strata-preserving morphism.

Strict fundamental domain. Let G be a group acting by strata-preserving morphisms on a stratified set $(X, \{X^\sigma\}_{\sigma \in \mathcal{P}})$. A subset $Y \subset X$ is called a *strict fundamental domain* for the action if it contains exactly one point from each orbit and if for each $\sigma \in \mathcal{P}$, there is a unique $p(\sigma) \in \mathcal{P}$ such that $g.X^\sigma = X^{p(\sigma)} \subset Y$ for some $g \in G$. (Note that Y is a union of strata and hence is closed).

2.2 Posets of groups

Let \mathcal{P} be a poset. A *poset of groups* over a poset \mathcal{P} is a functor $\mathbf{G}(\mathcal{P})$ from \mathcal{P} to the category of groups. More precisely, we have the following definition.

Definition 2.5. (Poset of Groups). A *poset of groups* $\mathbf{G}(\mathcal{P}) = (G_\sigma, \psi_{\sigma\tau})$ over a poset \mathcal{P} consists of the following data:

1. for each $\sigma \in \mathcal{P}$, a group G_σ , called the *local group* at σ ;
2. for each $\sigma < \tau$, an injective homomorphism $\psi_{\sigma\tau} : G_\sigma \rightarrow G_\tau$ such that if $\rho < \sigma < \tau$, then

$$\psi_{\rho\tau} = \psi_{\rho\sigma}\psi_{\sigma\tau}$$

We sometimes denote the homomorphism $\psi_{\sigma\tau}$ by ψ_a , where a stands for the morphism $\sigma < \tau$, we call a an *edge* and write the initial vertex: $i(a) = \sigma$ and the terminal vertex: $t(a) = \tau$.

The notion of poset of groups is similar to the notion of simple complex of groups, which is explained in [BH99]. Complex of groups arises naturally from an action of a group G by isometries on a complex X , which has a strict fundamental domain Y . There are local groups G_σ for each simplex $\sigma < Y$, which are the isotropy subgroups of G . And whenever

one cell σ is contained in another τ , we have injective $\psi_{\tau\sigma} : G_\tau \rightarrow G_\sigma$. There are inclusion maps $\phi_\sigma : G_\sigma \rightarrow G$ which are compatible with $\psi_{\tau\sigma}$: $\phi_\sigma\psi_{\tau\sigma} = \phi_\tau$ whenever $\sigma \subset \tau$. In the notion of complex of group, if σ is a face of τ in Y , then G_τ is a subgroup of G_σ . For the purpose of studying Artin groups, we want to reverse the order relation so that if σ is a face of τ , then G_σ is a subgroup of G_τ . Thus, we will use the notion of poset of groups instead of simple complex of groups as developed in [BH99].

2.2.1 Classifying space $BG(\mathcal{P})$ of a poset of groups

Category $CG(\mathcal{P})$. We associate to a poset of groups $\mathbf{G}(\mathcal{P})$ over \mathcal{P} the small category $CG(\mathcal{P})$ whose objects are the objects of \mathcal{P} and whose morphisms are the pairs (g, α) , where α is in the set of morphisms of the poset \mathcal{P} and $g \in G_{t(\alpha)}$. We define maps $i, t : CG(\mathcal{P}) \rightarrow V(\mathcal{P})$ by $i((g, \alpha)) = i(\alpha)$ and $t((g, \alpha)) = t(\alpha)$. The composition $(g, \alpha)(h, \beta)$ is defined if $i(\alpha) = t(\beta)$ and then it is equal to

$$(g, \alpha)(h, \beta) = (g\psi_\alpha(h), \alpha\beta).$$

The condition on ψ_α in the definition of a complex of groups implies the associativity of this law of composition. The map $(g, \alpha) \rightarrow \alpha$ is a functor p of $CG(\mathcal{P})$ on \mathcal{P} .

Classifying space $BG(\mathcal{P})$ of $\mathbf{G}(\mathcal{P})$. Recall that the nerve $N(C)$ of a small category C is a simplicial complex constructed from the objects and morphisms of C . There is a 0-simplex of $N(C)$ for each object of C . There is a 1-simplex for each morphism $f : x \rightarrow y$ in C . In general, the set $N(C)_k$ of k -simplexes of the nerve consists of the k -tuples of composable morphisms of C

$$x_0 \rightarrow x_1 \rightarrow x_2 \dots \rightarrow x_{k-1} \rightarrow x_k.$$

The geometric realization of the nerve of the category $CG(\mathcal{P})$ is denoted by $BG(\mathcal{P})$ and is called the *classifying space* of $G(\mathcal{P})$.

Example 2.6. If \mathcal{P} has only one object σ , then $BG(\mathcal{P})$ is the usual classifying space BG_σ of the group G_σ .

2.2.2 Groups associated to a poset of groups

Associated to any poset of groups $\mathbf{G}(\mathcal{P})$, there are two groups: one is the colimit of the system of groups and monomorphisms $(G_\sigma, \psi_{\sigma\tau})$:

$$G := \varinjlim_{\sigma \in \mathcal{P}} G_\sigma$$

The group G is obtained by taking the free product of the groups G_σ and making the identifications $\psi_{\sigma\tau}(h) = h, \forall h \in G_\sigma, \forall \sigma < \tau$.

The other group is the *fundamental group* $\pi_1 := \pi_1(G(\mathcal{P}), \sigma)$ of a poset of groups $G(\mathcal{P})$ on a poset \mathcal{P} based at a vertex σ of $|\mathcal{P}|$, which is the fundamental group of the geometric realization $BG(\mathcal{P})$ of the nerve of $CG(\mathcal{P})$ based at σ . A presentation of the fundamental group can be found explicitly [BH99]. Let $|\mathcal{P}|^k$ be the set of k -simplexes in $|\mathcal{P}|$. Consider the graph which is the 1-skeleton of $|\mathcal{P}|$ and choose a maximal tree T in this graph, i.e. a subgroup which is a tree containing all the vertices. Such a maximal tree is not unique in general: any subgraph T' of the graph which is a tree can be extended to a maximal tree.

Proposition 2.7. *The fundamental group $\pi_1(G(\mathcal{P}), \sigma)$ is isomorphic to the abstract group generated by the set*

$$\coprod_{\sigma \in |\mathcal{P}|^0} G_\sigma \coprod (|\mathcal{P}|^1)^\pm$$

subject to the relations

$$\left\{ \begin{array}{l} \text{the relations in the groups } G_\sigma \\ (a^+)^{-1} = a^- \text{ and } (a^-)^{-1} = a^+ \\ a^+b^+ = (ab)^+, \forall (a, b) \in |\mathcal{P}|^2 \\ \psi_a(g) = a^+ga^-, \forall g \in G_{i(a)} \\ a^+ = 1, \forall a \in T \end{array} \right\}$$

The subgroup of π_1 generated by $\{a^+ | a \in |\mathcal{P}|^1\}$ is isomorphic to the fundamental group of $|\mathcal{P}|$. It follows that if $|\mathcal{P}|$ is simply connected, then $\pi_1 = G$. In the case when L is not

simply connected, there is a natural surjection

$$\phi : \pi_1 \rightarrow G$$

by sending all the extra generators of π_1 to the identity in G .

Another way to look at the fundamental group π_1 of a poset of groups $\mathbf{G}(\mathcal{P})$ is that it is a group such that there exists a simply connected complex U with an action of π_1 such that the quotient U/π_1 is the underlying complex $|\mathcal{P}|$. The universal cover U can be defined using the basic construction, which we will outline in the next section. Here, we give one more description of the fundamental group as a semidirect product of a colimit group of a poset of groups and the fundamental group of $|\mathcal{P}|$. Let $\tilde{\mathcal{P}}$ be the poset with $|\tilde{\mathcal{P}}|$ is the universal cover of $|\mathcal{P}|$. $\mathbf{G}(\tilde{\mathcal{P}})$ is the induced poset of groups. Then,

$$\pi_1 = \lim \mathbf{G}(\tilde{\mathcal{P}}) \rtimes \pi_1(|\mathcal{P}|),$$

where $\lim \mathbf{G}(\tilde{\mathcal{P}})$ denotes the colimit group of the poset of groups $G(\tilde{\mathcal{P}})$. With this description, we also see in the case when $|\mathcal{P}|$ is simply connected, the fundamental group π_1 is exactly the colimit group G of $\mathbf{G}(\mathcal{P})$.

2.3 The basic construction

Let Q be a group. A *simple morphism* $\varphi = (\varphi_\sigma)$ from $G(\mathcal{P})$ to Q , written $\varphi : G(\mathcal{P}) \rightarrow Q$, is a map that associates to each $\sigma \in \mathcal{P}$ a homomorphism $\varphi_\sigma : G_\sigma \rightarrow Q$ such that if $\sigma < \tau$, then $\varphi_\sigma = \varphi_\tau \psi_{\sigma\tau}$. We say that φ is *injective on the local groups* if φ_σ is injective for each $\sigma \in \mathcal{P}$.

The natural homomorphisms $\iota_\sigma : G_\sigma \rightarrow G$ from local groups to the colimit group of the poset of groups give a canonical simple morphism $\iota : G(\mathcal{P}) \rightarrow G$, where $\iota = (\iota_\sigma)$. In general, ι_σ is not injective.

The complex of groups is *developable* if the natural map $G_\sigma \rightarrow G$ is injective for each $\sigma \in \mathcal{P}$.

Suppose we have a poset of groups $\mathbf{G}(\mathcal{P})$ over a poset \mathcal{P} , a stratified space $(Y, \{Y^\sigma\}_{\sigma \in \mathcal{P}})$

indexed by the poset \mathcal{P} and injective simple morphism $\varphi : G(\mathcal{P}) \rightarrow Q$ for some group Q , then we can build a stratified space on which the group Q acts on and with strict fundamental domain Y .

