KAC-MOODY GROUPS:
SPLIT AND RELATIVE THEORIES. LATTICES

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Abstract.— *In this survey article, we recall some facts about split Kac-Moody groups as defined by J. Tits, describe their main properties and then propose an analogue of Borel-Tits theory for a non-split version of them. The main result is a Galois descent theorem, i.e., the persistence of a nice combinatorial structure after passing to rational points. We are also interested in the geometric point of view, namely the production of new buildings admitting (nonuniform) lattices.*

CONTENTS

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INTRODUCTION

Historical sketch of Kac-Moody theory.— Kac-Moody theory was initiated in 1968, when V. Kac and R. Moody defined independently infinite-dimensional Lie algebras generalizing complex semisimple Lie algebras. Their definition is based on Serre's presentation theorem describing explicitly the latter (finite-dimensional) Lie algebras [Hu1, 18.3]. A natural question then is to integrate Kac-Moody Lie algebras as Lie groups integrate real Lie algebras, but this time in the infinite-dimensional setting. This difficult problem led to several propositions. In characteristic 0, a satisfactory approach consists in seeing them as subgroups in the automorphisms of the corresponding Lie algebras [KP1,2,3]. This way, V. Kac and D. Peterson developed the structure theory of Kac-Moody algebras in complete analogy with the classical theory: intrinsic definition and conjugacy results for Borel (resp. Cartan) subgroups, root decomposition with abstract description of the root system... Another aspect of this work is the construction of generalized Schubert varieties. These algebraic varieties enabled O. Mathieu to get a complete generalization of the character formula in the Kac-Moody framework [Mat1]. To this end, O. Mathieu defined Kac-Moody groups over arbitrary fields in the formalism of ind-schemes [Mat2].

Combinatorial approach.— Although the objects above – Kac-Moody groups and Schubert varieties – can be studied in a nice algebro-geometric context, we will work with groups arising from another, more combinatorial viewpoint. All of this work is due to J. Tits [T4,5,6,7], who of course contributed also to the previous problems. The aim is to get a much richer combinatorial structure for these groups. This led J. Tits to the notion of «Root Group Datum» axioms [T7] whose geometric counterpart is the theory of Moufang twin buildings. The starting point of the construction of Kac-Moody groups [T4] is a generalization of Steinberg's presentation theorem [Sp, Theorem 9.4.3] which concerns simply connected semisimple algebraic groups. In this context, the groups $\mathrm{SL}_n(\mathbf{K}[t,t^{-1}])$ are Kac-Moody groups obtained via Tits' construction.

Relative Kac-Moody theory in characteristic 0.— So far, the infinite-dimensional objects alluded to were analogues of split Lie algebras and split algebraic groups. Still, it is known that by far not all interesting algebraic groups are covered by the split theory – just consider the simplest case of the multiplicative group of a quaternion skewfield. This is the reason why Borel-Tits theory [BoT] is so important: it deals with algebraic groups over arbitrary fields $\mathbf{K}$, and the main results are a combinatorial structure theorem for $\mathbf{K}$-points, conjugacy theorems for minimal $\mathbf{K}$-parabolic subgroups (resp. for maximal $\mathbf{K}$-split tori). This theory calls for a generalization in the Kac-Moody setting: this work was achieved in the characteristic 0 case by G. Rousseau [Rou1,2,3; $\mathrm{B_3R}$].

Two analogies.— The example of the groups $\mathrm{SL}_n(\mathbf{K}[t,t^{-1}])$ is actually a good guideline since it provides at the same time another analogy for Kac-Moody groups. Indeed, over finite fields $\mathbf{K} = \mathbf{F}_q$, the former ones are arithmetic groups in the function field case. Following the first analogy – a Kac-Moody group is an infinite-dimensional reductive group, a large part of the present paper describes the author's thesis [Ré2] whose basic goal is to define $\mathbf{K}$-forms of Kac-Moody groups with no assumption on the groundfield $\mathbf{K}$. This prevents from using the viewpoint of automorphisms of Lie algebras as in the works previously cited, but the analogues of the main results of Borel-Tits theory are proved. This requires first to reconsider Tits' construction of Kac-Moody groups and to prove results in this (split) case, interesting in their own right. On the other hand, the analogy with arithmetic groups will also be discussed so as to see Kac-Moody groups as discrete groups – lattices for $\mathbf{F}_q$ large enough – of their geometries.

Tools.— Let us talk about tools now, and start with the main difficulty: no algebro-geometric structure is known for split Kac-Moody groups as defined by J. Tits. The idea is to replace this structure by two well-understood actions. The first one is not so mysterious since it is linear: it is in fact the natural generalization of the adjoint representation of algebraic groups. It plays a crucial role because it enables to endow a large family of subgroups with a structure of algebraic group. The second kind of action involves buildings: it is a usual topic in group theory to define a suitable geometry out of a given group to study it (use of Cayley graphs, of boundaries...) Kac-Moody groups are concerned by building techniques thanks to their nice combinatorial properties, refining that of BN-pairs (for an account on the general use of buildings in group theory, see [T2]). There exists some kind of (non-univoque) correspondence between buildings and BN-pairs [Ron §5], but it is too general to provide precise information in specific situations. Some refinements are to be adjusted accordingly – see the example of Euclidean buildings and «Valuated Root Data» in Bruhat-Tits theory [BrT1]. As already said, in the Kac-Moody setting the refinements consist in requiring the «Root Group Datum» axioms at the group level and in working with twin buildings at the geometric level. Roughly speaking, a twin building is the datum of two buildings related by opposition relations between chambers and apartments.

Organization of the paper.— This article is divided into four parts. Part 1 deals with group combinatorics in a purely abstract context. Since the combinatorial axioms will be satisfied by both split and non-split groups, it is an efficient way to formalize properties shared by them. There is described the geometry of twin buildings and the corresponding group theoretic axioms. The aim is to obtain two kinds of Levi decomposition of later interest. Part 2 describes the split theory of Kac-Moody groups. In particular, we explain why the adjoint representation can be seen as a substitute for a global algebro-geometric structure. An illustration of this is a repeatedly used argument combining negative curvature and algebraic groups arguments. Part 3 presents the relative theory of almost split Kac-Moody groups. A sketch of the proof of the structure theorem for rational points is given. The particular case of a finite groundfield is considered, as well as a classical example of a twisted group leading to a semi-homogeneous twin tree. At last, part 4 adopts the viewpoint of discrete groups. We first show that Kac-Moody theory enables to produce hyperbolic buildings (among many other possibilities) and justify why these geometries are particularly interesting. Then we show that Kac-Moody groups or their spherical parabolic subgroups over a finite groundfield are often lattices of their buildings. This leads to an analogy with arithmetic lattices over function fields.

The assumed knowledge for this article consists of general facts from building theory and algebraic groups. References for buildings are K. Brown's book [Br2] for the apartment systems viewpoint and M. Ronan's book [Ron] for the chamber systems one. Concerning algebraic groups, the recent books [Bo] and [Sp] are the main references dealing with relative theory.

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1. ABSTRACT GROUP COMBINATORICS AND TWIN BUILDINGS

The aim of this section is to provide all the abstract background we will need to study split and almost split Kac-Moody groups. In §1.1 is introduced the root system of a Coxeter group. These roots will index the group combinatorics of the «Twin Root Datum» axioms presented in §1.2. Then we describe in §1.3 the geometric side of the (TRD)-groups, that is the twin buildings on which they operate. Section 1.4 is dedicated to the geometric notions and realizations to be used later. At last, §1.5 goes back to group theory providing some semi-direct decomposition results for distinguished classes of subgroups.

1.1. Root system and realizations of a Coxeter group

The objects we define here will be used to index group theoretic axioms (1.2.A) and to describe the geometry of buildings (1.3.B).

A. Coxeter complex. Root systems. Let $M = [M_{st}]_{s,t \in S}$ denote a Coxeter matrix with associated Coxeter system (W, S). For our purpose, it is sufficient to suppose the canonical generating set S finite. The group W admits the following presentation:

$$W = \langle s \in S \mid (st)^{M_{st}} = 1 \text{ whenever } M_{st} < \infty \rangle.$$

We shall use the *length function* $\ell : W \to \mathbf{N}$ defined w.r.t. S.

The existence of an abstract simplicial complex acted upon by W is the starting point of the definition of buildings of type (W, S) in terms of apartment systems. This complex is called the *Coxeter complex* associated to W, we will denote it by $\Sigma(W, S)$ or Σ. It describes the combinatorial geometry of «slices» in a building of type (W, S) – the so-called *apartments*. The abstract complex Σ is made of translates of the *special subgroups* $W_J := \langle J \rangle$, $J \subset S$, ordered by inclusion [Br2, p.58-59].

The *root system* of (W, S) is defined by means of the length function ℓ [T4, §5]. The set W admits a W-action via left translations. *Roots* are distinguished halves of this W-set, whose elements will be called *chambers*.

Definition.— (i) *The* simple root *of index s is the half $\alpha_s := \{w \in W \mid \ell(sw) > \ell(w)\}$.*
(ii) *A* root *of W is a half of the form $w\alpha_s$, $w \in W, s \in S$. The set of roots will be denoted by Φ, it admits an obvious W-action.*
(iii) *A root is called* positive *if it contains* 1*; otherwise, it is called* negative. *Denote by Φ_+ (resp. Φ_-) the set of positive (resp. negative) roots of Φ.*
(iv) *The* opposite *of a root is its complement.*

Remark.— The opposite of the root $w\alpha_s$ is indeed a root since it is $ws\alpha_s$.

The next definitions are used for the group combinatorics presented in 1.2.A.

Definition.— (i) *A pair of roots $\{\alpha; \beta\}$ is called* prenilpotent *if both intersections $\alpha \cap \beta$ and $(-\alpha) \cap (-\beta)$ are nonempty.*
(ii) *Given a prenilpotent pair of roots $\{\alpha; \beta\}$, the* interval $[\alpha; \beta]$ *is by definition the set of roots γ with $\gamma \supset \alpha \cap \beta$ and $(-\gamma) \supset (-\alpha) \cap (-\beta)$. We also set $]\alpha; \beta[:= [\alpha; \beta] \setminus \{\alpha; \beta\}$.*

B. The Tits cone. Linear refinements. We introduce now a fundamental realization of a Coxeter group: the *Tits cone*. It was first defined in [Bbk, V.4] to which we refer for proofs. This approach will also allow to consider roots as linear forms [Hu2, II.5]. We keep the

Coxeter system (W, S) with Coxeter matrix $M = [M_{st}]_{s,t\in S}$. Consider the real vector space V over the symbols $\{\alpha_s\}_{s\in S}$ and define the *cosine matrix* A of W by $A_{st} := -\cos(\pi/M_{st})$: this is the matrix of a bilinear form B (w.r.t. the basis $\{\alpha_s\}_{s\in S}$). To each s of S is associated the involution $\sigma_s : \lambda \mapsto \lambda - 2B(\alpha_s, \lambda)\alpha_s$, and the assignment $s \mapsto \sigma_s$ defines a faithful representation of W. The *(positive) half-space* in V^* of an element λ in V is denoted by $D(\lambda)$: $D(\lambda) := \{x \in V^* \mid \lambda(x) > 0\}$. We denote its boundary – the kernel of λ – by $\partial\lambda$.

Definition.— (i) *The* standard chamber c *is the simplicial cone* $\displaystyle\bigcap_{s\in S} D(\alpha_s)$ *of elements of* V^* *on which all linear forms* α_s *are positive. A* chamber *is a* W*-translate of* c.
(ii) *The* standard facet of type J, $J \subset S$, *is* $F_I := \displaystyle\bigcap_{s\in I} \partial\alpha_s \cap \bigcap_{s\in S\setminus I} D(\alpha_s)$. *A* facet of type J *is a* W*-translate of* F_J.
(ii) *The* Tits cone $\overline{\mathcal{C}}$ *of* W *is the union of the closures of all chambers* $\displaystyle\bigcup_{w\in W} w\overline{c} = \bigsqcup_{w\in W, J\subset S} wF_J$.

A study of the action of W on $\overline{\mathcal{C}}$ shows that the type is well-defined. A facet is *spherical* if it is of type J with W_J finite. Facets of all types are represented here, as simplicial cones. The simplicial complex so obtained is not locally finite in general, but its interior $\mathcal{C}$ is. In fact, a facet is of nonspherical type if and only if it lies in the boundary of the Tits cone. Further properties of the cone $\overline{\mathcal{C}}$ are available in [V].

Example.— In the case of an affine reflexion group, the Tits cone is made of the union of an open half-space and the origin, which is the only nonspherical facet. The affine space in which the standard representation of the group is defined is just the affinisation of this cone [Hu2, II.6].

The viewpoint of linear forms for roots enables to introduce another – more restrictive – notion of interval of roots.

Definition.— *Given two roots* α *and* β, *the* linear interval *they define is the set* $[\alpha;\beta]_{\mathrm{lin}}$ *of positive linear combinations of them, seen as linear forms on* V.

Figure.—

The picture above illustrates a general fact: there are two ways to be prenilpotent. Either the walls of the roots intersect along a spherical facet – and the four pairs of roots of the form $\{\pm\alpha; \pm\beta\}$ are prenilpotent, or a root contains the other – and only two pairs among the four are prenilpotent.

Remark.— For a prenilpotent pair of roots, one has $[\alpha;\beta]_{\mathrm{lin}} \subset [\alpha;\beta]$, still the notions do not coincide, even in the Kac-Moody situation. G. Rousseau indicated an example of strict inclusion provided by a tesselation of the hyperbolic plane $\mathbb{H}^2$ [Ré2, 5.4.2].