Since the simple morphism $\varphi : G(\mathcal{P}) \rightarrow Q$ is injective on the local groups, we can identify each local group G_σ with its image $\varphi(G_\sigma)$ in Q . This basic construction can be defined as follows. Let

$$U(Q, Y) = Q \times Y / \sim,$$

the equivalence \sim is defined by

$$(g, y) \sim (g', y') \iff y = y' \text{ and } g^{-1}g' \in G_{\sigma(y)},$$

where $X^{\sigma(y)}$ is the smallest stratum containing y .

We write $[g, y]$ to denote the equivalence class of (g, y) . The group Q acts by strata preserving automorphisms according to the rule

$$g' \cdot [g, y] = [g'g, y].$$

And if we identify Y with the image of $\{1\} \times Y$ in $U(Q, Y)$ (where 1 is the identity element of Q), then Y is a strict fundamental domain for the action and the associated complex of groups is $G(\mathcal{P})$.

The simplest case of the basis construction is when we take $Q = G$ the colimit group and $Y = |\mathcal{P}|$. The construction gives us a complex $U(G, |\mathcal{P}|)$ with an action of G such that the quotient $U(G, |\mathcal{P}|)/G$ is $|\mathcal{P}|$. The universal cover of the poset of group U can be described as the cover of $U(G, |\mathcal{P}|)$ corresponding to the kernel of the map

$$\pi_1 \rightarrow G$$

There is a case of the basic construction which we will consider further; that is the basic construction with the group G which is the colimit of the system $\{G_\sigma\}$ of the complex of groups $\mathbf{G}(\mathcal{P})$. Group G can be the colimit of other systems of groups formed from the system $\{\mathbf{G}_\sigma\}$ in the two following settings:

1. Let $\mathcal{P} = \mathcal{P}(L)$ is the poset of (nonempty) cells of a simplicial complex L and $\mathcal{S} = \mathcal{S}(L) = \mathcal{P}(L) \cup \{\emptyset\}$. Then the geometric realization $|\mathcal{S}|$ is the cone on $|\mathcal{P}|$. We can form a complex of group $\mathbf{G}(\mathcal{S})$ from $\mathbf{G}(\mathcal{P})$ by defining $G_\emptyset := \{1\}$ and for other $\sigma \in \mathcal{S}$ the local group at σ of $\mathbf{G}(\mathcal{S})$ is the same as the local group at σ of $\mathbf{G}(\mathcal{P})$.
2. Suppose that L is a full subcomplex of L' . Define a new simple complex of groups $\mathbf{G}(\mathcal{P}(L'))$ over $\mathcal{P}(L')$ with groups $\{G'_\sigma\}_{\sigma \in \mathcal{P}(L')}$ by

$$G'_\sigma = \begin{cases} G_{\sigma \cap L} & \text{if } \sigma \cap L \neq \emptyset \\ \{e\} & \text{if } \sigma \cap L = \emptyset \end{cases}$$

It is easy to see that the system of groups in $\mathbf{G}(\mathcal{P}(L'))$ has the same colimit as $\mathbf{G}(\mathcal{P}(L))$. We then consider the basic construction with the same group G and with different stratified spaces.

Next we will study the relations of these constructions.

Lemma 2.8. *Suppose $\mathbf{G}(\mathcal{P})$ is a developable poset of groups over a poset \mathcal{P} . Let G be the colimit of the G_σ , $\sigma \in \mathcal{P}$. Assume the geometric realization of \mathcal{P} is simply connected. Then*

1. *The fundamental group of $\mathbf{G}(\mathcal{P})$ is G .*
2. *The universal cover of $\mathbf{G}(\mathcal{P})$ is $U(G, |\mathcal{P}|)$.*

Proof. If the geometric realization of \mathcal{P} is simply connected, then as we stated in section 2.2.2, the fundamental group π_1 of the poset of group is the same as the colimit G of the system $\{G_\sigma\}$. Since the two groups are the same, $U(G, |\mathcal{P}|)$ is the universal cover. \square

Lemma 2.9. *Suppose L is a contractible simplicial complex. Then $U(G, |\mathcal{P}(L)|)$ is G -equivariantly homotopy equivalent to $U(G, |\mathcal{S}(L)|)$.*

Proof. The geometric realization of $\mathcal{P}(L)$ is the barycentric subdivision of L . Thus, it is contractible. $|\mathcal{S}(L)|$ is the cone on $|\mathcal{P}(L)|$ and $U(G, |\mathcal{S}(L)|)$ is formed from $U(G, |\mathcal{P}(L)|)$ by coning off each copy of $|\mathcal{P}(L)|$. It follows that $U(G, |\mathcal{P}(L)|)$ and $U(G, |\mathcal{S}(L)|)$ are homotopy equivalent. \square

Let L and L' be simplicial complexes and L is a full subcomplex of L' . Let $\mathbf{G}(\mathcal{P}(L))$ be a poset of groups on the poset $\mathcal{P}(L)$. Define new posets of groups $\mathbf{G}(\mathcal{S}(L))$, $\mathbf{G}(\mathcal{P}(L'))$ and $\mathbf{G}(\mathcal{S}(L'))$ as above.

Lemma 2.10. *Suppose $L < L'$ is a full subcomplex and L' is contractible. Then $U(G, |\mathcal{S}(L)|)$ is G -equivariantly homotopy equivalent to $U(G, |\mathcal{P}(L')|)$.*

Proof. There is a deformation retraction $r : \mathcal{S}(L') \rightarrow \mathcal{S}(L)$ which sends each vertex of $L' - L$ to the cone point corresponding to $\emptyset \in \mathcal{S}(L)$. Recall that for each $x \in L$, we define $\sigma(x)$ to be the minimum simplex σ_0 such that x belongs to a unique (open) simplex corresponding to a chain $\sigma_0 < \sigma_1 < \dots < \sigma_k$ in \mathcal{P} . For any $x \in \mathcal{S}(L')$, we have $G'_{\sigma(x)} = G_{\sigma(r(x))}$. It follows that r induces a G -equivariant deformation retraction $U(G, |\mathcal{S}(L')|) \rightarrow U(G, |\mathcal{S}(L)|)$. By Lemma 2.9, $U(G, |\mathcal{S}(L')|)$ is G -equivariantly homotopy equivalent to $U(G, |\mathcal{P}(L')|)$. The lemma follows. \square

The following proposition is corollary 3.2.4 from [Hae92].

Proposition 2.11. *Suppose $G(\mathcal{P})$ is a developable poset of groups, and let G be the colimit of the G_σ . Suppose $|\mathcal{P}|$ is simply connected. Then $BG(\mathcal{P})$ is homotopy equivalent to the Eilenberg-MacLane space BG if and only if $U(G, |\mathcal{P}|)$ is contractible.*

Since $|\mathcal{P}|$ is simply connected, the fundamental group of $BG(\mathcal{P})$ is G . Let EG be the universal cover of BG and let $U = U(G, |\mathcal{P}|)$ be the basic construction. Haefliger proved that $BG(\mathcal{P})$ is homotopy equivalent to the Borel construction, $U \times_G EG$. This proves the proposition.

Chapter 3

POSETS OF SPACES

Associate with a poset of groups is a poset of spaces. Given a simplicial complex and a collection of spaces, we can construct a simplicial complex which gives rise to a poset of groups.

3.1 Realizations with aspherical complexes

Let \mathcal{P} be a poset. For each $\sigma \in \mathcal{P}$, define $P_\sigma := |\mathcal{P}_{\leq \sigma}|$ and call it the *cone* of σ . There is a natural inclusion $i_\sigma : P_\sigma \rightarrow |\mathcal{P}|$ and If $a = (\sigma < \tau)$ is an edge from σ to τ , then there is a natural inclusion $i_a : P_\sigma \rightarrow P_\tau$.

Remark 3.1. If $\mathcal{P} = \mathcal{P}(L)$ is the poset of cells in a cell complex L , then P_σ is the cell corresponding to the barycentric subdivision of σ . Similarly, if $\mathcal{P} = \mathcal{S}(L)$, then P_σ is the cone on the barycentric subdivision of σ . Also, $i_a : P_\sigma \rightarrow P_\tau$ is the inclusion of one cell as a face of the other.

Suppose Y is a space and $\pi : Y \rightarrow |\mathcal{P}|$ is a continuous map. For each $\sigma \in \mathcal{P}$, denote $Y(\sigma) = \pi^{-1}(\sigma)$. Let $Y(P_\sigma)$ be the subspace of $Y \times P_\sigma$ of all (y, x) such that $\pi(y) = i_\sigma(x)$. In other words, $Y(P_\sigma)$ is the graph of the restriction of π to $\pi^{-1}(P_\sigma)$. There are two projection maps:

$$\pi_\sigma : Y(P_\sigma) \rightarrow P_\sigma, \text{ defined by } (y, x) \rightarrow x,$$

and

$$Y(j_\sigma) : Y(P_\sigma) \rightarrow Y, \text{ defined by } (y, x) \rightarrow y.$$

We will identify the fiber $Y(\sigma) = \pi^{-1}(\sigma)$ with the fiber of π_σ above the vertex of P_σ mapped by i_σ on σ . For an edge $a = (\sigma < \tau)$, the inclusion map $i_a : P_\sigma \rightarrow P_\tau$ lifts to a map

$$Y(i_a) : Y(P_\sigma) \rightarrow Y(P_\tau), \text{ defined by } (x, y) \rightarrow (i_a(x), y).$$

Suppose that a section s of π over the 1-skeleton $|\mathcal{P}|^1$ of $|\mathcal{P}|$ is given. In particular each fiber $Y(\sigma)$ has a base point $s(\sigma)$. For each $\sigma \in \mathcal{P}$, this induces a section s_σ of π_σ over the 1-skeleton of P_σ .