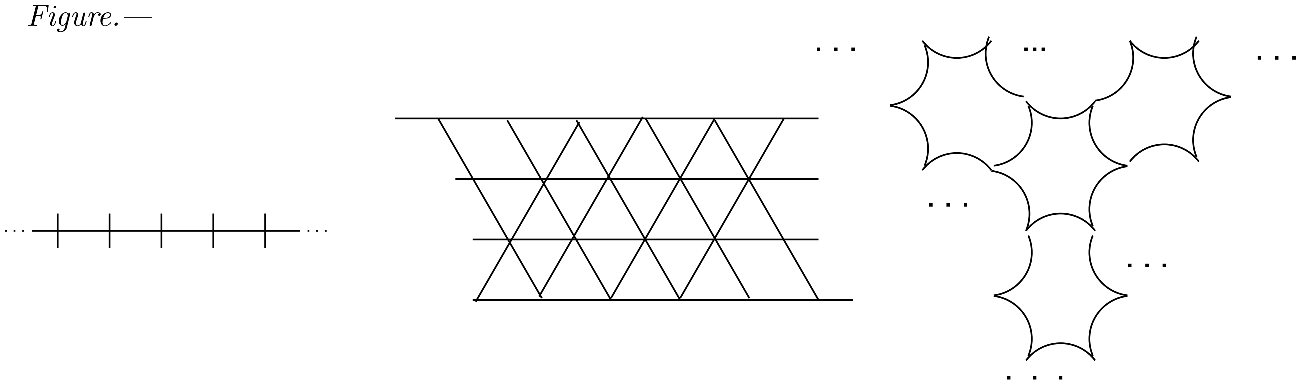
C. Geometric realizations. We will try to use geometry as much as possible instead of abstract set-theoretic structures. As an example, we will often represent Coxeter complexes by *polyhedral complexes* [BH]. We are interested in such spaces with the following additional properties.

(i) The complex is labelled by a fixed set of subsets of S – the *types*. We call *facets* the polyhedra inside. Codimension 1 facets are called *panels*, maximal facets are called *chambers*.
(ii) There is a countable family of codimension 1 subcomplexes – the *walls* – w.r.t. which are defined involutions – the *reflections*. A reflection fixes its wall and stabilizes the whole family of them.
(iii) For a chamber c, there is a bijection between the set of generators S and the walls supporting the panels of c. The corresponding reflections define a faithful representation of W by label-preserving automorphisms of the complex.
(iv) The W-action is simply transitive on chambers.

Remarks.— 1. The Tits cone satisfies all the conditions above, with simplicial cones as facets instead of polyhedra.
2. We may use geometric realizations where only spherical types appear. This is the case in the examples below.

Examples.— 1. The simplest example with infinite Coxeter group is given by the real line and its tesselation by the segments defined by consecutive integers. It is acted upon by the infinite dihedral group D_∞, and the corresponding buildings are trees.
2. Another famous example associated to an affine reflection group comes from the tesselation of the Euclidean plane by equilateral triangles. Buildings with this geometry as apartments are called $\widetilde{A}_2$- or *triangle buildings*. They are interesting because, even if they belong to the well-known class of *Euclidean buildings*, many of them do not come from Bruhat-Tits theory.
3. A well-known way to construct concretely a Coxeter group with a realization of its Coxeter complex is to apply Poincaré polyhedron theorem [Mas, IV.H.11]. This works in the framework of spherical, Euclidean or hyperbolic geometry; we just have to consider reflections w.r.t. to a suitable polyhedron. For a tiling of a hyperbolic space $\mathbb{H}^n$, the corresponding buildings are called *hyperbolic*, *Fuchsian* in the two-dimensional case.

Figure.—



1.2. Axioms for group combinatorics

We can now give axioms refining BN-pairs and adapted to the Kac-Moody situation.

A. The (TRD) axioms. The axioms listed below are indexed by the set of roots Φ of the Coxeter system (W, S). It is a slight modification of axioms proposed in [T7]. That it implies the group combinatorics introduced by V. Kac and D. Peterson [KP2] follows from [Ré2, Théorème 1.5.4], which elaborates on [Ch] and [T3].

Definition.— *Let G be an abstract group containing a subgroup H. Suppose G is endowed with a family $\{U_\alpha\}_{\alpha\in\Phi}$ of subgroups indexed by the set of roots Φ, and define the subgroups $U_+ := \langle U_\alpha \mid \alpha\in\Phi_+\rangle$ and $U_- := \langle U_\alpha \mid \alpha\in\Phi_-\rangle$. Then, the triple $\big(G,\{U_\alpha\}_{\alpha\in\Phi},H\big)$ is said to satisfy the* (TRD) axioms *if the following conditions are satisfied.*

(TRD0) *Each U_α is nontrivial and normalized by H.*

(TRD1) *For each prenilpotent pair of roots $\{\alpha;\beta\}$, the commutator subgroup $[U_\alpha,U_\beta]$ is contained in the subgroup $U_{]\alpha,\beta[}$ generated by the U_γ's, with $\gamma\in]\alpha,\beta[$.*

(TRD2) *For each s in S and u in $U_{\alpha_s}\setminus\{1\}$, there exist uniquely defined u' and u'' in $U_{-\alpha_s}\setminus\{1\}$ such that $m(u) := u'uu''$ conjugates U_β onto $U_{s\beta}$ for every root β. Besides, it is required that for all u and v in $U_{\alpha_s}\setminus\{1\}$, one should have $m(u)H = m(v)H$.*

(TRD3) *For each s in S, $U_{\alpha_s}\not\subset U_-$ and $U_{-\alpha_s}\not\subset U_+$.*

(TRD4) *G is generated by H and the U_α's.*

Such a group will be referred to as a (TRD)-group*. It will be called a* $(\mathrm{TRD})_{\mathrm{lin}}$-group *or said to satisfy the* $(\mathrm{TRD})_{\mathrm{lin}}$ axioms *if* (TRD1) *is still true after replacing intervals by linear ones.*

Remarks.— 1. A consequence of Borel-Tits theory [BoT, Bo, Sp] is that isotropic reductive algebraic groups satisfy $(\mathrm{TRD})_{\mathrm{lin}}$ axioms. We will see in 2.3.A that so do split Kac-Moody groups. The case of nonsplit Kac-Moody groups is the object of the Galois descent theorem – see 3.2.A.
2. The case of algebraic groups suggests to take into account more carefully proportionality relations between roots seen as linear forms. It is indeed possible to formalize the difference between reduced and non-reduced infinite root systems, and to derive refined $(\mathrm{TRD})_{\mathrm{lin}}$ axioms [Ré2, 6.2.5].

B. Main consequences. We can derive a first list of properties for a (TRD)-group G.

Two BN-pairs.— The main point is the existence of two BN-pairs in the group G. Define the *standard Borel subgroup of sign ϵ* to be $B_\epsilon := HU_\epsilon$. The subgroup $N < G$ is by definition generated by H and the $m(u)$'s of axiom (TRD2). Then, one has

$$H = \bigcap_{\alpha\in\Phi} N_G(U_\alpha) = B_+\cap N = B_-\cap N,$$

and (G,B_+,N,S) and (G,B_-,N,S) are BN-pairs sharing the same Weyl group $W = N/H$. As B_+ and B_- are not conjugate, the positive and the negative BN-pairs do not carry the same information. A conjugate of B_+ (resp. B_-) will be called a *positive (resp. negative) Borel subgroup.*

Refined Bruhat and Birkhoff decompositions.— A formal consequence of the existence of a BN-pair is a Bruhat decomposition for the group. In our setting, the decomposition for each sign can be made more precise. For each $w\in W$, define the subgroups $U_w := U_+\cap wU_-w^{-1}$ and $U_{-w} := U_-\cap wU_+w^{-1}$. The *refined Bruhat decompositions* are then [KP2, Proposition 3.2]:

$$G = \bigsqcup_{w\in W} U_w w B_+ \quad\text{and}\quad G = \bigsqcup_{w\in W} U_{-w} w B_-,$$

with uniqueness of the first factor. A third decomposition involves both signs and will be used to define the twinned structures (1.3.A). More precisely, the *refined Birkhoff decompositions* are [KP2, Proposition 3.3]:

$$G = \bigsqcup_{w\in W} (U_+ \cap wU_+w^{-1})wB_- = \bigsqcup_{w\in W} (U_- \cap wU_-w^{-1})wB_+,$$

once again with uniqueness of the first factors.

Other unique writings.— Another kind of unique writing result is valid for the groups $U_{\pm w}$. For each $z \in W$, define the (finite) sets of roots $\Phi_z := \Phi_+ \cap z^{-1}\Phi_-$ and $\Phi_{-z} := \Phi_- \cap z^{-1}\Phi_+$. Then, the group U_w (resp. U_{-w}) is in bijection with the set-theoretic product of the root groups indexed by $\Phi_{w^{-1}}$ (resp. $\Phi_{-w^{-1}}$) for a suitable (cyclic) ordering on the latter set [T4, proposition 3 (ii)], [Ré2, 1.5.2].

Two buildings.— Let us describe now how to construct a building out of each BN-pair, the twin structure relating them being defined in 1.3.A. Fix a sign ϵ and consider the corresponding BN-pair (G, B_ϵ, N, S). Let $d_\epsilon : G/B_\epsilon \times G/B_\epsilon \to W$ be the application which associates to $(gB_\epsilon, hB_\epsilon)$ the element w such that $B_\epsilon g^{-1}hB_\epsilon = B_\epsilon wB_\epsilon$. Then, d_ϵ is a W-*distance* making $(G/B_\epsilon, d_\epsilon)$ a building [Ron, 5.3]. The *standard apartment of sign* ϵ is $\{wB_\epsilon\}_{w\in W}$, the relevant apartment system being the set of its G-transforms. A *facet* of type J is a translate $gP_{\epsilon,J}$ of a *standard parabolic subgroup* $P_{\epsilon,J} := B_\epsilon W_J B_\epsilon$.

C. *Examples.—* The most familiar examples of groups enjoying the properties above are provided by Chevalley groups over Laurent polynomials. These groups are Kac-Moody groups of affine type. We briefly describe the case of the special linear groups $\mathrm{SL}_n(\mathbf{K}[t,t^{-1}]), n \geq 2$. From the Kac-Moody viewpoint, the groundfield is $\mathbf{K}$. The buildings involved are Bruhat-Tits. They are associated to the p-adic Lie groups $\mathrm{SL}_n\big(\mathbf{F}_q(\!(t)\!)\big)$ and $\mathrm{SL}_n\big(\mathbf{F}_q(\!(t^{-1})\!)\big)$ respectively in the case of a finite groundfield $\mathbf{F}_q$. The Weyl group is the affine reflection group $\mathcal{S}_n \ltimes \mathbf{Z}^{n-1}$. The Borel subgroups are

$$B_+ := \{M \in \begin{pmatrix} \mathbf{K}[t] & & \mathbf{K}[t] \\ & \ddots & \\ t\mathbf{K}[t] & & \mathbf{K}[t] \end{pmatrix} \mid \det M = 1\};$$

$$B_- := \{M \in \begin{pmatrix} \mathbf{K}[t^{-1}] & & t^{-1}\mathbf{K}[t^{-1}] \\ & \ddots & \\ \mathbf{K}[t^{-1}] & & \mathbf{K}[t^{-1}] \end{pmatrix} \mid \det M = 1\}.$$

As subgroup H, we take the standard Cartan subgroup T of $\mathrm{SL}_n(\mathbf{K})$ made of diagonal matrices with coefficients in $\mathbf{K}^\times$ and determinant 1. For $1 \leq i \leq n-1$, the monomial matrices with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in position $(i, i+1)$ on the diagonal (and 1's everywhere else on it) lift the $n-1$ simple reflexions generating the finite Weyl group of $\mathrm{SL}_n(\mathbf{K})$. The last reflexion «responsible for the affinisation» is lifted by

$$N_n := \begin{pmatrix} 0 & 0 & -t^{-1} \\ 0 & \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} & 0 \\ t & 0 & 0 \end{pmatrix}.$$

The situation is the same for root groups: besides the simple root groups of $\mathrm{SL}_n(\mathbf{K})$, one has to add

$$U_n := \{u_n(k) := \begin{pmatrix} 1 & 0 & 0 \\ & \ddots & \\ 0 & 1 & 0 \\ & & \ddots \\ kt & 0 & 1 \end{pmatrix} \mid k \in \mathbf{K}\},$$

to get the complete family of subgroups indexed by the simple roots.

1.3. Moufang twin buildings

The theory of *Moufang twin buildings* is the geometric side of the group combinatorics above. A good account of their general theory is [A1]. It was initiated in [T6], [T7] and *a posteriori* in [T1], whereas [RT2,3] deals with the special case of twin trees. References for the classification problem are [MR] for the unicity step, and then B. Mühlherr's work, in particular [Mü] and his forthcoming habilitationschrift.

A. Twin buildings. The definition of a *twin building* is quite similar to that of a building in terms of W-distance [T7, A1 §2].

Definition.— *A* twin building of type (W,S) *consists in two buildings* $(\mathcal{I}_+, d_+)$ *and* $(\mathcal{I}_-, d_-)$ *of type* (W,S) *endowed with a* $(W$-$)$codistance. *By definition, the latter is an application* $d^* : (\mathcal{I}_+ \times \mathcal{I}_-) \cup (\mathcal{I}_- \times \mathcal{I}_+) \to W$ *satisfying the following conditions for each sign* ϵ *and all chambers* x_ϵ *in* $\mathcal{I}_\epsilon$ *and* $y_{-\epsilon}$, $y'_{-\epsilon}$ *in* $\mathcal{I}_{-\epsilon}$.

(TW1) $d^*(y_{-\epsilon}, x_\epsilon) = d^*(x_\epsilon, y_{-\epsilon})^{-1}$.
(TW2) *If* $d^*(x_\epsilon, y_{-\epsilon}) = w$ *and* $d_{-\epsilon}(y_{-\epsilon}, y'_{-\epsilon}) = s \in S$ *with* $\ell(ws) < \ell(w)$, *then* $d^*(x_\epsilon, y'_{-\epsilon}) = ws$.
(TW3) *If* $d^*(x_\epsilon, y_{-\epsilon}) = w$ *then for each* $s \in S$, *there exists* $z_{-\epsilon} \in \mathcal{I}_{-\epsilon}$ *with* $d_{-\epsilon}(y_{-\epsilon}, z_{-\epsilon}) = s$ *and* $d^*(x_\epsilon, z_{-\epsilon}) = ws$.

From this definition can be derived two *opposition relations.* Two chambers are *opposite* if they are at codistance 1. Given an apartment $\mathbf{A}_\epsilon$ of sign ϵ, an *opposite* of it is an apartment $A_{-\epsilon}$ such that each chamber of $\mathbf{A}_\epsilon$ admits exactly one opposite in $\mathbf{A}_{-\epsilon}$. In this situation, the same assertion is true after inversion of signs. An apartment admits at most one opposite, and the set of apartments having an opposite forms an apartment system in the building [A1 §2].

Remark.— We defined two opposition relations, but we have to be careful with them. Whereas an element of the apartment system defined above admits by definition exactly one opposite, a chamber admits many opposites in the building of opposite sign. In the sequel, the bold letter $\mathbf{A}$ will refer to a pair $(\mathbf{A}_+, \mathbf{A}_-)$ of opposite apartments.

The connection with group combinatorics is folklore [Ré2, 2.6.4].

Proposition.— *Let G be a* (TRD)*-group. Then the buildings associated to the two BN-pairs of G are Moufang and belong to a twin building structure. In particular, two facets of opposite signs are always contained in a pair of opposite apartments. At last, G is transitive on pairs of opposite chambers.* □

So far, we explained how to derive two buildings from (TRD)-groups. The main consequences of Moufang property will be described in the next subsection. So what is left to do is to define the codistance. Whereas W-distance is deduced from Bruhat decomposition, the W-codistance can be made completely explicit thanks to the Birkhoff decomposition 1.2.B. Two chambers of opposite signs gB_+ and hB_- are at *codistance* w if and only if $g^{-1}h$ is in the Birkhoff class B_+wB_-. This definition does not depend on the choice of g and h in their class.

Examples.— 1. The class of twin trees has been studied in full generality in [RT2,3] where many properties are shown. For instance, the trees have to be at least semi-homogeneous. The proofs are made simpler than in the general case thanks to the use of an integral codistance which faithfully reflects the properties of the W-codistance.
2. For the groups $\mathrm{SL}_n(\mathbf{K}[t,t^{-1}])$ of 1.2.C, the opposition relation can be formulated more concretely. Two chambers gB_+ and hB_- are opposite if and only if their stabilizers – Borel subgroups of opposite signs – intersect along a conjugate of T. Over $\mathbf{F}_q$, this is equivalent to intersecting along a group of (minimal) order $(q-1)^{n-1}$.

B. Geometric description of the group action. Suppose we are given a (TDR)-group G and denote by $\mathbf{A}$ the standard pair of opposite apartments in its twin building, that is $\{wB_+\}_{w\in W} \sqcup \{wB_-\}_{w\in W}$. We want to give a geometric description of the G-action on its twin building.

Kernel of the action.— One has (i) $H = \mathrm{Fix}_G(\mathbf{A}_+) = \mathrm{Fix}_G(\mathbf{A}_-)$.

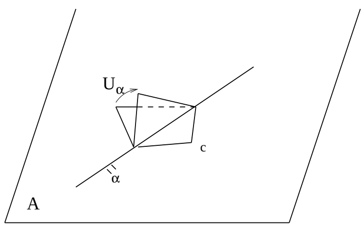
This result is the geometric formulation of [KP2, Corollary 3.4], it shows that the kernel of the action of G on its twin building is contained in $H < G$.

Moufang property.— Roughly speaking, requiring the Moufang property to a building is a way to make sure the latter admits a sufficiently large automorphism group, with well-understood local actions. We just state the main consequence of it, which will enable us to compute the number of chambers whose closure contains a given panel F. This integer will be referred to as the *thickness* at F of the building. Let us start with F_s the positive panel of type s in the closure of the standard chamber c. Then [Ron, (MO1) p.74]:

(ii) *The root group U_{α_s} fixes α_s and is normalized by H. It is simply transitive on the chambers containing F_s and $\neq c$.*

By homogeneity, point (ii) is true for every configuration panel-chamber-wall-root as above, and *mutatis mutandis* everything remains true for the negative building of G.

Figure.—



The Weyl group as a subquotient.— The following assertion is just a geometric formulation of axiom (TRD2). Consider the epimorphism $\nu : N \twoheadrightarrow W$ with $\mathrm{Ker}(\nu) = H$.

(iii) *The group N stabilizes $\mathbf{A}$; it permutes the U_α's by $nU_\alpha n^{-1} = U_{\nu(n)\alpha}$, and the roots of $\mathbf{A}$ accordingly.*

1.4. A deeper use of geometry

In this section, we exploit different geometric notions to study in further detail (TRD)-groups.

A. Convexity and negative curvature. To each of these notions corresponds a specific geometric realization, the *conical* and the *metric* realization respectively. These geometries can be defined for a single building, they basically differ by the way an apartment is represented inside. The representations of the whole building are obtained via standard glueing techniques [D2, §10].

The conical realization of a building.— An apartment of fixed sign is represented here by the Tits cone of the Weyl group (1.1.B). For twin buildings, a pair of opposite apartments is represented by the union of the Tits cone and its opposite in the ambient real vector space. We can then use the obvious geometric opposition because it is the restriction of the abstract opposition relation of facets. The point is that the shape of simplicial cone for each facet fits in convexity arguments.

The metric realization of a building.— This realization was defined by M. Davis and G. Moussong. Its main interest is that it is nonpositively curved. Actually, it satisfies the CAT(0) property: geodesic triangles are at least as thin as in the Euclidean plane. This enables to apply the following [BrT1, 3.2]

Theorem (Bruhat-Tits fixed point theorem).— *Every group of isometries of a* CAT(0)-*space with a bounded orbit has a fixed point.* □

This result generalizes a theorem applied by É. Cartan to Riemannian symmetric spaces to prove conjugacy of maximal compact subgroups in Lie groups. The metric realization only represents facets of spherical type. Indeed, consider the partially ordered set of spherical types $J \subset S$. A chamber is represented by the cone over the barycentric subdivision of this poset. G. Moussong [Mou] proved that this leads to a piecewise Euclidean cell complex which is locally nonpositively curved. A simple connectivity criterion by M. Davis [D1] proves the global CAT(0) property for Coxeter groups, and the use of retractions proves it at the level of buildings [D2].

B. Balanced subsets. In a single building, to fix a facet is the condition that defines the so-called family of parabolic subgroups. In the twin situation, we can define another family of subsets taking into account both signs. By taking fixators, it will also give rise to an interesting family of subgroups.

Definition.— *Call* balanced *a subset of a twin building contained in a pair of opposite apartments, intersecting the building of each sign and covered by a finite number of spherical facets.*

Suppose we are given a balanced subset Ω and a pair of opposite apartments $\mathbf{A}$ containing it. According to the Bruhat decompositions, for each sign ϵ the apartment $\mathbf{A}_\epsilon = \{wB_\epsilon\}_{w \in W}$ is isomorphic to the Coxeter complex of the Weyl group W. We can then define interesting sets of roots associated to the inclusion $\Omega \subset \mathbf{A}$, namely:

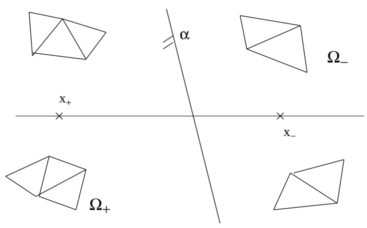
$$\Phi^u(\Omega) = \{\alpha \in \Phi \mid \overline{\alpha} \supset \Omega \quad \partial\alpha \not\supset \Omega\}, \qquad \Phi^m(\Omega) = \{\alpha \in \Phi \mid \partial\alpha \supset \Omega\},$$
$$\text{and} \qquad \Phi(\Omega) = \{\alpha \in \Phi \mid \overline{\alpha} \supset \Omega\} = \Phi^u(\Omega) \sqcup \Phi^m(\Omega).$$

So as to work in an apartment of fixed sign, we shall use the terminology of separation and strong separation. Using the Tits cone realization, we denote by Ω_+ (resp. Ω_-) the subset $\Omega \cap \mathbf{A}_+$ (resp. the subset of opposites of points in $\Omega \cap \mathbf{A}_-$). This enables to work only in $\mathbf{A}_+$. The root α is said to *separate* Ω_+ from Ω_- if $\overline{\alpha} \supset \Omega_+$ while $-\overline{\alpha} \supset \Omega_-$; α *strongly separates* Ω_+ from Ω_- if it separates Ω_+ from Ω_- and if not both of Ω_+ and Ω_- are contained in the wall $\partial\alpha$. Then $\Phi^u(\Omega)$ is the set of roots strongly separating Ω_+ from Ω_-, $\Phi(\Omega)$ that of roots separating Ω_+ from Ω_-. We will often omit the reference to Ω when it is obvious. Here is an important lemma which precisely makes use of a convexity argument [Ré2, 5.4.5].

Lemma.— *The sets of roots Φ^u, Φ^m and Φ are finite.*

Idea of proof.— The set Φ^m is obviously stable under opposition, it is finite since there is only a finite number of walls passing through a given spherical facet. Choose a point in each facet meeting Ω_+ (resp. Ω_-) and denote the barycenter so obtained by x_+ (resp. x_-). The point $x_\pm$ is contained in a spherical facet $F_\pm$. Connect the facets by a minimal gallery Γ, and denote by $d_\pm$ the chamber whose closure contains $F_\pm$ and at maximal distance from the chamber of Γ whose closure contains $F_\pm$. By convexity, a root in Φ^u has to contain d_+ but not d_-, so $\#\Phi^u$ is bounded by the length of a gallery connecting these chambers.

Figure.—



□

Examples.— According to 1.3.A, a pair of spherical points of opposite signs is a simple but fundamental example of balanced subset. Let us describe the sets Φ^m and Φ^u in such a situation. Set $\Omega_\pm := \{x_\pm\} \subset \mathbf{A}_+$ so that $\Omega = \{-x_-; x_+\}$, and draw the segment $[x_-; x_+]$ in the Tits cone. The set Φ^m is empty if $[x_-; x_+]$ intersects a chamber. The roots of Φ^u are the ones that strongly separate x_+ from x_-.

Assume first that each point lies in (the interior of) a chamber, which automatically implies that $\Phi^m(\Omega)$ is empty. First, by transitivity of G on the set of pairs of opposite apartments (1.3.A), we may assume that $\mathbf{A}$ is the standard one. Then thanks to the W-action, we can suppose that the positive point x_+ lies in the standard positive chamber c_+; there is a w in W such that x_- lies in wc_+. Choose a minimal gallery between c_+ and wc_+. Then Φ^u is the set $\Phi_{w^{-1}}$ of the $\ell(w)$ positive roots whose wall is crossed by it. The first picture below is an example of type $\widetilde{A}_2$ with $\Phi^m = \varnothing$ and $\#\Phi^u = 5$.

Now we keep this $\widetilde{A}_2$ example, fix a wall $\partial\alpha$ and consider pairs of points $\{x_\pm\}$ with $[x_-; x_+] \subset \partial\alpha$. This implies that Φ^m will be always equal to $\{\pm\alpha\}$. Still, as in the previous case, the size of Φ^u will increase when so will the distance between x_- and x_+. So the second picture below is an example of type $\widetilde{A}_2$ with $\#\Phi^m = 2$ and $\#\Phi^u = 8$.

Figure.—

1.5. Levi decompositions

This section is dedicated to the statement of two kinds of decomposition results, each being an abstract generalization of the classical Levi decompositions.

A. Levi decompositions for parabolic subgroups. The first class of subgroups distinguished in a group with a BN-pair is that of parabolic subgroups. The existence of root groups in the (TRD) case suggests to look for more precise decompositions [Ré2, 6.2].

Theorem (Levi decomposition for parabolic subgroups).— *Let F be a facet. Suppose F is spherical or G satisfies* $(\mathrm{TRD})_{\mathrm{lin}}$. *Then for every choice of a pair $\mathbf{A}$ of opposite apartments containing F, the corresponding parabolic subgroup $\mathrm{Fix}_G(F)$ admits a semi-direct product decomposition*

$$\mathrm{Fix}_G(F) = M(F, \mathbf{A}) \ltimes U(F, \mathbf{A}).$$

The group $M(F, \mathbf{A})$ is the fixator of $F \cup -F$, where $-F$ is the opposite of F in $\mathbf{A}$; it is generated by H and the root groups U_α with $\partial\alpha \supset F$. It satisfies the (TRD) *axioms, and the* $(\mathrm{TRD})_{\mathrm{lin}}$ *refinement if so does G. The group $U(F, \mathbf{A})$ only depends on F, it is the normal closure in $\mathrm{Fix}_G(F)$ of the root groups U_α with $\alpha \supset F$.* □

Remarks.— 1. It is understood in the statement that all the roots considered above are subsets of $\mathbf{A}$. By transitivity of G on pairs of opposite apartments, the root groups are conjugates of that in the (TRD) axioms.
2. In the case of an infinite Weyl group, Levi decompositions show that the action of a (TRD)-group on a single building cannot be discrete since the fixator of a point contains infinitely many root groups.

B. Levi decompositions for small subgroups. As mentioned in 1.4.B, the twin situation leads to the definition of another class of interesting subgroups.