Definition 3.2. A *poset of spaces* over \mathcal{P} is a space Y together with a projection map $\pi : Y \rightarrow |\mathcal{P}|$ and a section $s : |\mathcal{P}|^1 \rightarrow Y$ over the 1-skeleton of $|\mathcal{P}|$ so that

- For each $\sigma \in \mathcal{P}$, $Y(\sigma)$ is path-connected and there is a retraction $r_\sigma : Y(P_\sigma) \rightarrow Y(\sigma) \subset Y(P_\sigma)$ (the identification is indicated above) which is homotopic to the identity relatively to $Y(\sigma)$ and which is compatible with s , i.e., $r_\sigma s_\sigma(x) = s(x)$ for all x in the 1-skeleton of P_σ .
- Given an edge a from σ to τ , let $\theta : Y(\sigma) \rightarrow Y(\tau)$, where $\theta(y) := r_\tau Y(i_a)(y)$. If $G_\sigma = \pi_1(Y(\sigma), s(\sigma))$ and if a is an edge from $\sigma \rightarrow \tau$, then the homomorphism $\phi_a : G_\sigma \rightarrow G_\tau$ induced by θ , is injective.

The groups G_σ and the homomorphisms $\phi_a : G_{i(a)} \rightarrow G_{t(a)}$ determine a poset of groups $\mathbf{G}(\mathcal{P})$ [Hae92]. We say that $Y \rightarrow |\mathcal{P}|$ is a *realization* of $\mathbf{G}(\mathcal{P})$.

Definition 3.3. Suppose Y is a poset of CW complexes over \mathcal{P} with associated poset of groups $\mathbf{G}(\mathcal{P})$. Then Y is an *realization with aspherical complexes* of $\mathbf{G}(\mathcal{P})$ if each $Y(\sigma)$ has the homotopy type of the classifying space BG_σ .

Claim 1. *The classifying space $BG(\mathcal{P})$ of the poset of groups $\mathbf{G}(\mathcal{P})$ is a realization with aspherical complexes of $\mathbf{G}(\mathcal{P})$.*

Proof. Recall that the classifying space $BG(\mathcal{P})$ is the geometric realization of the nerve of the associated category $CG(\mathcal{P})$. If \mathcal{P} has only one object σ associated with group G_σ , then $BG(\mathcal{P})$ is just the usual BG_σ . In general, the projection $CG(\mathcal{P}) \rightarrow \mathcal{P}$ induces a map

$\pi : BG(\mathcal{P}) \rightarrow |\mathcal{P}|$. The fiber over $\sigma \in \mathcal{P}$ is the classifying space BG_σ . Thus, $BG(\mathcal{P})$ is a realization with aspherical complexes of $\mathbf{G}(\mathcal{P})$. \square

3.2 Explicit description

Let $\mathbf{G}(\mathcal{P})$ be a poset of groups over a poset \mathcal{P} . For each $\sigma \in \mathcal{P}$, let D_σ denote the dual cone $|\mathcal{P}_{\geq\sigma}|$ and choose a model for BG_σ . By using the mapping cylinder, we may assume that if $\sigma < \tau$, then BG_σ is a subcomplex of BG_τ . Thus, when $\sigma < \tau$, we have

$$BG_\sigma < BG_\tau \text{ and } D_\tau < D_\sigma$$

To form the realization of the complex of groups $\mathbf{G}(\mathcal{P})$, we glue together the spaces $BG_\sigma \times D_\sigma$ in the following fashion: whenever $\sigma < \tau$ glue $BG_\sigma \times D_\sigma$ to $BG_\tau \times D_\tau$ by identifying them along the common subspace $BG_\sigma \times D_\tau$.

Claim 2. *The resulting CW-complex Y is a realization with aspherical complexes of $\mathbf{G}(\mathcal{P})$.*

Proof. First of all, we need to prove that the resulting CW-complex is a poset of spaces over \mathcal{P} . There is a projection $\pi : Y \rightarrow |\mathcal{P}|$ induced by the projections of $BG_\sigma \times D_\sigma$ on the second factor. The gluing pattern on the second factors of each component of Y is the same as the one for making $|\mathcal{P}|$. Hence, the projection map is continuous.

For each $\sigma \in \mathcal{P}$, choose a base point $s(\sigma)$ in each Eilenberg-MacLane space BG_σ ; if $\sigma < \tau$ then $s(\tau) = s(\sigma)$. We then can find a section $s : |\mathcal{P}|^1 \rightarrow Y$ over the 1-skeleton of $|\mathcal{P}|$. Put $Y(\sigma) := \pi^{-1}(s(\sigma)) = BG_\sigma$ - a path-connected space. $Y(P_\sigma)$ is the graph of the resection of π to $\pi^{-1}(P_\sigma)$ where P_σ is $|\mathcal{P}_{\leq\sigma}|$. P_σ is a cone with the cone point σ . For each $\nu < \sigma$, $BG_\nu = \pi^{-1}(\nu)$ is a subcomplex of BG_σ . Thus, there is a retraction $Y(P_\sigma) \rightarrow Y(\sigma)$, which maps each space BG_ν for each $\nu < \sigma$ to BG_σ by injection map. And hence, Y is a realization of $G(\mathcal{P})$. \square

The following lemma is from [\[Hae92\]](#).

Lemma 3.4. *Any realization with aspherical complexes is homotopy equivalent to the classifying space $BG(\mathcal{P})$ of the poset of groups $\mathbf{G}(\mathcal{P})$.*

Next we will study the conditions of the poset of groups $\mathbf{G}(\mathcal{S}(L))$ so that its realization with aspherical complexes is indeed aspherical.

Lemma 3.5. (*[CD95b]*) *Suppose that L is a simplicial complex and $\mathbf{G}(\mathcal{S}(L))$ is a developable poset of groups over the poset $\mathcal{S}(L)$. Let G denote the colimit of $\mathbf{G}(\mathcal{S}(L))$ and let $Y \rightarrow |\mathcal{S}(L)|$ be a realization with aspherical complexes. If L is a flag complex, then $U(G, |\mathcal{S}(L)|)$ is contractible and Y is aspherical.*

Proof. Since $|\mathcal{S}(L)|$ is a cone, it is contractible (and thus, simply connected), by Proposition 2.11 it suffices to prove that $U(G, |\mathcal{S}(L)|)$ is contractible.

Suppose L is a flag complex. We shall prove by induction on the number of vertices of L . If L is a simplex σ , then $U(G_\sigma, |\mathcal{S}_{\leq\sigma}|)$ is a cone, so it is contractible. Now assume that for any full subcomplex L' of L , the space $U(G', |\mathcal{S}(L')|)$ is contractible (where G' denotes the colimit for $\mathcal{S}(L')$). Denote by $\langle T \rangle$ the full subcomplex of L for a subset T of the vertex set of L , $V(L)$. If L is not a simplex, there are two vertices v_1 and v_2 which are not connected by an edge (the condition that L is flag complex implies the existence of these two vertices). Put

$$L_0 = \langle V(L) - \{v_1, v_2\} \rangle,$$

$$L_1 = \langle V(L) - \{v_1\} \rangle,$$

$$L_2 = \langle V(L) - \{v_2\} \rangle.$$

Since L_i are proper subcomplexes of L , the lemma holds for each. For $i = 0, 1, 2$, put

$$G_i = \lim_{\sigma \in \mathcal{S}(L_i)} G_\sigma$$

Then G is the amalgamated product of G_1 and G_2 along G_0 . Let T be the associated Bass-Serre tree, i.e. a tree T such that G acts on T with orbit space an edge, with vertex stabilizers G_1 and G_2 and with edge stabilizer G_0 . Therefore, we get a tree of spaces, where the tree is T , the vertex spaces are copies of $U(G_i, |\mathcal{S}(L_i)|)$ for $i = 1, 2$, and the edge spaces are copies of $U(G_0, |\mathcal{S}(L_0)|)$. Since each of these spaces is contractible (by induction), $U(G, |\mathcal{S}(L)|)$ is contractible. The lemma follows. \square

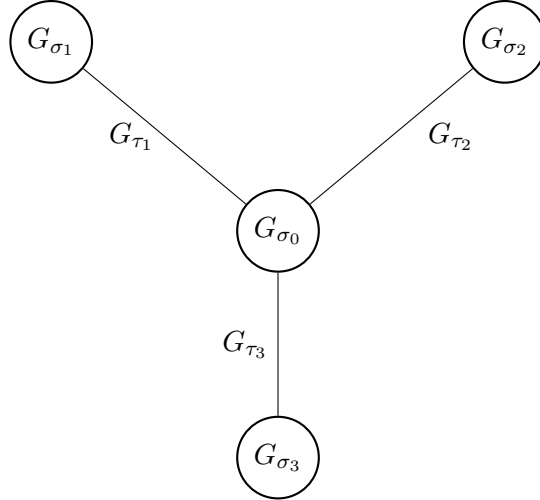


Figure 3.1: A poset of groups

Consider the poset of groups over the poset of simplices of a graph L consisting of 4 vertices and 3 edges (see figure 3.1).

The dual cones of the edges are points, the dual cones of vertices with local groups G_{σ_1} , G_{σ_2} and G_{σ_3} are segments; while the dual cone of the vertex with local group G_{σ_0} is a cone on 3 points, which has the form of letter Y.

The gluing pattern is described in figure 3.2.

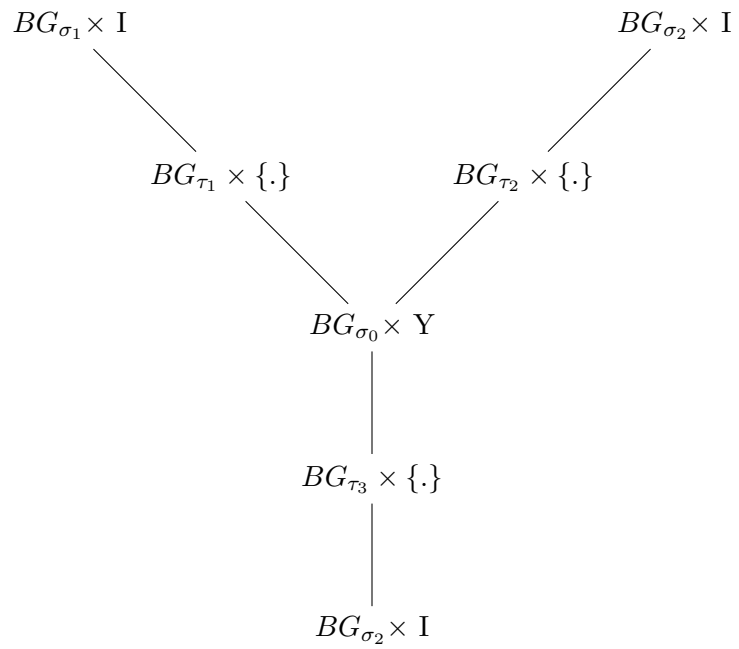


Figure 3.2: Gluing pattern

Chapter 4

ARTIN GROUPS

The class of Artin groups contains a wide range of groups such as free groups, free abelian groups and braid groups. Artin groups are defined abstractly and have close relation with Coxeter groups. In fact, a Coxeter group is the quotients of the associated Artin group. In this chapter, we provide the definition of Artin groups, the geometric and action dimensions of irreducible spherical Artin groups.

4.1 Definition and examples

Definition 4.1. We repeat some of the material in Chapter 1. An Artin group is a group with a presentation by a system of generators a_i , $i \in I$, and relations

$$a_i a_j a_i \cdots = a_j a_i a_j \cdots; \quad i, j \in I$$

where the words on each side of these relations are sequences of m_{ij} ($= m_{ji}$) alternating letters a_i and a_j . We set $m_{ij} = \infty$ to mean that there is no relation between a_i and a_j , and $m_{ii} = 1$.

The matrix $M = (m_{ij})_{i,j \in I}$ is a *Coxeter matrix* on I . Corresponding to a Coxeter matrix is a *Coxeter group*, which is defined as the Artin group but with additional relations $a_i^2 = 1$ for all $i \in I$. Coxeter group and Artin group can also be determined by a graph Γ , which has the set of vertices is I and any two vertices i and j are connected if and only if $m_{ij} \geq 3$. If Γ is connected, we say that the Artin group (and Coxeter group) is *irreducible*.

Related to an Artin group (and Coxeter group) is a simplicial complex, called the nerve

L of the Artin group (correspondingly, the Coxeter group). The definition of the nerve is following: the vertices of L are indexed by I , the index set of generators of the Artin group. Any collection of k vertices of L forms a $k - 1$ simplex if and only if the Coxeter subgroup generated by those k generators is finite (by theorem 2, Section 1.8, Chapter IV in [Bou68], the group generated by any subset of I with corresponding relations is actually a subgroup of the Coxeter group). Clearly, L is a simplicial complex. We usually denote the Artin group and Coxeter group corresponding to the nerve L by A_L and W_L , respectively. When the nerve L is a simplex σ , we have a *spherical* Artin group A_σ .

Examples of Artin group include braid groups, free groups and free abelian groups.

4.2 Artin group as a poset of groups

We recall in this section a description of a poset of Artin groups given in [CD95a].

Associated with a simplicial complex L with edge-labels by integers ≥ 2 are a Coxeter group W_L and an Artin group A_L . For each simplex σ in L , the Coxeter group generated by vertices of σ with corresponding relations is a subgroup of W_L , and for simplices $\sigma \subset \tau$, W_σ is a subgroup of W_τ . The same statement is also true for A_L . This was proved by Deligne [Del72] for Artin groups of finite type (spherical Artin groups) and by van der Lek [vdL83] for general Artin groups.

Now consider two posets $\mathcal{P}(L)$ and $\mathcal{S}(L)$ where $\mathcal{P}(L)$ is the poset of simplices in L and $\mathcal{S}(L)$ is $\mathcal{P}(L)$ union with an empty simplex; the partial order in both posets is defined by inclusion. We can define two posets of groups: $\mathbf{G}(\mathcal{P}(L))$ and $\mathbf{G}(\mathcal{S}(L))$. The first poset $\mathbf{G}(\mathcal{P}(L))$ is defined as follows: for each σ in $\mathcal{P}(L)$ the local group G_σ is the Artin group A_σ . For $\sigma < \tau$ there is an injection $A_\sigma \rightarrow A_\tau$ and the maps are compatible as in the definition of poset of groups. For $\mathbf{G}(\mathcal{S}(L))$, the local groups for $\sigma \neq \emptyset$ is the same as the local group in $\mathbf{G}(\mathcal{P}(L))$, the local group at \emptyset is trivial group.

It is easy to see that the colimit group of both posets of groups $\mathbf{G}(\mathcal{P}(L))$ and $\mathbf{G}(\mathcal{S}(L))$ is the Artin group A_L . But the fundamental groups are different if L is not simply connected. For any L the fundamental group of $\mathbf{G}(\mathcal{S}(L))$ is A_L because $|\mathcal{S}(L)|$ is simply connected (it

is a cone on the barycentric subdivision of L).

4.3 The $K(\pi, 1)$ -Conjecture

A central question of the studying of Artin groups is the $K(\pi, 1)$ -Conjecture, which was firstly stated by Brieskorn [Bri71] for some spherical Artin groups in a Boubaki seminar in 1971. After hearing the talk, Deligne [Del72] worked on the problem for other spherical Artin groups. The conjecture was then extended to a more general setting, which according to van der Lek [vdL83], is due to Arnold, Pham and Thom. The $K(\pi, 1)$ -Conjecture states that a certain complex (defined from complement of hyperplane arrangement) is a Eilenberg-MacLane space for the Artin group.

Charney and Davis [CD95a] describe a poset of groups over the poset $\mathcal{P} = \mathcal{S}(L)$ of the nerve L of an Artin group A_L . As is pointed out in [CD95a], this poset of groups is developable and the fundamental group (which is also the colimit group) is the Artin group A_L . It is proved in [CD95a] that the $K(\pi, 1)$ -Conjecture for Artin groups is equivalent to the following conjecture.

Conjecture 1. *The universal cover, $U(A_L, |\mathcal{S}(L)|)$, is contractible.*

$U(A_L, |\mathcal{S}(L)|)$ is the basic construction which we described in Chapter 2. One of the main results of [CD95a] is a proof of Conjecture 1 when L is a flag complex (recall that a flag complex is a complex which can be determined by its 1-skeleton, i.e., a collection of k vertices of the complex forms a simplex if and only if every pair of those vertices is connected by an edge)..

4.4 A $K(\pi, 1)$ manifold model for spherical Artin groups

In this section, we study a manifold for spherical Artin groups. This manifold comes from complement of hyperplane arrangement in sphere. To define it, we will need some preliminary facts.

4.4.1 Removing a tubular neighborhood

We will describe a procedure of blowing up a sub-manifold in a smooth manifold. This is equivalent to removing a tubular neighborhood of the sub-manifold, but the advantage of blowing up is that we do not need to specify the neighborhood of the submanifold. The discussion of this section follows Davis [Dav78] and Janich [Jan68].

Suppose that E is a smooth vector bundle over a manifold A . Let E_0 denote the complement of the zero-section. The positive real number \mathbb{R}_+ act on E_0 by fiber wise scalar multiplication. Define *the nonnegative cylinder bundle* as

$$C_+E := E_0 \times_{\mathbb{R}_+} [0, \infty),$$

where a positive real number s acts on $(x, t) \in E_0 \times [0, \infty)$ by $s \cdot (x, t) = (xs^{-1}, st)$. Denote the image of (x, t) in C_+E by $[x, t]$. The boundary of C_+E is called the *sphere bundle*, ΣE ; it is the subset

$$\Sigma E = E_0 \times_{\mathbb{R}_+} \{0\} \cong E_0/\mathbb{R}_+.$$

There is a canonical map $c : C_+E \rightarrow E$ defined by $c([x, t]) = tx$, which takes $C_+E - \Sigma E$ diffeomorphically onto E_0 and ΣE onto A via the projection mapping.

Remark 4.2. If we pick a metric for E , then the map $[x, 0] \rightarrow x/|x|$ identifies ΣE with the set of vectors of unit length in E .

Suppose that M is a smooth manifold and A is a submanifold and a closed subset. Let N be the normal bundle of A in M . Let $T : N \rightarrow M$ be a *tubular map* which is an embedding such that:

- $T|_A$ is the inclusion and
- $T_* : N \rightarrow N$ is the identity (where T_* is the map from the normal bundle of A in N to the normal bundle of A in M induced by the differential).