Definition.— *A subgroup of G is called* small *if it fixes a balanced subset.*

To describe the combinatorial structure of small subgroups, we need to require the additional (NILP) axiom. It is not really useful to state it here explicitly, because our basic object of study is the class of Kac-Moody groups which satisfy it. We have [Ré2, 6.4]:

Theorem (Levi decomposition for small subgroups).— *Suppose G is a* $(\mathrm{TRD})_{\mathrm{lin}}$-*group satisfying* (NILP), *and let Ω be a balanced subset. Then for every choice of pair of opposite apartments $\mathbf{A}$ containing Ω, the small subgroup $\mathrm{Fix}_G(\Omega)$ admits a semi-direct product decomposition*

$$\mathrm{Fix}_G(\Omega) = M(\Omega, \mathbf{A}) \ltimes U(\Omega, \mathbf{A}),$$

where both factors are closely related to the geometry of Ω *(and of* $\mathbf{A}$*, w.r.t. which all roots are defined). Namely,* $M(\Omega,\mathbf{A})$ *satisfies the* $(\mathrm{TRD})_{\mathrm{lin}}$ *axioms for the root groups* U_α *with* $\partial\alpha \supset \Omega$*, that is* $\alpha \in \Phi^m(\Omega)$*; and* $U(\Omega,\mathbf{A})$ *is in bijection with the set-theoretic product of the root groups* U_α *with* $\alpha \in \Phi^u(\Omega)$ *for any given order.* □

Remark.— In the Kac-Moody case, it can be shown that $U(\Omega,\mathbf{A})$ only depends on Ω and not on $\mathbf{A}$, so that we will use the notation $U(\Omega)$ instead.

Examples.— 1. As a very special case, consider in the standard pair of opposite apartments the balanced subset $\Omega := \{c_+; wc_-\}$, where $c_\pm$ are the standard chambers. Then, thanks to the first case of example 1.4.B, we see that $\mathrm{Fix}_G(\Omega) = T \ltimes U_w$, with U_w as defined in 1.2.B. Set-theoretically, the latter group is in particular a product of $\ell(w)$ root groups.
2. Another special case is provided by the datum of two opposite spherical facets $\pm F$ in $\mathbf{A}$. Then $U(F \cup -F) = \{1\}$ and the fixator of the pair is at the same time the M-factor of the Levi decomposition above and in the previous sense for the parabolic subgroups $\mathrm{Fix}_G(F)$ and $\mathrm{Fix}_G(-F)$.

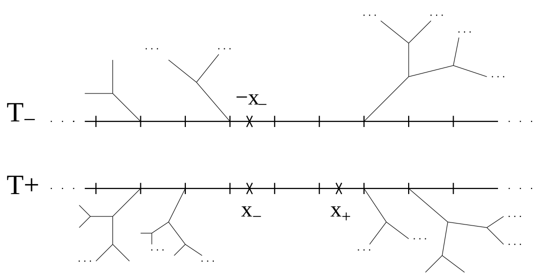
C. *Further examples.—* The examples below appear in the two simplest cases of special linear groups over Laurent polynomials (1.2.C). They will illustrate both kinds of decomposition. Examples of hyperbolic twin buildings will be treated in §4.1.

The $\mathrm{SL}_2(\mathbf{F}_q[t,t^{-1}])$ *case.—* The Bruhat-Tits buildings involved are both isomorphic to the homogeneous tree T_{q+1} of valency $q+1$. A pair of opposite apartments is represented by two parallel real lines divided by the integers.

Let us consider first the fixator of a single point. If this point is in an open edge, then its fixator is a Borel subgroup isomorphic to the group $\mathrm{SL}_2\begin{pmatrix} \mathbf{F}_q[t] & \mathbf{F}_q[t] \\ t\mathbf{F}_q[t] & \mathbf{F}_q[t] \end{pmatrix}$. If the point is a vertex, then it is isomorphic to the Nagao lattice $\mathrm{SL}_2(\mathbf{F}_q[t])$.

Let us consider now pairs of points $\{x_+; -x_-\}$ of opposite signs. We use the notations of the examples of 1.4.B. If the points are opposite vertices, then Φ^u is empty and Φ^m consists of two opposite roots: the corresponding small subgroup is equal to the M-factor, a finite group of Lie type and rank one over $\mathbf{F}_q$. If the points are non opposite middles of edges e_+ and $-e_-$, and if the edges e_+ and e_- in the positive straight line are at W-distance w, then the corresponding small fixator is the semi-direct product of T (isomorphic to $\mathbf{F}_q^\times$) by a commutative group isomorphic to the additive group of the $\mathbf{F}_q$-vector space $\mathbf{F}_q^{\ell(w)}$.

Figure.—



The $\mathrm{SL}_3(\mathbf{F}_q[t,t^{-1}])$ *case.—* The case of single points is quite similar. For instance, let us fix a vertex x. The set L of chambers whose closure contains x can naturally be seen as the building of $\mathrm{SL}_3(\mathbf{F}_q)$. The fixator P of x is isomorphic to $\mathrm{SL}_3(\mathbf{F}_q[t])$. The M-factor in the Levi decomposition of P is a finite group of Lie type A_2 over $\mathbf{F}_q$ and its action on the small building L is the natural one. The infinite U-factor fixes L pointwise.

Suppose now we are in the second case of example 1.4.B: the pair of points $\{x_+; -x_-\}$ lies in a wall $\partial\alpha$. We already now that Φ^m contains the two opposite roots $\pm\alpha$, so that the M-factor of $\mathrm{Fix}(\{x_+; -x_-\})$ is a rank one finite group of Lie type over $\mathbf{F}_q$. If we assume α to be the first simple root of $\mathrm{SL}_3(\mathbf{F}_q)$, then M is the group of matrices $\begin{pmatrix} X & 0 \\ 0 & \det X^{-1} \end{pmatrix}$, with $X \in \mathrm{GL}_2(\mathbf{F}_q)$. The U-factor is a (commutative) p-group, with $p = \mathrm{char}(\mathbf{F}_q)$. It is isomorphic to the group of matrices $\begin{pmatrix} 1 & 0 & P \\ 0 & 1 & Q \\ 0 & 0 & 1 \end{pmatrix}$, where P and Q are polynomials of $\mathbf{F}_q[t]$ of degree bounded the number of vertices between x_- and x_+.

2. SPLIT KAC-MOODY GROUPS

In 1987, J. Tits defined group functors whose values over fields we will call *(split) Kac-Moody groups.* In §2.1, we give the definition of these groups, which involves some basic facts from Kac-Moody algebras to be recalled. Section 2.2 relates Kac-Moody groups to the abstract combinatorics previously studied, providing afterwards some more information. The three last sections deal then with more specific properties. In §2.3 is presented the adjoint representation and §2.4 combine it with Levi decompositions to endow each small subgroup with a structure of algebraic group. At last, §2.5 presents an argument occuring several times afterwards. It is used for example to prove a conjugacy theorem for Cartan subgroups.

2.1. Tits functors and Kac-Moody groups

We sum up here the step-by-step definition of split Kac-Moody groups by J. Tits. It is also the opportunity to indicate how the notions defined in §1.1 for arbitrary Coxeter groups arise in the Kac-Moody context. All the material here is contained in [T4, §2].

A. Definition data. Functorial requirements. The data needed to define a Tits group functor are quite similar to that needed to define a Chevalley-Demazure group scheme.

Definition.— (i) *A* generalized Cartan matrix *is an integral matrix $A = [A_{s,t}]_{s,t\in S}$ satisfying: $A_{s,s} = 2$, $A_{s,t} \leq 0$ when $s \neq t$ and $A_{s,t} = 0 \Leftrightarrow A_{t,s} = 0$.*
(ii) *A* Kac-Moody root datum *is a 5-tuple $\big(S, A, \Lambda, (c_s)_{s\in S}, (h_s)_{s\in S}\big)$, where A is a generalized Cartan matrix indexed by the finite set S, Λ is a free $\mathbf{Z}$-module (whose $\mathbf{Z}$-dual is denoted by $\Lambda^\vee$). Besides, the elements c_s of Λ and h_s of $\Lambda^\vee$ are required to satisfy $\langle c_s \mid h_t\rangle = A_{ts}$ for all s and t in S.*

Thanks to these elementary objects, it is possible to settle abstract functorial requirements generalizing the properties of functors of points of Chevalley-Demazure group schemes [T4, p.545]. The point is that *over fields* a problem so defined does admit a unique solution, concretely provided by a group presentation [T4, Theorem 1]. This is the viewpoint we will adopt in the next subsection and for the rest of our study. Let us first define further objects only depending on a Cartan generalized matrix.

Definition.— (i) *The* Weyl group *of a generalized Cartan matrix A is the Coxeter group*

$$W = \langle s\in S \mid (st)^{M_{st}} = 1 \text{ whenever } M_{st} < \infty\rangle.$$

with $M_{st} = 2, 3, 4, 6$ or ∞ according to whether $A_{st}A_{ts} = 0, 1, 2, 3$ or is ≥ 3.
(ii) *The* root lattice *of A is the free $\mathbf{Z}$-module $Q := \bigoplus_{s\in S} \mathbf{Z}a_s$ over the symbols a_s, $s\in S$. We also set $Q_+ := \bigoplus_{s\in S} \mathbf{N}a_s$.*

The group W operates on Q via the formulas $s.a_t = a_t - A_{st}a_s$. This allows to define the *root system* Δ^{re} of A by $\Delta^{re} := W.\{a_s\}_{s\in S}$. This set has all the abstract properties of the set of real roots of a Kac-Moody algebra. In particular, one has $\Delta^{re} = \Delta^{re}_+ \sqcup \Delta^{re}_-$, where $\Delta^{re}_\epsilon := \epsilon Q_+ \cap \Delta^{re}$. Let us explain now how to recover the notions of §1.1. Recall that given a Coxeter matrix M, we defined its cosine matrix so as to introduce the Tits cone on which acts the group. The starting point here is the generalized Cartan matrix which plays the role of the cosine matrix. This allows to define the Tits cone as in 1.1.B, and all the objects related to it.

Instead of V, the real vector space we use here is $Q_{\mathbf{R}} := Q \otimes_{\mathbf{Z}} \mathbf{R}$. This way, we get a bijection between Δ^{re} and the root system of the Weyl group as defined in 1.1.A, for which the notions of prenilpotence and intervals coincide. General references for abstract infinite root systems are [H2] and [Ba]. Note that in the Kac-Moody setting, linear combinations of roots giving roots only involve integral coefficients. This leads to the following

Convention.— *In Kac-Moody theory, the root system Δ^{re} will play the role of the root system of the Weyl group of A, and the roots will be denoted by latin letters.*

B. Lie algebras. Universal envelopping algebras. Z-forms. As already said, a more constructive approach consists in defining groups by generators and relations. The idea is to mimic a theorem by Steinberg giving a presentation of a split simply connected semisimple algebraic group over a field. So far, so good. The point is that the relations involving root groups are not easily available. One has to reproduce some kind of Chevalley's construction in the Kac-Moody context. This implies to introduce a quite long series of «tangent» objects such as Lie algebras, universal envelopping algebras and at last $\mathbf{Z}$-forms of them.

Kac-Moody algebras.— From now on, we suppose we are given a Kac-Moody root datum $\mathcal{D} = \big(S, A, \Lambda, (c_s)_{s\in S}, (h_s)_{s\in S}\big)$.

Definition.— *The* Kac-Moody algebra *associated to $\mathcal{D}$ is the $\mathbf{C}$-Lie algebra $\mathfrak{g}_{\mathcal{D}}$ generated by $\mathfrak{g}_0 := \Lambda^{\vee} \otimes_{\mathbf{Z}} \mathbf{C}$ and the sets $\{e_s\}_{s\in S}$ and $\{f_s\}_{s\in S}$, submitted to the following relations.*

$$[h, e_s] = \langle c_s, h\rangle e_s \quad \textit{and} \quad [h, f_s] = -\langle c_s, h\rangle f_s \quad \textit{for } h\in\mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_0] = 0,$$

$$[e_s, f_s] = -h_s \otimes 1, \quad [e_s, f_t] = 0 \quad \textit{for } s \neq t \textit{ in } S,$$

$$(\mathrm{ad} e_s)^{-A_{st}+1} e_t = (\mathrm{ad} f_s)^{-A_{st}+1} f_t = 0 \quad - \quad \textit{Serre relations.}$$

This Lie algebra admits an abstract Q-gradation by declaring the element e_s (resp. f_s) of degree a_s (resp. $-a_s$) and the elements of $\mathfrak{g}_0$ of degree 0. Each degree for which the corresponding subspace is nontrivial is called a *root* of $\mathfrak{g}_{\mathcal{D}}$.

For each $s\in S$, the derivations $\mathrm{ad}e_s$ and $\mathrm{ad}f_s$ are locally nilpotent, which enables to define the automorphisms $\exp(\mathrm{ad}e_s)$ and $\exp(\mathrm{ad}f_s)$ of $\mathfrak{g}_{\mathcal{D}}$. The automorphisms $\exp(\mathrm{ad}e_s).\exp(\mathrm{ad}f_s).\exp(\mathrm{ad}e_s)$ and $\exp(\mathrm{ad}f_s).\exp(\mathrm{ad}e_s).\exp(\mathrm{ad}f_s)$ are equal and will be denoted by s^*. We are interested in s^* because if we define W^* to be the group of automorphisms of $\mathfrak{g}_{\mathcal{D}}$ generated by the s^*'s, then $s^* \mapsto s$ lifts to a homomorphism $\nu : W^* \twoheadrightarrow W$. Considering the W^*-images of the spaces $\mathbf{C}e_s$ and $\mathbf{C}f_s$ makes Δ^{re} appear as a subset of roots of $\mathfrak{g}_{\mathcal{D}}$. A basic fact about real roots a in Kac-Moody theory is that the corresponding homogeneous subspaces $\mathfrak{g}_a \subset \mathfrak{g}_{\mathcal{D}}$ are one-dimensional, so that the W-action on Δ^{re} is lifted by the W^*-action on the homogeneous spaces $\mathfrak{g}_a$, $a\in\Delta^{re}$. Besides, for $s\in S$ and $w^*\in W^*$, the pair of opposite elements $w^*(\{\pm e_s\})$ only depends on the root $a := \nu(w^*)a_s$; we denote it by $\{\pm e_a\}$.