We now can define a smooth manifold with boundary $M \odot A$. As a set, $M \odot A$ is the disjoint union of $M - A$ and ΣN , where ΣN is defined as above. The tubular map T induces a map

$\tau : C_+N \rightarrow (M - A) \cup \Sigma N$ defined by

$$\tau([x, t]) = \begin{cases} T(tx) & t \neq 0 \\ [x, 0] & t = 0. \end{cases}$$

$(M - A) \cup \Sigma N$ has exactly one smooth structure as a manifold with boundary which agrees with the original smooth structure on $M - A$ and with respect to which τ is a diffeomorphism onto a neighborhood of ΣN in $(M - A) \cup \Sigma N$, see [Jan68]. This manifold with boundary is denoted by $M \odot A$. The smooth structure on $M \odot A$ is independent of the choice of the tubular map T and the construction is functorial as pointed out in [Dav78].

Moreover, if M is a smooth G -manifold and A is an invariant submanifold, then we can choose an equivariant tubular map $T : N \rightarrow M$ and define $M \odot A$ as before. Since G acts smoothly on M and since the construction is functorial, it acts smoothly on $M \odot A$.

4.4.2 A $K(\pi, 1)$ manifold model and its boundary

We have a natural map from an Artin group A_L to a Coxeter group W_L , by sending generators of A_L to corresponding generators of W_L . The kernel of this map is a normal subgroup of A_L and it is called the *pure Artin group*:

$$0 \rightarrow PA_L \rightarrow A_L \rightarrow W_L \rightarrow 0.$$

Suppose for now that $L = \sigma$ is a simplex (A_L is spherical Artin group), thus $W_L = W_\sigma$ is a finite Coxeter group. This Coxeter group W_σ acts as real linear reflection group on \mathbb{R}^{d+1} , where d is dimension of σ . The reflections in W_σ are the orthogonal reflections across the hyperplanes H . Complexification gives a linear action on $\mathbb{C}^{d+1} = \mathbb{R}^{d+1} \otimes \mathbb{C}$. Denote by \mathcal{A} the (finite) set of linear hyperplanes, $\mathcal{A} = \{H \otimes \mathbb{C}\}$. A *subspace* G of \mathcal{A} is an intersection of hyperplanes in \mathcal{A} . It is a fixed point set of some subgroup of the Coxeter group. Denote by $L(\mathcal{A})$ the poset of all subspaces of \mathcal{A} . The union of all subspaces of \mathcal{A} is the set of points with nontrivial isotropy group. This set can be viewed as stratified space with strata

indexed by the poset $\mathcal{P} = \mathcal{P}(\sigma)$. Define $M(\mathcal{A})$, the *complement* of \mathcal{A} , by

$$M(\mathcal{A}) = \mathbb{C}^{d+1} - \bigcup_{H \in \mathcal{A}} H.$$

Also denote $S(\mathcal{A}) = M(\mathcal{A})/\mathbb{R}_+ \cong M(\mathcal{A}) \cap S^{2d+1}$. Since

$$M(\mathcal{A}) \cong S(\mathcal{A}) \times \mathbb{R}_+,$$

we see that $S(\mathcal{A})$ is homotopy equivalent to $M(\mathcal{A})$. Deligne [Del72] proved that $M(\mathcal{A})$ is homotopy equivalent to $K(PA_\sigma, 1)$, and therefore, so is $S(\mathcal{A})$. Our goal is to define a bordification of $M(\mathcal{A})$ by a manifold with faces $\bar{M}(\mathcal{A})$, with its faces indexed by $L(\mathcal{A})$, as well as a compactification $\bar{S}(\mathcal{A})$ of $S(\mathcal{A})$, so that the interior of M is $M(\mathcal{A})$ and the interior of $\bar{S}(\mathcal{A})$ is $S(\mathcal{A})$.

Given a subspace $G \in L(\mathcal{A})$, define a subarrangement \mathcal{A}_G of \mathcal{A} , and a hyperplane arrangement \mathcal{A}^G in G :

$$\mathcal{A}_G := \{H \in \mathcal{A} \mid G \leq H\}$$

$$\mathcal{A}^G := \{G \cap H \mid H \in \mathcal{A} - \mathcal{A}_G\}.$$

Denote by G^\perp the perpendicular complement of G in \mathbb{C}^{d+1} , i.e., $\mathbb{C}^{d+1} = G \oplus G^\perp$, then \mathcal{A}_G is naturally identified with a hyperplane arrangement in G^\perp , namely,

$$\mathcal{A}_{G^\perp} = \{H \cap G^\perp \mid H \in \mathcal{A}_G\}$$

We will blowup \mathbb{C}^{d+1} along all subspaces of \mathcal{A} starting with the subspaces of smallest dimension. This will result in a sequence of manifolds with faces, $\mathbb{C}^{d+1} = M(-1) < M(0) < \dots < M(d)$, where $M(i)$ is obtained from $M(i-1)$ by blowing up the subspaces of dimension $2i$. The first step of the blowing up procedure is blowing up the smallest subspace $F = \{0\}$. And the result is

$$M(0) = S^{2d+1} \times [0, \infty).$$

The face $\partial_F M(0)$ corresponding to $F = \{0\}$ is $S^{2d+1} \times 0$. Similarly, each subspace G of \mathcal{A} is blown up to a manifold with boundary $G(0)$ with $\partial_F G(0) = S^{2d+1} \cap G$. Suppose,

by induction, that $M(i-1)$ has been defined and for each subspace G of dimension $\geq 2i$, submanifold with faces $G(i-1)$ also has been defined. Let G be a subspace of dimension $2i$. Blowup $M(i-1)$ along $G(i-1)$ to get $M(i-1) \odot G(i-1)$. In the process, we introduce a new face indexed by G , $\partial_G M(i-1) \odot G(i-1)$. Similarly, we can blow up all other $2i$ -dimensional subspaces to obtain $M(i)$. Then we can define for each subspace G of dimension $\geq 2(i+1)$: $G(i) := M(i) \cap G$. We continue this process to $i = d$ and obtain $\bar{M}(\mathcal{A}) := M(d)$. Finally, define

$$\bar{S}(\mathcal{A}) := \partial_0 M(d),$$

where $\partial_0(\cdot)$ denotes the blowup corresponding to a minimum face $F = \{0\}$.

Next, we will study closely the boundary of $\bar{M}(\mathcal{A})$ and $\bar{S}(\mathcal{A})$. Let $\rho < \sigma$ be a simplex in σ , there is the Coxeter group $W_\rho < W_\sigma$. Let G be the fixed subspace of W_ρ . When we blow up all the subspaces contained in G (which are the elements of \mathcal{A}^G), we get a submanifold $\bar{M}(\mathcal{A}^G)$ in an appropriate blowup N of \mathbb{C}^{d+1} . The normal bundle of $\bar{M}(\mathcal{A}^G)$ in N is trivial with fiber $\mathbb{C}^{c(\rho)}$, where $c(\rho)$ means the codimension of ρ in σ . Consider G^\perp , the hyperplane arrangement in G^\perp is the reflection arrangement of W_ρ . If we continue blow up N along subspaces for the reflection arrangement of W_ρ , we obtain a part of $\bar{M}(\mathcal{A})$, which is

$$\bar{M}(\mathcal{A}_{G^\perp}) \times \bar{M}(\mathcal{A}^G).$$

A part of the boundary of this submanifold is

$$\partial_G \bar{M}(\mathcal{A}) = \bar{S}(\mathcal{A}_{G^\perp}) \times \bar{M}(\mathcal{A}^G)$$

and a corresponding face in $\bar{S}(\mathcal{A})$

$$\partial_G \bar{S}(\mathcal{A}) = \bar{S}(\mathcal{A}_{G^\perp}) \times \bar{S}(\mathcal{A}^G)$$

Denote the manifold $\bar{S}(\mathcal{A})$ by M_σ . We summarize the above argument into the following proposition.

Proposition 4.3. *Suppose A_σ is a spherical Artin group of dimension d . Then there is a $(2d+1)$ -dimensional manifold with boundary M_σ with the following properties.*

1. M_σ is homotopy equivalent to BA_σ (i.e., M_σ is aspherical and its fundamental group is isomorphic to A_σ).
2. if $\rho < \sigma$ is a proper face, then $M_\rho \subset \partial M_\sigma$. Moreover, the normal bundle of M_ρ in ∂M_σ is trivial.

The following lemma is needed for our construction in Chapter 6.

Lemma 4.4. *The boundary of M_σ contains a collection of disjoint manifolds M_ρ for every $\rho < \sigma$.*

Proof. From the Proposition 4.3 we know that M_ρ is contained in the boundary of M_σ . We will prove that we can choose M_ρ such that it has no intersection with other submanifolds.

Since the statement of the Proposition 4.3 is true for any simplex σ , it follows that if ρ is simplex of codimension 1 of σ , then the boundary of M_ρ contains submanifolds M_π for any $\pi < \rho < \sigma$. Thus, it is enough to choose a disjoint collection of M_ρ of M_σ where ρ is simplex of codimension 1 (one manifold M_ρ can have multiple disjoint copies, one for each $\pi < \rho$, in the collection).

Let ρ_1 and ρ_2 be two simplexes of σ of codimension 1. We will study the manifold M_{ρ_1} and M_{ρ_2} in the hyperplane complement and in its quotient by the Coxeter group W_σ . Let G_1 and G_2 be the fixed subspaces of W_{ρ_1} and W_{ρ_2} . G_1 and G_2 are planes in \mathbb{C}^{d+1} (2-dimensional vector spaces). They intersect at the origin (the fixed subspace of W_σ). After blowing up the origin, these two subspaces have no intersection. Thus, the two manifolds $\bar{S}(\mathcal{A}_{G_1})$ and $\bar{S}(\mathcal{A}_{G_2})$ are totally disjoint because they sit in the sphere bundle of each G_1 and G_2 . If the group W_σ does not map G_1 to G_2 , then these 2 manifolds remain disjoint in the quotient of the hyperplane complement by the group W_σ . If the group W_σ does map G_1 to G_2 , then the two manifold are the same in the quotient. But since the normal bundle of this manifold in the boundary of M_σ is trivial, we can be able to take as many as copies of M_{ρ_1} (or M_{ρ_2}) as we want. the lemma is proved. \square

4.5 Geometric and action dimensions

Let Γ be a discrete torsion-free group. We have the following notions of dimensions for Γ .