Divided powers and $\mathbf{Z}$-forms.— We want now to construct a $\mathbf{Z}$-form of the universal envelopping algebra $\mathcal{U}\mathfrak{g}_{\mathcal{D}}$ in terms of divided powers. Subrings of this ring will be used for algebraic differential calculus on some subgroups of Kac-Moody groups. For each $u\in\mathcal{U}\mathfrak{g}_{\mathcal{D}}$ and each $n\in\mathbf{N}$, $u^{[n]}$ will denote the divided power $(n!)^{-1}u^n$ and $\binom{u}{n}$ will denote $(n!)^{-1}u(u-1)...(u-n+1)$. For each $s\in S$, $\mathcal{U}_{\{s\}}$ (resp. $\mathcal{U}_{\{-s\}}$) is the subring $\sum_{n\in\mathbf{N}} \mathbf{Z}e_s^{[n]}$ (resp. $\sum_{n\in\mathbf{N}} \mathbf{Z}f_s^{[n]}$) of $\mathcal{U}\mathfrak{g}_{\mathcal{D}}$. We denote by $\mathcal{U}_0$ the subring of $\mathcal{U}\mathfrak{g}_{\mathcal{D}}$ generated by the (degree 0) elements of the form $\binom{h}{n}$, with $h\in\Lambda^{\vee}$ and $n\in\mathbf{N}$.

Definition.— *We denote by $\mathcal{U}_{\mathcal{D}}$ the subring of $\mathcal{U}\mathfrak{g}_{\mathcal{D}}$ generated by $\mathcal{U}_0$ and the $\mathcal{U}_{\{s\}}$ and $\mathcal{U}_{\{-s\}}$ for s in S. It contains the ideal $\mathcal{U}_{\mathcal{D}}^+ := \mathfrak{g}_{\mathcal{D}}.\mathcal{U}\mathfrak{g}_{\mathcal{D}} \cap \mathcal{U}_{\mathcal{D}}$.*

The point about $\mathcal{U}_{\mathcal{D}}$ is the following

Proposition.— *The ring $\mathcal{U}_{\mathcal{D}}$ is a* $\mathbf{Z}$-form *of $\mathcal{U}\mathfrak{g}_{\mathcal{D}}$ i.e., the natural map $\mathcal{U}_{\mathcal{D}} \otimes_{\mathbf{Z}} \mathbf{C} \to \mathcal{U}\mathfrak{g}_{\mathcal{D}}$ is a bijection.*

Remark.— In the finite-dimensional case – when $\mathfrak{g}_{\mathcal{D}}$ is a semisimple Lie algebra, $\mathcal{U}_{\mathcal{D}}$ is the ring used to define Chevalley groups by means of formal exponentials [Hu1, VII].

C. Generators and Steinberg relations. Split Kac-Moody groups. We can now start to work with groups.

First step: root groups and integrality result.— Let us consider a root c in Δ^{re}. We denote by $\mathfrak{U}_c$ the $\mathbf{Z}$-scheme isomorphic to the additive one-dimensional group scheme $\mathbf{G}_a$ and whose Lie algebra is the $\mathbf{Z}$-module generated by $\{\pm e_c\}$. For each root c, the choice of a sign defines an isomorphism $\mathbf{G}_a \xrightarrow{\sim} \mathfrak{U}_c$. These choices are analogues of the ones defining an «épinglage» in the finite-dimensional case. We denote for short $\mathfrak{U}_s$ (resp. $\mathfrak{U}_{-s}$) the $\mathbf{Z}$-group scheme $\mathfrak{U}_{a_s}$ (resp. $\mathfrak{U}_{-a_s}$), and x_s (resp. x_{-s}) the isomorphism $\mathbf{G}_a \xrightarrow{\sim} \mathfrak{U}_s$ (resp. $\mathbf{G}_a \xrightarrow{\sim} \mathfrak{U}_{-s}$) induced by the choice of e_s (resp. f_s) in $\{\pm e_s\}$ (resp. $\{\pm f_s\}$). Let us consider now $\{a;b\}$ a prenilpotent pair of roots. The direct sum of the homogeneous subspaces $\mathfrak{g}_c$ of degree $c \in [a;b]_{\mathrm{lin}}$ is a nilpotent Lie subalgebra $\mathfrak{g}_{[a;b]_{\mathrm{lin}}}$ of $\mathfrak{g}_{\mathcal{D}}$, which defines a unique unipotent complex algebraic group $U_{[a;b]_{\mathrm{lin}}}$, by means of formal exponentials.

Proposition.— *There exists a unique $\mathbf{Z}$-group scheme $\mathfrak{U}_{[a;b]_{\mathrm{lin}}}$ containing the $\mathbf{Z}$-schemes $\mathfrak{U}_c$ for $c \in [a;b]_{\mathrm{lin}}$, whose value over $\mathbf{C}$ is the complex algebraic group $U_{[a;b]_{\mathrm{lin}}}$ with Lie algebra $\mathfrak{g}_{[a;b]_{\mathrm{lin}}}$. Moreover, the product map*

$$\prod_{c \in [a;b]_{\mathrm{lin}}} \mathfrak{U}_c \to \mathfrak{U}_{[a;b]_{\mathrm{lin}}}$$

is an isomorphism of $\mathbf{Z}$-schemes for every order on $[a;b]_{\mathrm{lin}}$.

Remark.— This result is an abstract generalization of an integrality result of Chevalley [Sp, Proposition 9.2.5] concerning the commutation constants between root groups of split semisimple algebraic groups. The integrality here is expressed by the existence of a $\mathbf{Z}$-structure extending the $\mathbf{C}$-structure of the algebraic group. We will make these $\mathbf{Z}$-group schemes a bit more explicit when defining the adjoint representation (in 2.3.A).

Second step: Steinberg functors.— Now we see the $\mathbf{Z}$-schemes $\mathfrak{U}_{[a;b]_{\mathrm{lin}}}$ just as group functors over rings. We will amalgamate them so as to obtain a big group functor containing all the relations between the root groups. This is the step of the procedure where algebraic structures are lost. The relations $c \in [a;b]_{\mathrm{lin}}$ give rise to an inductive system of group functors with injective transition maps $\mathfrak{U}_c(R) \hookrightarrow \mathfrak{U}_{[a;b]_{\mathrm{lin}}}(R)$ over every ring R.

Definition.— *The* Steinberg functor *– denoted by* St_A *– is the limit of the inductive system above.*

This group functor only depends on the generalized Cartan matrix A, not on the whole datum $\mathcal{D}$. Besides, the W^*-action on $\mathfrak{g}_{\mathcal{D}}$ extends to a W^*-action on St_A: for each w in W, we will still denote by w^* the corresponding automorphism.

Third step: the tori and the other relations.— What is left to do at this level is to introduce a torus, a split one by analogy with Chevalley groups. In fact, it is determined by the lattice Λ of the Kac-Moody root datum $\mathcal{D}$. As in the previous step, we are only interested in group functors. Denote by $\mathcal{T}_\Lambda$ the functor of points $\mathrm{Hom}_{\mathbf{Z}-\mathrm{alg}}(\mathbf{Z}[\Lambda],-) = \mathrm{Hom}_{\mathrm{groups}}(\Lambda,-^\times)$ defined over the «category» $\mathbf{Z}$-alg of rings. For a ring R, r an invertible element in it and h an element of $\Lambda\check{}$, the notation r^h corresponds to the element $\lambda \mapsto r^{\langle\lambda|h\rangle}$ of $\mathcal{T}_\Lambda(R)$. We will also denote $r^{\langle\lambda|h\rangle}$ by $\lambda(r^h)$. This way we can see elements of Λ as characters of $\mathcal{T}_\Lambda$. We finally set $\tilde{s}(r) := x_s(r)x_{-s}(r^{-1})x_s(r)$ for each invertible element r in a ring R, and $\tilde{s} := \tilde{s}(1)$.

Definition.— *For each ring R, we define the group $\underline{G}_\mathcal{D}(R)$ as the quotient of the free product $\mathrm{St}_A(R) * \mathcal{T}_\Lambda(R)$ by the following relations.*

$$tx_s(q)t^{-1} = x_s(c_s(t)q); \quad \tilde{s}(r)t\tilde{s}(r)^{-1} = s(t);$$
$$\tilde{s}(r^{-1}) = \tilde{s}r^{h_s}; \quad \tilde{s}u\tilde{s}^{-1} = s^*(u);$$

with $s\in S, c\in\Delta^{re}, t\in\mathcal{T}_\Lambda(R), q\in R, r\in R^\times, u\in\mathfrak{U}_c(R)$.

We define this way a group functor denoted by $\underline{G}_\mathcal{D}$ and called the Tits functor *associated to $\mathcal{D}$. The value of this functor on the field $\mathbf{K}$ will be called the* split Kac-Moody group *associated to $\mathcal{D}$ and defined over $\mathbf{K}$.*

Remark.— This is the place where we use the duality relations between Λ and $\Lambda\check{}$ required in the definition of $\mathcal{D}$. This duality occurs in the expression $c_s(t)$, and allows to define a W-action on Λ by sending s to the involution of Λ: $\lambda \mapsto \lambda - \langle\lambda \mid h_s\rangle c_s$. This yields a W-action on $\mathcal{T}_\Lambda$ thanks to which the expression $s(t)$ makes sense in the defining relations. The expression $s^*(u)$ comes from the W^*-action on the Steinberg functor.

Examples.— 1. As a first example, we can consider a Kac-Moody root datum where A is a Cartan matrix and $\Lambda\check{}$ is the $\mathbf{Z}$-module freely generated by the h_s's. Then what we get over fields is the functor of points of the simply connected Chevalley scheme corresponding to A. This can be derived from the theorem by Steinberg alluded to in 2.1.A [Sp, Theorem 9.4.3]. One cannot expect better since as already explained, algebraic structures are lost during the amalgam step defining St_A.
2. More generally, the case where Λ (resp. $\Lambda\check{}$) is freely generated by the c_s's (resp. h_s's) will be referred to as the *adjoint* (resp. *simply connected*) case.
3. The groups $\mathrm{SL}_n(\mathbf{K}[t,t^{-1}])$ of 1.2.C can be seen as split Kac-Moody groups over $\mathbf{K}$. The generalized Cartan matrix is the matrix $\widetilde{A}_{n-1}$ indexed by $\{0;1;...n-1\}$, characterized for $n \geq 3$ by $A_{ii} = 2$, $A_{ij} = -1$ for i and j consecutive modulo n and $A_{ij} = 0$ elsewhere. In rank 2, $\widetilde{A}_2$ is just $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. The $\mathbf{Z}$-module Λ (resp. $\Lambda\check{}$) is then the free $\mathbf{Z}$-module of rank $n-1$ generated by the simple roots c_1, ... c_{n-1} of the standard Cartan subgroup T of $\mathrm{SL}_n(\mathbf{K})$ (resp. the corresponding one-parameter multiplicative subgroups). To be complete, one has to add the character $c_0 := -\sum_{1\leq i\leq n-1} c_i$ (resp. the cocharacter $h_0 := -\sum_{1\leq i\leq n-1} h_i$). These examples show in particular that neither Λ nor $\Lambda\check{}$ need to contain the c_s's or the h_s's as free families.

2.2. Combinatorics of a Kac-Moody group

This section states the connection with the abstract group combinatorics presented in §1, then stresses some specific properties (often due to splitness). We are given here a split Kac-Moody group G defined over $\mathbf{K}$ which acts on its twin building $(\mathcal{I}_+, \mathcal{I}_-, d^*)$.

A. Main statement. As the (TRD) axioms were adjusted to the Kac-Moody situation, it is not surprising to get the following result [Ré2, 8.4.1].

Proposition.— *Let G be a Kac-Moody group, namely the value of a Tits group functor over a field $\mathbf{K}$. Then G is a* $(\mathrm{TRD})_{\mathrm{lin}}$*-group for the root groups of its definition, which also satisfies also the axiom* (NILP) *for Levi decomposition of small subgroups* (1.5.B)*. The standard Cartan subgroup T plays the role of the normalizer of the U_α's.*

Sketch of proof.— In the defining relations of a Kac-Moody group (2.1.C), everything seems to be done so as to conform to the (TRD) axioms. Still, we have to be a bit more careful with the non degeneracy requirements. Indeed, for the first half of axiom (TRD0) and axiom (TRD3), we have to use the adjoint representation defined later. This point is harmless since the definition of this representation does not need group combinatorics. □

Remark.— This result can be used to prove the unicity over fields of Kac-Moody groups. Actually, the original group combinatorics involved in the proof by J. Tits [T4, §5] was more general.

B. Specific properties. As Tits functors generalize Chevalley-Demazure schemes, one can expect further properties for at least two reasons. The first one is the «algebraicity» of Kac-Moody groups i.e., the fact that the precise defining relations are analogues of that for algebraic groups. The second reason is splitness.

«Algebraicity» of Tits functors.— First, we know that the biggest group normalizing all the root groups is the standard Cartan subgroup T: this is a purely combinatorial consequence of the (TRD) axioms due to the fact that in BN-pairs, Borel subgroups coincide with their normalizer. Besides we have a completely explicit description of the T-action on root groups – by characters. This leads to the following result [Ré2, 10.1.3].

Proposition.— *If the groundfield $\mathbf{K}$ has more than 3 elements, then the fixed points under T in the buildings are exactly the points in $\mathbf{A}$ (the standard pair of opposite apartments).* □

Remark.— What this proposition says is that this set is not bigger than $\mathbf{A}$. This result has to be related to another one, where the same hypothesis $\#\mathbf{K} > 3$ is needed: in this case, N is exactly the normalizer of T in G [Ré2, 8.4.1].

Another algebraic-like consequence is about Levi decompositions and residues in buildings. Consider the standard facet $\pm F_J$ of spherical type $J \subset S$ and sign $\pm$. Then the M-factor (see 1.5.B, example 2) is the Kac-Moody group corresponding to the datum where A is restricted to $J \times J$, and only the c_s's and h_s's with $s \in J$ are kept. This is a root datum abstractly defining points of a reductive group. Moreover, the union of the sets of chambers whose closure contains $\pm F_J$ – the *residue* of $\pm F_J$ – is the twin building of the reductive group, on which it operates naturally. The generalization to arbitrary facets is straightforward.