Definition 4.5. The *geometric dimension*, $\text{gdim}(\Gamma)$, of Γ is the minimal n such that Γ admits a properly discontinuous action on a contractible n -complex, in other words, Γ has a finite n -dimensional classifying space $B\Gamma$.

Definition 4.6. The *action dimension*, $\text{actdim}(\Gamma)$, is the smallest integer n such that Γ admits a properly discontinuous action on a contractible n -manifold. If no such n exists, then $\text{actdim}(\Gamma) = \infty$.

Notice that if Γ admits a proper action on a contractible manifold M and if Γ is torsion-free, then the action is free and M/Γ is a finite dimensional model for $B\Gamma$. A theorem of Stallings [Sta65] says that every n -complex is homotopy equivalent to a complex that embeds in \mathbb{R}^{2n} . It follows that for a torsion free group Γ we have

$$\text{actdim}(\Gamma) \leq 2\text{gdim}(\Gamma),$$

We could see that by immersing an n -complex (a $K(\Gamma, 1)$ model) into \mathbb{R}^{2n} and taking its regular neighborhood.

For a RAAG, its action dimension is studied in [ADOS16]. In particular, the authors prove:

Theorem 4.7. *Suppose L is a k -dimensional flag complex and A_L is the RAAG. Then, if $H_k(L; \mathbb{Z}/2) \neq 0$, then $\text{actdim}(A_L) = 2k + 2$. If $H_k(L; \mathbb{Z}/2) = 0$ and $k \neq 2$, then $\text{actdim}(A_L) \leq 2k + 1$.*

Let us discuss some of the ideas of the proof of the above theorem; for more details, see [ADOS16]. To prove the theorem, the authors introduce the so-called *octahedralization*, OL of a simplicial complex L . If V is the set of vertices of L , then the set of vertices of OL is $V \times \{\pm 1\}$; and if a collection of vertices $\{v_0, \dots, v_k\}$ forms a simplex in L , then any subset of the form $\{(v_0, \epsilon_0), \dots, (v_k, \epsilon_k)\}$ of $V \times \{\pm 1\}$ spans a k -simplex. The octahedralization

OL of the nerve L of a RAAG A_L sits inside the boundary $\partial_\infty A_L$ of A_L and plays a role of obstructor complex ([BKK02]) for the RAAG. There are two dimensions of OL : one is the *embedding dimension*, which is the minimum dimension m in which OL can embed piecewise linearly in S^m . The other is the *van Kampen dimension* - the maximum m such that some particular cohomology class of the configuration space of OL is nonzero. The action dimension of A_L is related to the embedding dimension and the van Kampen dimension of OL in the right-angled case by the following inequalities:

$$\text{embdim}OL + 1 \geq \text{actdim}A_L \geq \text{vkdim}OL + 2.$$

The main result of [ADOS16] concerns $\text{vkdim}OL$ for a flag complex L (recall that for a RAAG, the nerve L is a flag complex). In the case when the top homology H_k of L with $\mathbb{Z}/2$ -coefficients is nonzero, the van Kampen dimension of OL is $2k$; consequently, the action dimension of A_L is $2k + 2$.

4.5.1 Geometric dimension of spherical Artin groups

For spherical Artin group A_σ , we have the following theorem about its geometric dimension.

Theorem 4.8. *The geometric dimension of a spherical Artin group A_σ is $\dim(\sigma) + 1$.*

Proof. It was proven in [Del72] that spherical Artin groups admit finite dimensional classifying space. Alternatively, we can see that spherical Artin groups has the nerve L a simplex, which is obviously a flag complex. Then, by the result of [CD95a], these groups satisfy the $K(\pi, 1)$ Conjecture. Charney and Davis proved that there is a finite Eilenberg-Maclane space of dimension $\sigma + 1$ for the Artin group. Thus, $\text{gdim}(A_\sigma) \leq \dim(\sigma) + 1$.

On the other hand, the cohomology of A_σ with \mathbb{Z} -coefficients is nonzero in the top dimension, which is equal to $\dim(\sigma) + 1$ ([Dav78]). Since the cohomology dimension of a group is a lower bound for geometric dimension, the theorem follows. \square

4.5.2 Action dimension of irreducible spherical Artin groups

The finite (irreducible) Coxeter groups were classified by Coxeter in 1935, see [Cox35]. They consist of three one-parameter families of increasing rank A_n, B_n, D_n , one one-parameter family of dimension two, $I_2(p)$ and six exceptional groups H_3, H_4, F_4, E_6, E_7 and E_8 .

Since the geometric dimension of a spherical Artin group A_σ is $\dim(\sigma) + 1$, the action dimension of A_σ is less than or equal to $2\dim(\sigma) + 2$. In fact, the action dimension of irreducible spherical Artin group is $2\dim(\sigma) + 1$.

Theorem 4.9. *Let Γ be the Artin group corresponding to Coxeter groups of type A_n, B_n or D_n . Then the action dimension of Γ is $2n - 1$.*

Proof. The number n in the notation of A_n, B_n or D_n is the number of generators of each Artin group, which is also the geometric dimension of the (spherical) Artin group. For the Artin group corresponding to the Coxeter groups of type A_n, B_n and D_n we have an aspherical manifold model (from hyperplane arrangement, see Section 4.4) of dimension $2n - 1$. Thus, $\text{actdim}(\Gamma) \leq 2n - 1$. We need to prove the opposite direction, i.e., $\text{actdim}(\Gamma) \geq 2n - 1$.

Type A_n . In [BKK02] the authors defined a notion of obstructor dimension for a group, $\text{obdim}(\Gamma)$, which relates to a minimum dimension m in which certain simplicial complexes can be embedded into an m -dimensional sphere (embedding dimension). The obstructor dimension is a lower bound for action dimension, i.e., for any group Γ , we have $\text{actdim}(\Gamma) \geq \text{obdim}(\Gamma)$. The authors also proved that $\text{obdim}(\Gamma) = 2n - 1$ for Γ is the braid group on $n+1$ strands. It follows that $\text{actdim}(\Gamma) \geq 2n - 1$.

Type B_n . There is a decomposition of the Artin group of type B_n , $A(B_n)$, as a semidirect product of free group and braid group on n strands \mathcal{B}_n ([CP05], proposition 2.1):

$$A(B_n) \cong F_n \rtimes \mathcal{B}_n.$$

Corollary 27 from [BKK02] states that if groups H and Q are finitely generated, H is weakly convex and if $G = H \rtimes Q$, then $\text{obdim}G \geq \text{obdim}H + \text{obdim}Q$. Examples of weakly convex groups are hyperbolic, CAT(0) or semi hyperbolic groups. Since both F_n and \mathcal{B}_n are finitely

generated and F_n is hyperbolic, we have

$$\text{obdim}(A(\mathcal{B}_n)) \geq \text{obdim}(F_n) + \text{obdim}(\mathcal{B}_n)$$

Obstructor dimension of a free group is 2, $\text{obdim}(\mathcal{B}_n) = 2(n-1) - 1 = 2n - 3$. Thus,

$$\text{obdim}(A(\mathcal{B}_n)) \geq 2 + 2n - 3 = 2n - 1.$$

So,

$$\text{actdim}(A(\mathcal{B}_n)) \geq 2n - 1.$$

Type D_n . The Artin group of type D_n can also be decomposed as a semidirect product of free group and braid group on n strands for $n \geq 4$ ([CP05], proposition 2.3)

$$A(D_n) \cong F_{n-1} \rtimes \mathcal{B}_n.$$

By the same argument as above, we have $\text{actdim}(A(D_n)) \geq 2n - 1$. □

For groups $A(I_2(p))$, we have $\text{actdim}(A(I_2(p))) = 3$. This is from [Gor04] based on previous work [Dro87]. In [Gor04], the Artin groups with 1-dimensional nerve, which are 3-manifold group, were classified. The 3-manifold Artin groups are the ones which have the nerve is a tree or a triangle with all labels 2.

Theorem 4.10. *We have the following equalities:*

- $\text{actdim}(A(H_3)) = 5$,
- $\text{actdim}(A(F_4)) = 7$,
- $\text{actdim}(A(H_4)) = 7$,
- $\text{actdim}(A(E_6)) = 11$,
- $\text{actdim}(A(E_7)) = 13$,
- $\text{actdim}(A(E_8)) = 15$.

Proof. In the upcoming work by Davis and Huang [DH16], an analog of OL as in the RAAG case for general Artin groups is defined. They do so by studying the so called "standard abelian subgroups" of spherical Artin groups. They are the maximal abelian subgroups of each (standard) parabolic subgroup of a particular Artin group. This collection of abelian subgroups can be encoded by a complex, which is a subdivision of the nerve L . For irreducible spherical Artin groups, this complex is just the barycentric subdivision of $L = \sigma$ (σ is a simplex).