Remark.— The analogy with reductive groups can fail on some points. Consider for instance the case of the centralizer of the standard Cartan subgroup T. It is classical in the algebraic setting that the centralizer of a Cartan subgroup is not bigger than the subgroup itself. Consider once again the groups $\mathrm{SL}_n(\mathbf{K}[t, t^{-1}])$ of 1.2.C. Then the centralizer of T – the standard Cartan subgroup of $\mathrm{SL}_n(\mathbf{K})$ – is infinite-dimensional since it is the group of determinant 1 diagonal matrices with monomial coefficients.

Splitness.— The main analogy with split reductive groups is that the torus is a quotient of a finite numbers of copies of $\mathbf{K}^\times$ and each root group is isomorphic to the additive group of the groundfield $\mathbf{K}$. Combined with a geometric consequence of the Moufang property, the latter property of root groups says:

Lemma.— *Every building arising from a split Kac-Moody group is* homogeneous*, that is thickness is constant* (*equal to* $\#\mathbf{K}+1$) *over all panels.* $\square$

The buildings arising from a split Kac-Moody group will be referred to as *(split) Kac-Moody buildings.*

2.3. The adjoint representation

Besides the geometry of buildings, an important tool to study Kac-Moody groups is a linear representation which generalizes the adjoint representation of algebraic groups. Let us fix a Kac-Moody root datum $\mathcal{D}$ and the corresponding Tits functor $\underline{G}_{\mathcal{D}}$.

A. Formal sums and distribution algebras. Let us go back to the situation of 2.1.C; we work in particular with the prenilpotent pair of roots $\{a;b\}$. In the $\mathbf{Z}$-form $\mathcal{U}_{\mathcal{D}}$, we consider the subring $\mathcal{U}_{[a;b]_{\text{lin}}} := \mathcal{U}_{\mathcal{D}} \cap \mathcal{U}\mathfrak{g}_{[a;b]_{\text{lin}}}$. Let us define the R-algebra $\big(\widehat{\mathcal{U}_{[a;b]_{\text{lin}}}}\big)_R$ of formal sums $\displaystyle\sum_{c\in\mathbf{N}a+\mathbf{N}b} r_c u_c$, with u_c homogeneous of degree c in $\mathcal{U}_{\mathcal{D}}$ and $r_c \in R$.

Definition.— *Denote by* $\mathfrak{U}_{[a;b]_{\text{lin}}}$ *the group functor which to each ring R associates the subgroup* $\langle \exp(re_c) \mid r\in R, c\in \Delta^{re}\cap(\mathbf{N}a+\mathbf{N}b)\rangle < \big(\widehat{\mathcal{U}_{[a;b]_{\text{lin}}}}\big)_R^{\times}$. *Here* $\exp(re_c)$ *is the formal exponential* $\sum_{n\geq 0} r^n e_c^{[n]}$ *and* $\big(\widehat{\mathcal{U}_{[a;b]_{\text{lin}}}}\big)_R^{\times}$ *is the multiplicative group of the R-algebra* $\big(\widehat{\mathcal{U}_{[a;b]_{\text{lin}}}}\big)_R$.

There is a little abuse of notation when denoting by $\mathfrak{U}_{[a;b]_{\text{lin}}}$ a new object, but we will justify it soon. Thanks to a generalized Steinberg commutator formula, one can prove first:

Proposition.— *The group functor* $\mathfrak{U}_{[a;b]_{\text{lin}}}$ *is the functor of points of a smooth connected group scheme of finite type over* $\mathbf{Z}$*, with Lie algebra* $\mathfrak{g}_{[a;b]_{\text{lin}}} \cap \mathcal{U}_{\mathcal{D}}$*. Its value over* $\mathbf{C}$ *is the complex algebraic group* $U_{[a;b]_{\text{lin}}}$ *and we have* $\mathrm{Dist}_e U_{[a;b]_{\text{lin}}} \cong \mathcal{U}\mathfrak{g}_{[a;b]_{\text{lin}}}$. $\square$

The notation Dist_e is for the *algebra of distributions supported at unity* of an algebraic group. This proposition is the place where a bit of algebraic differential calculus [DG, II.4] comes into play. We will not detail the use of it, but just say that it allows to prove that the group functors above and of proposition 2.1.C coincide [Ré2, 9.3.3]. In other words, this result leads to a concrete description of the group functors occuring in the amalgam defining St_A.

B. The adjoint representation as a functorial morphism. For every ring R, the extension of scalars from $\mathbf{Z}$ to R of $\mathcal{U}_{\mathcal{D}}$ is denoted by $\big(\mathcal{U}_{\mathcal{D}}\big)_R$. Denote by $\mathrm{Aut}_{filt}(\mathcal{U}_{\mathcal{D}})(R)$ the group of automorphisms of the R-algebra $\big(\mathcal{U}_{\mathcal{D}}\big)_R$ preserving its filtration arising form that of $\mathcal{U}_{\mathcal{D}}$ and its ideal $\mathcal{U}_{\mathcal{D}}^+ \otimes_{\mathbf{Z}} R$. The naturality of scalar extension makes the assignement $R \mapsto \mathrm{Aut}_{filt}(\mathcal{U}_{\mathcal{D}})(R)$ a group functor. We have [Ré2, 9.5.3]:

Theorem.— *There is a morphism of group functors* $\mathrm{Ad} : \underline{G}_{\mathcal{D}} \to \mathrm{Aut}_{filt}(\mathcal{U}_{\mathcal{D}})$ *characterized by:*

$$\mathrm{Ad}_R\big(x_s(r)\big) = \sum_{n\geq 0} \frac{(\mathrm{ad} e_s)^n}{n!} \otimes r^n, \quad \mathrm{Ad}_R\big(x_{-s}(r)\big) = \sum_{n\geq 0} \frac{(\mathrm{ad} f_s)^n}{n!} \otimes r^n,$$

$$\mathrm{Ad}_R\big(\mathcal{T}_\Lambda(R)\big) \text{ fixes } \big(\mathcal{U}_0\big)_R \quad \text{and} \quad \mathrm{Ad}_R(h)(e_a \otimes r) = c_a(h)(e_a \otimes r),$$

for s in S, for each ring R, each h in $\mathcal{T}_\Lambda(R)$ and each r in R.

Sketch of proof.— No use of group combinatorics is needed for this result. Roughly speaking, to define the adjoint representation just consists in inserting «ad» in the arguments of the formal exponentials for the root groups, and in making the torus operate diagonally w.r.t. the duality given by $\mathcal{D}$. One has to follow all the steps of the definition of Tits functors, and to verify that all the defining relations involved are satisfied by the partial linear actions of the characterization. Proposition 2.3.B enables to justify that the first formula defines a representation of the Steinberg functor. Then, the torus must be taken into account, but the verification concerning the rest of the defining relations of $\underline{G}_{\mathcal{D}}$ is purely computational. The naturality of the representation is an easy point. $\square$

C. The adjoint representation over fields. Restriction to values of Tits functors over fields brings further properties, precisely because this allows the use of group combinatorics according to 2.2.A – see [Ré2, 9.6] for proofs.

Proposition.— (i) *Over fields, the kernel of the adjoint representation is the center of the Kac-Moody group G. This center is the intersection of kernels of characters of T, namely the centralizers in T of the root groups.*
(ii) *If the groundfield is algebraically closed, the image of* Ad *is again a Kac-Moody group.* $\square$

2.4. Algebraic subgroups

One of the main interests of the adjoint representation is that it provides algebro-geometric structures for the family of small subgroups of a Kac-Moody group.

A. Statement. The following result is, up to technicalities, the combination of the abstract Levi decompositions and restrictions of source and target of the adjoint representation [Ré2, 10.3].

Theorem (Algebraic structures).— *For every balanced subset Ω, denote by $G(\Omega)$ the quotient group $\mathrm{Fix}_G(\Omega)/Z\big(\mathrm{Fix}_G(\Omega)\big)$ of the corresponding small subgroup by its center. Then $G(\Omega)$ admits a natural structure of algebraic group, arising from the adjoint representation.* $\square$

Remark.— An important question then is to know how far the quotient $G(\Omega)$ is from the small subgroup $\mathrm{Fix}_G(\Omega)$ itself. To be more precise, let us fix a pair $\mathbf{A}$ of opposite apartments containing the balanced subset Ω. Thanks to a suitable g in G, we may assume $\mathbf{A}$ is the standard pair of opposite apartments. At the group level, this corresponds to conjugating the small subgroup so as to make it contain the standard Cartan subgroup T. Then the center of $\mathrm{Fix}_G(\Omega)$ is the intersection of the kernels (in T) of the characters associated to the roots in $\Phi(\Omega)$. (These roots index the root groups generating the small subgroup together with T). This shows that the quotient is always by a group abstractly isomorphic to a diagonalisable algebraic group. This point will be important when we handle Cartan subgroups, because preimages of diagonalisable groups will stay diagonalisable.

Examples.— Let us consider some examples from 1.5.C, i.e., from some small subgroups of $\mathrm{SL}_2(\mathbf{K}[t,t^{-1}])$ and $\mathrm{SL}_3(\mathbf{K}[t,t^{-1}])$. There will be no quotient in the SL_2 case: the standard Cartan subgroup is one-dimensional. By taking a pair of opposite middles of edges, one gets a small subgroup isomorphic to the M-factor of the last example in 1.5.C. This group consists of the matrices $\begin{pmatrix} X & 0 \\ 0 & \det X^{-1} \end{pmatrix}$, with $X \in \mathrm{GL}_2(\mathbf{F}_q)$. Its center is one-dimensional and this is the corresponding quotient that actually admits an algebraic structure by the procedure above.

B. *Sketch of proof.—* The main tools for this proof are basic results from algebraic geometry or algebraic group theory, such as constructibility of images of morphisms, use of tangent maps to justify that a bijective algebraic morphism is an algebraic isomorphism... Let us make the same reductions as in the remark above. In particular, Ω is in the standard pair of opposite apartments. Recall also the Levi decomposition $\mathrm{Fix}_G(\Omega) = M(\Omega, \mathbf{A}) \ltimes U(\Omega, \mathbf{A})$, with $M(\Omega, \mathbf{A})$ satisfying the abstract (TRD) axioms.

The first step is the existence of a finite-dimensional subspace $W(\Omega)_{\mathbf{K}} \subset \big(\mathcal{U}_{\mathcal{D}}\big)_{\mathbf{K}}$ such that the resctriction Ad_Ω of the adjoint representation is the center $Z\big(\mathrm{Fix}_G(\Omega)\big)(< T)$ [Ré2, 10.3.1]. Consequently, Ad_Ω is faithful on $U(\Omega)$ and on the subgroup of $M(\Omega, \mathbf{A})$ generated by a positive half of Φ^m. The latter group is the «unipotent radical» of an abstract Borel subgroup of $M(\Omega, \mathbf{A})$. Then, it is proved that the isomorphic images of these two groups are unipotent closed $\mathbf{K}$-subgroups of $\mathrm{GL}\big(W(\Omega)_{\mathbf{K}}\big)$ [Ré2, 10.3.2].

The next step is to show that the abstract image $M_{\Omega,\mathbf{A}} := \mathrm{Ad}_\Omega\big(M(\Omega, \mathbf{A})\big)$ is also a closed subgroup. This is done by proving that its Zariski closure in $\mathrm{GL}\big(W(\Omega)_{\mathbf{K}}\big)$ is reductive and comparing the classical decompositions so obtained. The proof of this fact works as follows. We know at this level that the image by Ad_Ω of the abstract Borel subgroup of $M(\Omega, \mathbf{A})$ is closed. By Bruhat decomposition and local closedness of orbits, this gives a bound on the dimension of the Zariski closure. The use of Cartier dual numbers allows to compute the Lie algebras involved and to show that the bound is reached. The triviality of the unipotent radical follows from singular tori arguments [Ré2, 10.3.3].

The final step is to show that the semi-direct product so obtained in $\mathrm{GL}\big(W(\Omega)_{\mathbf{K}}\big)$ is an algebraic one: this is proved by Lie algebras (tangent maps) arguments [Ré2, 10.3.4]. □

2.5. Cartan subgroups

We present now a general argument combining negative curvature and algebraic groups. Then we apply it to give a natural (more intrinsic) definition of *Cartan subgroups*, previously defined by conjugation. Recall that a group of isometries is *bounded* if it acts with bounded orbits [Br2, p.160].

A. A typical argument. We suppose we are in the following setting: a group H acts in a compatible way on a Kac-Moody group G and on its twin building. (For instance, H may be a subgroup of G but this is not the only example.)

Argument.— 1. *Justify that H is a bounded group of isometries of both buildings and apply the Bruhat-Tits fixed point theorem to get a balanced H-fixed subset Ω.*
2. *Apply results from algebraic groups to the small H-stable subgroup* $\mathrm{Fix}_G(\Omega)$.

Remark.— In subsection 3.2.B, the argument is applied to the Galois actions arising from the definition of almost split forms.

B. Cartan subgroups. Recall that the *standard Cartan subgroup* is the value T of the split torus $\mathcal{T}_{\mathcal{D}}$ on the grounfield $\mathbf{K}$. The provisional definition of Cartan subgroups describes them as conjugates of T. We want to give a more intrinsic characterization of them, by means of their behaviour w.r.t. the adjoint representation Ad. We suppose here that the groundfield $\mathbf{K}$ is infinite.

A criterion for small subgroups.— This criterion precisely involves the adjoint representation. Let H be a subgroup of the Kac-Moody group G. Call H Ad-*locally finite* if its image by Ad is locally finite, that is the linear span of the H-orbit of each point in $(\mathcal{U}_{\mathcal{D}})_{\mathbf{K}}$ is finite-dimensional.