Let $L = \sigma$ be the simplex of correct dimension n for each Artin group of type from the list $\{H_3, F_4, H_4, E_6, E_7, E_8\}$ ($n = 2$ for $A(H_3)$, $n = 3$ for $A(F_4)$ and so on). The barycenter, denoted by v , of L has the link is a triangulation of the sphere S^{n-1} . The link of v , $\text{Lk}(v)$, which is the boundary of the barycentric subdivision of L , is a flag complex. And thus, by the main result of [ADOS16], the van Kampen dimension of the octahedralization of $\text{Lk}(v)$ is $2(n - 1)$. Now, the barycentric subdivision of L is the join of v and its link: $v * \text{Lk}(v)$; by the definition of the octahedralization of a complex, we have the octahedralization of $v * \text{Lk}(v)$ is the suspension of the octahedralization of the link of v . Thus, the van Kampen dimension of the octahedralization of $v * \text{Lk}(v)$ is $2(n - 1) + 1 = 2n - 1$. Since the action dimension of A_L is greater or equal to the van Kampen dimension of any of its obstructor complexes + 2, it follows that $\text{actdim}(A_L) \geq 2n - 1 + 2 = 2n + 1$.

Since we already proved the other direction of the inequality, the theorem follows. □

Remark 4.11. The above proof works for all irreducible spherical Artin groups.

Remark 4.12. For Artin group which is the direct product of irreducible spherical Artin groups, its action dimension is the sum of the action dimensions of its factors. This follows from the result proven in [Yoo02].

Chapter 5

CONTRACTIBLE COMPLEXES

This chapter deals with the question of when a simplicial complex L can be embedded into a contractible complex of the same dimension; and from that, we will define a new poset of groups on the poset of a contractible complex.

5.1 Embedding into contractible complexes

The following lemma is standard. See [Fen70], for example, for references.

Lemma 5.1. *Suppose L is a (connected) finite simplicial complex of dimension d , $d \neq 2$. Then L can be embedded in a contractible complex C of the same dimension if and only if $H^d(L, \mathbb{Z}) = 0$.*

Proof. The necessity follows from long exact sequence:

$$H^d(C, L; \mathbb{Z}) \rightarrow H^d(C; \mathbb{Z}) \rightarrow H^d(L; \mathbb{Z}) \rightarrow H^{d+1}(C, L; \mathbb{Z}).$$

Suppose $H^d(L, \mathbb{Z}) = 0$; this implies that $H_d(L, \mathbb{Z}) = 0$ and $H_{d-1}(L, \mathbb{Z})$ is free by the universal coefficient theorem for cohomology. If $d = 1$, then L is a tree and the result is obvious. When $d \geq 3$, let $K = L \cup \text{Cone}(L^{d-2})$ where $\text{Cone}(L^{d-2})$ is the cone on the $(d-2)$ -skeleton of L . It is easy to see that $H^d(K, \mathbb{Z}) = H^d(L, \mathbb{Z}) = 0$. By the Hurewicz homomorphism:

$$\pi_{d-1}(K) \simeq H_{d-1}(L, \mathbb{Z}).$$

So $\pi_{d-1}(K)$ is a free abelian group. Adding d -cells to kill $\pi_{d-1}(K)$ embeds K and hence,

L , in a contractible d -complex. □

We treat the case $d = 2$ separately.

Lemma 5.2. *Suppose L is 2-dimensional simplicial complex. If $H^2(L, \mathbb{Z}) = 0$ and $\pi_1(L)$ is normally generated by r elements where $r = \text{rk}H_1(L, \mathbb{Z})$, then L can be embedded in a contractible complex C of the same dimension.*

Proof. The case $d = 2$ is special and the condition $H^2(L, \mathbb{Z}) = 0$ is not enough because L can be acyclic 2-complex (i.e., all homology groups are trivial) with non-trivial fundamental group. Thus, we need the assumption that $\pi_1(L)$ is normally generated by r elements where $r = \text{rk}H_1(L, \mathbb{Z})$. In this case, we can attack r 2-cells to kill $\pi_1(L)$ and $H_1(L)$ without introducing anything into $H_2(L)$. The resulting 2-complex is contractible. □

Note that the complexes in Lemmas 5.1 and 5.2 can be made to be simplicial. This is because of the **Simplicial Approximation Theorem** [Hat02].

Theorem 5.3. *If K is a finite simplicial complex and L is an arbitrary simplicial complex, then any map $f : K \rightarrow L$ is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of K .*

5.2 A new poset of groups

Let L be the nerve of an Artin group A_L . We have a poset of group $\mathbf{G}(\mathcal{P}(L))$ as described in Chapter 2. The colimit group of this poset of groups is the Artin group. Let C be a contractible simplicial complex, which contains L as subcomplex. We will define a new complex of groups $\mathbf{G}(\mathcal{P}(C))$, which has the same colimit groups A_L .

A new poset. For each $\sigma \in \mathcal{P}(C)$, if σ is also in $\mathcal{P}(L)$, define the local group at σ in $\mathbf{G}(\mathcal{P}(C))$ is the same as the local group at σ in $\mathbf{G}(\mathcal{P}(L))$. If σ is not in $\mathcal{P}(L)$, let ρ be the maximal simplex in $\mathcal{P}(L)$ such that $\rho < \sigma$, then let the local group at σ is the local group G_ρ of $\mathbf{G}(\mathcal{P}(L))$. If there is no such ρ , then put the local group at σ is a trivial group.

Obviously, the two complexes of groups have the same colimit group, which is the Artin group. We will use this complex of groups in the construction in chapter 6.

Aspherical manifolds for local groups. For each σ of dimension i in $\mathcal{P}(L)$, there is a $K(A_\sigma, 1)$ manifold model of dimension $2i + 1$ (Proposition 4.3). If σ is not in $\mathcal{P}(L)$ and if the local group at σ is trivial, define $M_\sigma = D^{2i+1}$. If the local group at σ is A_ρ for ρ a maximal simplex in $\mathcal{P}(L)$ which is contained in σ , then let $M_\sigma = M_\rho \times D^{2i-2h}$, where h is the dimension of ρ . Thus, for each σ in $\mathcal{P}(L_1)$, we have a manifold M_σ of dimension $2i + 1$.

It is easy to see that the statements in Proposition 4.3 and lemma 4.4 remain valid for these new manifolds.

Now suppose that the Artin group A_L satisfies the $K(\pi, 1)$ Conjecture. As we discussed in Section 4.3, this is equivalent to say that the basic construction $U(A_L, |\mathcal{S}(L)|)$ is contractible. It follows that, by Lemma 2.10, $U(G, |\mathcal{P}(C)|)$ is also contractible.

We will consider the complex of groups $\mathbf{G}(\mathcal{P}(C))$ in the next following sections. But for convenience, we will use L instead of C . Thus, we have a complex of group $\mathbf{G}(\mathcal{P}(L))$ with $U(G, |\mathcal{P}(L)|)$ contractible and for each local group G_σ of $\sigma \in \mathcal{P}(L)$, there is a $K(G_\sigma, 1)$ manifold model M_σ of dimension $2i + 1$, such that the statements in Proposition 4.3 and lemma 4.4 hold.

Chapter 6

CONSTRUCTION OF THE MANIFOLDS

For a nerve L of the Artin group A_L , if L is embedded into a contractible simplicial complex C , we defined a new poset of groups $\mathbf{G}(\mathcal{P}(C))$ in Section 5.2. This poset of group has the Artin group as its colimit and fundamental groups. We will construct an aspherical manifold with A_L as its fundamental group using some realization of the poset of groups $\mathbf{G}(\mathcal{P}(C))$. For each local group at simplex σ of dimension i of the poset of groups $\mathbf{G}(\mathcal{P}(C))$, there is an aspherical manifold M_σ of dimension $2i + 1$. We will need to replace the dual cone of each simplex in the explicit description of a realization of a poset of groups by some manifold. For convenience, we will use the notation L instead of C , and consider the poset of groups $\mathbf{G}(\mathcal{P}(L))$ with L contractible.

6.1 Thickening of a simplicial complex

The notion of thickening of a simplicial complex is well studied in a broad view in [Maz63] and [Wal66], where a *thickening* (more precisely, an *m-thickening*) of a finite d dimensional CW-complex K^d with base point $*$ is defined by a simple homotopy equivalence map $\varphi : K^d \rightarrow M^m$ of K^d and a manifold M^m of dimension $m \geq d + 3$, such that

- $\varphi(*) \in \partial M^m$,
- the inclusion $i : \partial M^m \rightarrow M^m$ induces an isomorphism $i_* : \pi_1(\partial M^m) \rightarrow \pi_1(M^m)$ of fundamental groups,
- the tangent space to M^m at $*$ is oriented.

Here, we will consider only a simple version of thickening, a thickening as a regular neighborhood of a complex of dimension d in a manifold of dimension $2d + 1$ or $2d$. In the following subsections, we will denote a thickening of L by $\text{Th}L$ or $\text{Th}(L, M)$ if we want to specify the manifold M .

6.1.1 Basic case.

It is well known that for any simplicial complex L of dimension d , we have a simplicial embedding of L as a subcomplex of some PL (piecewise linear) triangulation of S^{2d+1} . Thus, we have a thickening $\text{Th}L$ of L , which is a submanifold of S^{2d+1} . The restriction of this thickening to each i -simplex σ is a disk of the form $\sigma \times D^{2d+1-i}$, where D^{2d+1-i} is a disk of dimension $2d + 1 - i$.

The *thickening of the cone on L* of dimension d , denoted by $\text{Th}(\text{Cone}L)$, is the disk D^{2d+2} , together with the distinguished submanifold of $\partial D^{2d+2} = S^{2d+1}$, which is defined as the thickening of L in S^{2d+1} . (Note: the cone on L has dimension $d + 1$, thus it has an usual thickening in S^{2d+3} . But since it is a cone, we can find a thickening of lower dimension.)

6.1.2 Thickened dual cones

Let L be a simplicial complex and σ is a simplex in L . Recall that the dual cone, D_σ , of σ is the cone on the barycentric subdivision of the link of σ in L . In other words, let $\mathcal{P} = \mathcal{P}(L)$, then $D_\sigma = |\mathcal{P}_{\geq \sigma}|$ is its dual cone. Both σ and its dual cone D_σ are subcomplexes of the barycentric subdivision of L . If the codimension of σ in L is k , then

- $\dim Lk'_\sigma = k - 1$,
- $\text{Th}Lk'_\sigma \subset S^{2k-1}$ is a submanifold of codimension 0,
- and $\text{Th}D_\sigma$ is a disk of dimension $2k$.

Next suppose that $\tau > \sigma$ is a coface of σ of codimension l in L (i.e., σ is a subsimplex of τ and $l < k$). In Lk'_σ , τ is a vertex and the thickening of τ in the thickening of Lk'_σ , denoted by $T(\tau, Lk'_\sigma)$, is a disk of dimension $2k - 1$. We have

- $D_\tau \subset Lk'_\sigma$,
- $\text{Th}D_\tau$ is a $2l$ -disk, which can be embedded in the $(2k - 1)$ -disk, $T(\tau, Lk'_\sigma)$.

Thus, $T(\tau, Lk'_\sigma)$ splits as product of 2 disks

$$T(\tau, Lk'_\sigma) = D^{2(k-l)-1} \times D^{2l}.$$

6.2 The construction

For each simplex σ of dimension i of the poset of groups $\mathbf{G}(\mathcal{P}(L))$, there is an aspherical manifold M_σ of dimension $2i + 1$. The fundamental group of M_σ is the local group at σ .

Define

$$M(D_\sigma) := M_\sigma \times \text{Th}D_\sigma.$$

Since the thickening of the dual cone is a disk, $M(D_\sigma)$ contracts to M_σ . If the dimension of L is d , then the codimension of σ is $k = d - i$. Hence, the dimension of $\text{Th} D_\sigma$ is $2k = 2(d - i)$. And the dimension of $M(D_\sigma)$ is $2i + 1 + 2(d - i) = 2d + 1$. The dimension of $M(D_\sigma)$ does not depend on σ .

So we define a manifold of dimension $2d + 1$, which contracts to M_σ for σ is a simplex in L . We will use these manifolds to construct a manifold for the Artin group.

Gluing. The idea of the construction is that we will glue together the $M(D_\sigma)$ along parts of their boundaries in the same way the dual cones are glued together to get the barycentric subdivision of L .

Let $\sigma < \tau$ are simplices in L , $\dim \sigma = i$, $\dim \tau = j$. We have the boundaries of $M(D_\sigma) = M_\sigma \times \text{Th}D_\sigma$ and $M(D_\tau) = M_\tau \times \text{Th}D_\tau$ are

$$\partial M(D_\sigma) = \partial M_\sigma \times \text{Th}D_\sigma \cup M_\sigma \times \partial \text{Th}D_\sigma$$

$$\partial M(D_\tau) = \partial M_\tau \times \text{Th}D_\tau \cup M_\tau \times \partial \text{Th}D_\tau$$

Consider two parts of these boundaries: $M_\sigma \times \partial \text{Th}D_\sigma$ and $\partial M_\tau \times \text{Th}D_\tau$, they have a

common codimension 0 submanifold

$$M_\sigma \times D^{2(j-i)-1} \times D^{2(d-j)}.$$

This follows from the following observations:

- $T(\tau, Lk'_\sigma)$ is a $2(d-i) - 1$ -disk in $\partial \text{Th}D_\sigma$ and there is the splitting $T(\tau, Lk'_\sigma) = D^{2(j-i)-1} \times D^{2(d-j)}$,
- $\text{Th} D_\tau = D^{2(d-j)}$, and $M_\sigma \times D^{2(j-i)-1} \subset \partial M_\tau$.

We can do this procedure for all simplices in L and construct a manifold

$$M(L) := \left(\coprod_{\sigma \in \mathcal{P}} M(D_\sigma) \right) / \sim .$$

Claim 3. *The result of the above construction is a manifold.*

Proof. The only problem that prevents the construction result to be a manifold is that the same piece (or some part of it) of boundary of some local manifold is glued more than one time in $\partial M_\tau \times \text{Th}D_\tau$. But this is not the case. By lemma 4.4, we can choose a collection of disjoint sub-manifolds M_σ for M_τ . And for each such M_σ , choose a small disk around it to get $M_\sigma \times D^{2(j-i)-1}$.

□

The main theorem of this dissertation, Theorem 1.1, follows from the following claim.

Claim 4. *The manifold $M(L)$ is aspherical.*

Proof. We know that $U(G, |\mathcal{P}(L)|)$ is contractible. Since L is simply connected (it is contractible), by proposition 2.11, $BG(\mathcal{P}(L))$ is homotopy equivalent to the Eilenberg-MacLane space BG . And any realization is homotopy equivalent to $BG(\mathcal{P}(L))$ by Lemma 3.4. Thus, the realization by aspherical manifolds M is aspherical.

□

Since the manifold $M(L)$ we constructed is aspherical, its universal cover is contractible and it admits a proper action from the Artin group. This implies that the action dimension of the Artin group is less than or equal to the dimension of $M(L)$, which is $2n + 1$.

Remark on Corollary 1.2. When the nerve of the Artin group is a flag complex, we can prove Corollary 1.2 directly using lemma 3.5 and the fact that we can embed L into a contractible flag complex of the same dimension.

Proof of Corollary 1.3. From Chapter 4, we know that the action dimension of an irreducible spherical Artin group with the nerve σ a simplex of dimension n is $2n + 1$. We also know that for any subgroup G of a group Γ , we have

$$\text{actdim}(G) \leq \text{actdim}(\Gamma).$$

Thus, if an Artin group A_L has a subgroup which is an irreducible spherical Artin group A_σ with $\dim(L) = \dim(\sigma) = n$, then $\text{actdim}(A_L) \geq \text{actdim}(A_\sigma) = 2n + 1$. The main theorem 1.1 implies that $\text{actdim}(A_L) = 2n + 1$.

6.3 Discussion

The case when the standard spherical subgroups of A_L are not irreducible has considerable difficulty in finding the action dimension. Let take the RAAGs as examples. In this case, by Theorem 4.7, we know that the action dimension is the maximal $(2d + 2)$ when the top homology group of L with $\mathbb{Z}/2$ -coefficients is non-trivial, otherwise the action dimension is $\leq 2d + 1$. The case when the dimension of L is 1 is fully classified in general case (for Artin groups) by Gordon [Gor04]. The main result of [Gor04] states that an Artin group is 3-manifold group if and only if it is \mathbb{Z}^3 or L is a tree. If L is a graph and has a cycle, then the action dimension of the Artin group is 4. Let consider the case when dimension of L is 2 and A_L is a RAAG. If L is 2-simplex, then $A_L = \mathbb{Z}^3$. Thus, $\text{actdim}(A_L) = 3$. The link of a vertex of L is a graph. If L contains a vertex, whose link has nontrivial first homology (thus, it has a cycle). Then the action dimension is at least 5 because A_L has a subgroup of the form $\mathbb{Z} \times A_\Gamma$, when Γ is a graph and not a tree (A_Γ is a 4-manifold group by the result of Gordon [Gor04]). It seems that for other cases, the action dimension of A_L is 4, but it is not clear how to construct an aspherical 4-manifold for A_L (if we use $T^3 \times I$ as a local manifold for each 2-simplex, where $T^3 = S^1 \times S^1 \times S^1$ - 3-dimensional torus and I is

an interval, there is not enough components of its boundary to do the gluing).

We may try to reduce the dimension of the manifold in our construction. For this, we need to reduce the dimension of thickening of the dual cells and the dimension of the local manifolds. We must maintain the condition that for $\sigma < \tau$, M_σ sits in the boundary of M_τ . In the case of RAAGs, this means that the manifold corresponding to an edge should be $T^2 \times I$ (so it contains S^1 in its boundary), the manifold corresponding to a triangle should be $T^3 \times D^2$ (then it contains $T^2 \times I$ in its boundary). In general, the manifold corresponding to an n -simplex should be $T^{n+1} \times D^n$. This makes the dimension of the local manifold of the top simplex in L in the RAAGs case is exactly $2n + 1$.

In the upcoming work [DH16], Davis and Huang prove that if $H^n(L, \mathbb{Z}/2) \neq 0$, where n is the dimension of L , then the action dimension of A_L is $2(n + 1)$. We prove that if $H^n(L, \mathbb{Z}) = 0$ then the action dimension of A_L is $2n + 1$. There is a gap between the case $H^n(L, \mathbb{Z}) \neq 0$ and $H^n(L, \mathbb{Z}/2) \neq 0$. We can have $H^n(L, \mathbb{Z}) \neq 0$ but $H^n(L, \mathbb{Z}/2) = 0$. Thus, we have the following question:

Question 1. Can we replace the $\mathbb{Z}/2$ by \mathbb{Z}/p for p - a prime in the result of [DH16]? i.e., does it true that if $H^n(L, \mathbb{Z}/p) \neq 0$, where n is the dimension of L , then the action dimension of A_L is $2(n + 1)$.

Another direction we can investigate is that if we have a situation like in the Artin group case, e.g., if there is a collection of aspherical manifolds, one is contained in the boundary of the other, exactly like the manifolds for spherical Artin groups, we can construct a manifold for the fundamental group of the poset of groups. One candidate we have in mind is the mapping class groups $\text{Mod}(S_{g,n})$ of $S_{g,n}$ - a Riemann surface of genus g with n punctures. The action dimension of the mapping class groups was studied in [Des06].

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