Theorem.— *The following assertions are equivalent.*

(i) *H is* Ad-*locally finite.*
(ii) *H intersects a finite number of Bruhat double classes for each sign $\pm$.*
(iii) *H is contained in the intersection of two fixators of spherical facets of opposite signs.*

Remark.— 1. The importance of this theorem lies in the fact that it is a connection between the two G-actions we chose to study G: the adjoint representation and the action on the twin building.
2. This result is due to V. Kac and D. Peterson [KP3, Theorem 1] who proved a lemma relating the Q-gradation and the Bruhat decompositions. This lemma was used by them to prove conjugacy of Cartan subgroups over $\mathbf{C}$, with Kac-Moody groups defined as automorphism groups of Lie algebras. No argument of negative curvature appears in their work, they invoke instead convexity properties of a certain distance function on the Tits cone.

Definition and conjugacy of $\mathbf{K}$*-split Cartan subgroups.*— We say that a subgroup is Ad-*diagonalizable* if its image by the adjoint representation is a diagonalisable group of automorphisms of $(\mathcal{U}_{\mathcal{D}})_{\mathbf{K}}$. Until the end of the section, we will assume the groundfield $\mathbf{K}$ to be infinite so as to characterize algebraic groups by their points.

Definition.— *A subgroup of G is a* ($\mathbf{K}$-split) Cartan subgroup *if it is* Ad-*diagonalizable and maximal for this property.*

We will omit the prefix «$\mathbf{K}$-split» when the groundfield is separably closed.

Theorem.— *The* ($\mathbf{K}$*-split*) *Cartan subgroups are all conjugates by G of the standard Cartan subgroup T.*

Sketch of proof. [Ré2, 10.4.2]— Basically, this is the application of argument 2.5.A, thanks to the smallness criterion above. An Ad-diagonalisable subgroup has to be small and its image in the algebraic subgroup is a Cartan subgroup in the algebraic sense. We have then to use transitivity of G on pairs of opposite apartments and the algebraic conjugacy theorem of Cartan subgroups to send our subgroup in T thanks to a suitable element of G. □

C. Connection with geometry. Combining this theorem with proposition 2.2.B and the fact that G is transitive on the pairs of opposite apartments, we get [Ré2, 10.4.3 and 10.4.4]:

Corollary.— *Suppose the groundfield $\mathbf{K}$ is infinite. Then, there is a natural G-equivariant dictionary between the pairs of opposite apartments and the $\mathbf{K}$-split Cartan subgroups of G. Moreover, given a facet F, this correspondence relates pairs of opposite apartments containing F and $\mathbf{K}$-split Cartan subgroups of G contained in the parabolic subgroup* $\mathrm{Fix}_G(F)$. □

We saw in the abstract study of (TRD)-groups that parabolic subgroups do not form the only interesting class of subgroups. Let us consider a balanced subset Ω of the twin building of G. A natural question is to know what the correspondence says in this case. First we choose a pair of opposite apartments $\mathbf{A}$ containing Ω. We work in the conical realization, that is see $\mathbf{A}$ as the union $\overline{\mathcal{C}} \cup -\overline{\mathcal{C}}$ of a Tits cone and its opposite in the real vector space V^*. Then it makes sense to define the *convex hull* of Ω – denoted by $\mathrm{conv}(\Omega)$ – to be the trace on $\overline{\mathcal{C}} \cup -\overline{\mathcal{C}}$

of the convex hull determined in V^*. The same trick enables to define the *vectorial extension* $\mathrm{vect}(\Omega, \mathbf{A})$. The difference of notation is justified by [Ré2, 10.4.5]:

Proposition.— (i) $\mathbf{K}$*-split Cartan subgroups of* G *contained in* $\mathrm{Fix}_G(\Omega)$ *are the preimages of the maximal* $\mathbf{K}$*-split tori of the algebraic group* $G(\Omega)$*. Besides, they are in one-to-one* G*-equivariant correspondence with the pairs of opposite apartments containing* Ω*.*
(ii) *The subspace* $\mathrm{conv}(\Omega)$ *and the subgroup* $U(\Omega)$ *only depend on* Ω*.*

Remark.— The independence of $\mathbf{A}$ for the convex hull is actually useful for the Galois descent.

3. RELATIVE KAC-MOODY THEORY

We describe now an analogue of Borel-Tits theory. Section 3.1 defines the *almost split forms* of Kac-Moody groups. This gives the class of groups which are concerned by the theory. In §3.2, we state the structure theorem for rational points and try to give an idea of its quite long proof. The geometric method for the Galois descent follows faithfully the lines of G. Rousseau's work, who used it for the characteristic 0 case [Rou1,2]. Section 3.3 introduces the class of *quasi-split* groups and states that this is the only class of almost split Kac-Moody groups over finite fields. In §3.4, we focus on relative links and apartments, so as to compute thicknesses for instance, and in §3.5 we use the classical example of the unitary group SU_3 to show how things work concretely.

3.1. Definition of forms

The requirements are basically of two kinds. We have to generalize conditions from algebraic geometry, and then to ask for isotropy conditions. We suppose we are given a groundfield $\mathbf{K}$, we choose a separable closure $\mathbf{K}_s$ (resp. an algebraic closure $\overline{\mathbf{K}}$) of it. We work with the group $G = \underline{G}_{\mathcal{D}}(\overline{\mathbf{K}})$ determined by $\overline{\mathbf{K}}$ and the Kac-Moody root datum $\mathcal{D}$. To $\mathcal{D}$ is also associated the $\mathbf{Z}$-form $\mathcal{U}_{\mathcal{D}}$; if R is any ring, then the scalar extension of $\mathcal{U}_{\mathcal{D}}$ from $\mathbf{Z}$ to R will be denoted by $(\mathcal{U}_{\mathcal{D}})_R$.

A. Functorial forms. Algebraic forms. Here is the part of the conditions aiming to generalize algebraic ones. Unfortunately, $\mathbf{K}$-forms of Kac-Moody groups are not defined by a list of conditions stated once and for all at the beginning of the study. One has to require a condition, then to derive some properties giving rise to the notions involved in the next requirement, and so on. We sum up here the basic steps of this procedure.

Use of functoriality.— A Kac-Moody group is a value of a group functor over a field. Consequently, it is natural to make occur a group functor as main piece of a $\mathbf{K}$-form. A *functorial* $\mathbf{K}$*-form* of G is a group functor defined over field extensions of $\mathbf{K}$ which coincides with the Tits functor $\underline{G}_{\mathcal{D}}$ over extensions of $\overline{\mathbf{K}}$. This is a really weak requirement since in the finite-dimensional case, it just takes into account the functor of points of a scheme. Still, this condition enables to make the Galois group $\Gamma := \mathrm{Gal}(\mathbf{K}_s/\mathbf{K})$ operate on G [Ré2, 11.1.2]. We explained in 2.2.A that the adjoint representation is a substitute for a global algebro-geometric structure on a Kac-Moody group. It is quite natural then to work with this representation to define forms [Ré2, 11.1.3].

Definition.— *A* prealgebraic $\mathbf{K}$-form *of G consists in the datum of a functorial $\mathbf{K}$-form $\underline{G}$ and of a filtered $\mathbf{K}$-form $\mathcal{U}_{\mathbf{K}}$ of the $\overline{\mathbf{K}}$-algebra $(\mathcal{U}_{\mathcal{D}})_{\overline{\mathbf{K}}} := \mathcal{U}_{\mathcal{D}} \otimes_{\mathbf{Z}} \overline{\mathbf{K}}$, both satisfying the following conditions.*

(PREALG1) *The adjoint representation* Ad *is Galois-equivariant.*
(PREALG2) *The value of $\underline{G}$ on a field extension gives rise to a group embedding.*

Remark.— It is understood in *filtered* that $\mathcal{U}_{\mathbf{K}}$ is a direct sum $\mathbf{K} \oplus \mathcal{U}^+_{\mathbf{K}}$, where $\mathcal{U}^+_{\mathbf{K}}$ is a $\mathbf{K}$-form of the ideal $(\mathcal{U}^+_{\mathcal{D}})_{\overline{\mathbf{K}}}$. The existence of a filtered $\mathbf{K}$-form of the $\overline{\mathbf{K}}$-algebra $(\mathcal{U}_{\mathcal{D}})_{\overline{\mathbf{K}}}$ enables this time to define a Γ-action on $(\mathcal{U}_{\mathcal{D}})_{\overline{\mathbf{K}}}$, and finally on the automorphism group $\mathrm{Aut}_{filt}(\mathcal{U}_{\mathcal{D}})(\overline{\mathbf{K}})$.

Example.— As a fundamental example, we can consider the datum of the restriction of the Tits functor to the $\mathbf{K}$-extensions (contained in $\overline{\mathbf{K}}$) and the $\mathbf{K}$-algebra $(\mathcal{U}_{\mathcal{D}})_{\mathbf{K}}$. This will be referred to as the *split form* of G.

Algebraicity conditions.— We suppose now we are given a *splitting extension* $\mathbf{E}/\mathbf{K}$ of the prealgebraic form $(\underline{G}, \mathcal{U}_{\mathbf{K}})$, that is a field over extensions of which the form is the split one. We also assume that *the field* $\mathbf{E}$ *is infinite* and that *the extension* $\mathbf{E}/\mathbf{K}$ *is normal.* The definition of Cartan subgroups in terms of Ad and the Γ-equivariance of this representation provides [Ré2, 11.2.2]:

Lemma.— *The image of an* $\mathbf{E}$*-split Cartan subgroup by a Galois automorphism is still an* $\mathbf{E}$*-split Cartan subgroup.* $\square$

In view of the conjugacy of Cartan subgroups, this lemma suggests to rectify Galois automorphisms so as to make them stabilize the standard Cartan subgroup T. If γ denotes a Galois automorphism such that $\gamma T = gTg^{-1}$, then $\bar{\gamma} := \mathrm{int} g^{-1} \circ \gamma$ will denote such a rectified automorphism. At the $\mathbf{E}$-algebras level, the symbol $\bar{\gamma}$ will denote the rectified (semilinear) automorphism $\mathrm{Ad} g^{-1} \circ \gamma$, where γ is the Galois automorphism of $(\mathcal{U}_{\mathcal{D}})_{\mathbf{E}}$ arising from the form. The choice of g modulo N will be harmless. Another class of interesting subgroups is that of root groups, for which things are not so nice. Still, we have [Ré2, 11.2.3]:

Lemma.— *If the* c_s*'s are free in the lattice* Λ *of the Kac-Moody root datum* $\mathcal{D}$*, then the image of a root group (w.r.t. to* T*) under a rectified Galois automorphism is still a root group (w.r.t. to* T*).* $\square$

Note that the action on the group G has no reason to respect its nice combinatorics. The additional requirements are then [Ré2, 11.2.5]:

Definition.— *An* algebraic $\mathbf{K}$-form *of* G *is a prealgebraic form* $(\underline{G}, \mathcal{U}_{\mathbf{K}})$ *such that for each* $\gamma \in \Gamma$*, one has:*

(ALG0) *the rectified automorphism* $\bar{\gamma}$ *stabilizes the family of root groups (w.r.t. to* T*).*
(ALG1) $\bar{\gamma}$ *respects the abstract* Q*-gradation of* $(\mathcal{U}_{\mathcal{D}})_{\mathbf{E}}$*, inducing a permutation of* Q *still denoted by* $\bar{\gamma}$ *and satisfying the homogeneity condition* $\bar{\gamma}(na) = n\bar{\gamma}(a)$*,* $a \in \Delta^{re}$*,* $n \in \mathbf{N}$*.*
(ALG2) $\bar{\gamma}$ *respects the algebraic characters and cocharacters of* T *among all the abstract ones.*

Remarks.— The algebraic characters (resp. cocharacters) are the ones arising from Λ (resp. $\Lambda^\vee$): (ALG2) is indeed an algebraicity condition. Concerning (ALG1), the justification comes from algebraic differential calculus. If G admitted an algebro-geometric structure – a topological Hopf algebra of regular functions for instance – with U_a as a closed subgroup, then the subring $\displaystyle\bigoplus_{j \leq n} \mathbf{E} \frac{e_a^j}{j!}$ would be the $\mathbf{E}$-algebra of invariant distributions tangent to U_a. So it is natural to require a rectified Galois automorphism (stabilizing T) to respect the order of these distributions. Here are now the main consequences of algebraicity [Ré2, 11.2.5 and 11.3.2].

Proposition.— (i) $\bar{\gamma}$ *is a group automorphism of* Q*, it stabilizes* Δ^{re} *and induces the same permutation as the one defined via the root groups.*
(ii) $\bar{\gamma}$ *stabilizes* N *inducing an automorphism of* W *sending the reflexion w.r.t.* ∂a *to the reflection w.r.t.* $\partial\bar{\gamma}(a)$*.*
(iii) *If the Dynkin diagram of* A *is connected, each Galois automorphism sends a Borel subgroup on a Borel subgroup, possibly after opposition of its sign.* $\square$

In the non connected case, oppositions of signs can occur componentwise. At this step, there is no reason why the BN-pair of fixed sign should be respected. This is a requirement to be done.

B. Almost splitness. This condition is the «isotropy part» of the requirements [Rou3].

Definition.— *The algebraic form* $(\underline{G}, \mathcal{U}_{\mathbf{K}})$ *is* almost split *if the Galois action stabilizes the conjugacy of Borel subgroups of each sign.*

The stability of conjugacy classes of Borel subgroups enables to rectify each Galois automorphisms γ in such a way that $\gamma^* := {\mathrm{intg}_\sigma}^{-1} \circ \gamma$ stabilizes B_+ and B_-. This way, the element g_σ is defined modulo $T = B_+ \cap B_-$. The assignement $\gamma \mapsto \gamma^*$ defines an Γ-action on W stabilizing S and called the *$*$-action*. The terminology is justified by the analogy with the action considered by J. Tits in the classification problem of semisimple groups. Now we can turn for the first time to the other interesting G-space, namely the twin building of the group [Ré2, 11.3.3 and 11.3.4].

Proposition.— (i) *Via group combinatorics, the Galois group operates on the buildings by automorphisms, up to permutation of types by the $*$-action defined above.*
(ii) *The Galois group acts by isometries on the metric realizations of the building of each sign.* Γ*-orbits are finite.* □

The last assertion is the starting point of the argument in 2.5.A: there exist Γ-fixed balanced subsets. In fact, we can prove [Ré2, 11.3.5]:

Theorem.— *Let us assume now* $\mathbf{E}/\mathbf{K}$ *Galois and let us consider* Ω*, balanced and Galois-fixed. Then the subspace* $W(\Omega)_{\mathbf{E}}$ *is defined over* $\mathbf{K}$*, the homomorphism* Ad_Ω *is* Γ*-equivariant and its image is a closed* $\mathbf{K}$*-subgroup of* $\mathrm{GL}\big(W(\Omega)_{\mathbf{E}}\big)$. □

So to speak, this is the first step of the Galois descent: looking at Γ-fixed balanced subsets and at the corresponding small subgroups (resp. algebraic groups).

3.2. The Galois descent theorem

This section states the main structure result about rational points of almost split Kac-Moody $\mathbf{K}$-groups. The geometric side is at the same time the main tool and an interesting result in its own right.

A. Statement [Ré2, 12.4.3]. Here is the result which justifies the analogy between the theory of almost split forms of Kac-Moody groups and Borel-Tits theory for isotropic reductive groups. We keep the almost split $\mathbf{K}$-form $(\underline{G}, \mathcal{U}_{\mathbf{K}})$ of the previous section. We suppose now $\mathbf{E} = \mathbf{K}_s$, i.e., that the form is split over the separable closure $\mathbf{K}_s$. A Kac-Moody group endowed with such a form will be called an *almost split Kac-Moody* $\mathbf{K}$*-group*. By definition, the *group of rational points* of the form is the group of Galois-fixed points $G(\mathbf{K}) := G^\Gamma$.

Theorem (Galois descent). — *Let G be a Kac-Moody group, almost split over* $\mathbf{K}$*. Then*

(i) *The set of* Γ*-fixed points in the twin building of G is still a twin building* $\mathcal{J}^\natural$.
(ii) *The group of rational points* $G(\mathbf{K})$ *is a* $(\mathrm{TRD})_{\mathrm{lin}}$*-group for a natural choice of subgroups suggested by the geometry of a pair of opposite apartments in* $\mathcal{J}^\natural$*. Indeed,* $\mathcal{J}^\natural$ *is a geometric realization of the twin building abstractly associated to* $G(\mathbf{K})$.
(iii) *Maximal* $\mathbf{K}$*-split tori are conjugate in* $G(\mathbf{K})$. □

Remark.— The assumption of splitness over $\mathbf{K}_s$ is not necessary in the algebraic case, for which we can apply the combination of a theorem due to Cartier [Bo, Theorem 18.7] and a theorem due to Chevalley-Rosenlicht-Grothendieck [Bo, Theorem 18.2].

B. *Sketch of proof.*— Let us start with notations and terminology. The symbol $\Omega^\natural$ will always denote a Γ-fixed balanced subset, and Ω will denote the Γ-stable union of spherical facets covering it. We are working now in the conical realization of the twin building so as to apply convexity arguments. A *generic subspace* is a subspace of a pair of opposite apartments meeting a spherical facet.

Relative geometric objects.— We define now relative objects with suggestive terminology. This does not mean that we know at this step that they form a building.

Definition.— (i) *A* (spherical) $\mathbf{K}$-facet *is the subset of* Γ*-fixed points of a* (*spherical*) Γ*-stable facet. A* $\mathbf{K}$-chamber *is a spherical* $\mathbf{K}$*-facet of maximal closure. A* $\mathbf{K}$-panel *is a* $\mathbf{K}$*-facet of codimension one in the closure of a* $\mathbf{K}$*-chamber.*
(ii) *A* $\mathbf{K}$-apartment *is a generic* Γ*-fixed subspace, maximal for these properties.*

Remark.— These geometric definitions also give sense to an opposition relation on $\mathbf{K}$-facets of oppposite signs, which will be called *opposite* if they form a symmetric subset of a pair of opposite apartments.

Let us turn now to the other application of argument 2.5.A, with the Galois group as bounded isometry group [Ré2, 12.2.1 and 12.2.3].

Theorem.— *For each* Γ*-fixed balanced subset* $\Omega^\natural$*, there exists a* Γ*-stable pair of opposite apartments containing* Ω*. Since the Galois action preserves barycenters,* $\mathrm{conv}(\Omega^\natural)$ *is* Γ*-fixed.*

Sketch of proof.— Theorem 3.1.B says that $\mathrm{Ad}_\Omega\big(\mathrm{Fix}_G(\Omega)\big)$ is a $\mathbf{K}$-group, so it admits a Cartan subgroup defined over $\mathbf{K}$. The Γ-equivariant dictionnary of 2.5.B says that this corresponds geometrically to a Γ-stable pair of opposite apartments containing Ω. The last assertion follows from the fact that the convex hull can be determined in any apartment containing $\Omega^\natural$. □

The last assertion is the starting point of convexity arguments, which enable to prove the following facts [Ré2, 12.2.4]:

– Two $\mathbf{K}$-facets of opposite signs are always contained in a $\mathbf{K}$-apartment.
– There is an integer d such that $\mathbf{K}$-apartments and $\mathbf{K}$-chambers are all d-dimensional.
– The convex hull of two opposite $\mathbf{K}$-chambers is a $\mathbf{K}$-apartment, and each $\mathbf{K}$-apartment can be constructed this way. In particular, $\mathbf{K}$*-apartments are symmetric double cones.*
– Each $\mathbf{K}$-facet is in the closure of a spherical one. This enables to define $\mathbf{K}$-chambers as $\mathbf{K}$-facets (*a priori* not spherical) of maximal closure.

Rational transitivity results.— The next step is to study $\mathscr{I}^\natural$ as a $G(\mathbf{K})$-set [Ré2, 12.3.2].

Theorem.— (i) *The group of rational points* $G(\mathbf{K})$ *is transitive on the* $\mathbf{K}$*-apartments.*
(ii) *Two* $\mathbf{K}$*-facets of arbitrary signs are always contained in a* $\mathbf{K}$*-apartment.* □

Remarks.— The proof of point (ii) requires a Levi decomposition argument for a Γ-fixed balanced subset made of three points, so Levi decompositions are useful for balanced subset more general than pairs of spherical points of opposite signs.

Relative roots. Anisotropic kernel.— To introduce a relative (TRD)-structure on the rational points, we need of course to define a relative Coxeter system $(W^\natural, S^\natural)$. We can make the choice of a pair of $\mathbf{K}$-facets $\pm F^\natural$ of opposite signs and of a pair $\pm c$ of opposite chambers in such a way that:

– $F^\natural$ and $-F^\natural$ define a $\mathbf{K}$-apartment $\mathbf{A_K}$ by $\mathbf{A_K} := \mathrm{conv}(F^\natural \cup -F^\natural)$;
– the Γ-stable facet $\pm F$ such that $\pm F^\natural = (\pm F)^\Gamma$ is in the closure of the chamber $\pm c$;
– the pair of opposite apartments $\mathbf{A}$ defined by $\pm c$ is Γ-stable.

We see $\mathbf{A}$ as the union of the Tits cone and its opposite in the real vector space V^*. At last, we denote by $L^\natural$ the linear span of $\mathbf{A_K}$ in V^*.

Definition.— *The restriction $a^\natural := a\,|_{L^\natural}$ of a root a of $\mathbf{A}$ (seen as a linear form on V^*) is a* $\mathbf{K}$-root *of $\mathbf{A_K}$ if the trace $D(a^\natural) := \mathbf{A_K} \cap D(a)$ is generic. In this case, $D(a^\natural)$ is called the* $\mathbf{K}$-halfspace *of $a^\natural$.*

For a given $\mathbf{K}$-halfspace $D(a^\natural)$, we consider the set $\Phi_{a^\natural}$ of roots b with $D(b) \cap \mathbf{A_K} = D(a^\natural)$. This set is stable under Γ and so is the group generated by the root groups U_b, $b \in \Phi_{a^\natural}$. Let us denote by $V_{D(a^\natural)}$ the group of its Γ-fixed points.

Definition.— (i) *The group $V_{D(a^\natural)}$ is the* relative root group *associated to the $a^\natural$.*
(ii) *The fixator* $\mathrm{Fix}_G(\mathbf{A_K})$ *– which we denote by $Z(\mathbf{A_K})$ – is the* anisotropic kernel *associated to $\mathbf{A_K}$.*

Rational group combinatorics.— The next step is to prove that $G(\mathbf{K})$ admits a nice combinatorial structure, stronger than BN-pairs but weaker than the $(\mathrm{TRD})_{\mathrm{lin}}$-axioms. This group combinatorics is due to V. Kac and D. Peterson: the *refined Tits systems* [KP2]. This is an abstract way to obtain the relative Coxeter system $(W^\natural, S^\natural)$ we are looking for, via BN-pairs. We have [Ré2, 12.4.1 and 12.4.2]:

Lemma.— *The group $W^\natural$ is a subquotient of $G(\mathbf{K})$, namely the quotient of the stabilizer in $G(\mathbf{K})$ of $\mathbf{A_K}$ by its fixator. The half of each sign of the double cone $\mathbf{A_K}$ is a geometric realization of the Coxeter complex of $W^\natural$, and the latter group is generated by reflections w.r.t. to $F^\natural$. The $\mathbf{K}$-halfspaces $D(a^\natural)$ are in $W^\natural$-equivariant bijection with the abstract set of roots $\Phi_{\mathbf{K}}$ of $(W^\natural, S^\natural)$.* $\square$

We have now an index set for the $(\mathrm{TRD})_{\mathrm{lin}}$-axioms. What is still unclear is how to define a linear interval of $\mathbf{K}$-halfspaces. For $D(a^\natural)$ and $D(b^\natural)$, we set

$$[D(a^\natural); D(b^\natural)]_{\mathrm{lin.}} := \{D(c^\natural) \mid \exists \lambda, \mu \in \mathbf{R}_+ \quad c^\natural = \lambda a^\natural + \mu b^\natural\}.$$

With all the objects defined above, point (i) of the Galois descent theorem more precisely says that *the group of rational points $G(\mathbf{K})$ – endowed with the $\mathbf{K}$-points $Z(\mathbf{K})^\Gamma$ of the anisotropic kernel and the family $\{V_{D(a^\natural)}\}_{D(a^\natural) \in \Phi_{\mathbf{K}}}$ – satisfies the* $(\mathrm{TRD})_{\mathrm{lin}}$*-axioms.* Point (ii) says that *the set of Γ-fixed points in the twin building is the realization of a twin building.* Besides, all the suggestive terminology for relative objects is justified.

Maximal $\mathbf{K}$-split tori.— In the formulation above, we forgot the maximal $\mathbf{K}$-split tori. According to 2.5.C, it is natural to expect a relative dictionary between $\mathbf{K}$-apartments and maximal

$\mathbf{K}$-split tori. A $\mathbf{K}$-*split torus* of G is a $\mathbf{K}$-split subgroup of a Γ-stable Cartan subgroup (this definition makes sense since the latter groups admit natural structures of $\mathbf{K}$-tori). Let us keep the $\mathbf{K}$-apartment $\mathbf{A_K}$, and $Z(\mathbf{K}) = \mathrm{Fix}_G(\mathbf{A_K})$.

Proposition.— *The anisotropic kernel $Z(\mathbf{A_K})$ is the fixator of any pair of opposite $\mathbf{K}$-chambers $\pm F^\natural$ in $\mathbf{A_K}$. It is a Γ-stable small subgroup and the Levi factor of the Γ-stable parabolic subgroup $\mathrm{Fix}_G(\pm F^\natural)$. Its associated algebraic group is semisimple anisotropic.*

Idea of proof. [Ré2, 12.3.2]— The first assertion follows from example 2 in 1.5.B, and the anisotropy part is justified as in the proof of Lang's theorem below (3.3.A). $\square$

As the quotient of a small subgroup defining its associated algebraic group is by a diagonalisable group, the anisotropy assertion gives rise to a $G(\mathbf{K})$-equivariant map

$$\begin{array}{ccc} \{\mathbf{K}\text{-apartments}\} & \longrightarrow & \{\text{Maximal } \mathbf{K}\text{-split tori}\} \\ \mathbf{A_K} & \mapsto & Z_d^0(\mathbf{A_K}), \end{array}$$

where $Z_d^0(\mathbf{A_K})$ is the connected $\mathbf{K}$-split part of the center of $Z(\mathbf{A_K})$. A more precise statement of point (iii) of the Galois descent theorem is that *each $\mathbf{K}$-split torus can be conjugated by $G(\mathbf{K})$ into $Z_d^0(\mathbf{A_K})$*. In particular, the map above is surjective. In fact, it is bijective for $\mathbf{K}$ large enough.

3.3. A Lang type theorem

We describe here the specific situation in which the groundfield is finite.

A. Quasi-splitness. An algebraic $\mathbf{K}$-group is *quasi-split* if it contains a Borel subgroup defined over $\mathbf{K}$, so in the twin situation, it is natural to introduce the following

Definition.— *A Kac-Moody $\mathbf{K}$-group is* quasi-split *if it contains two opposite Borel subgroups stable under the Galois group Γ.*

Remarks.— Geometrically, this means that two opposite chambers are Galois-stable.

For comparison, the classical Lang theorem [Bo, 16.6] asserts that over a finite field, an algebraic group has to be quasi-split.

Theorem (Lang's theorem for Kac-Moody groups).— *Let G be a Kac-Moody group, almost split over a finite field $\mathbf{F}_q$. Then G is in fact quasi-split over $\mathbf{F}_q$.*

Sketch of proof. [Ré2, 13.2.2]— Actually, the proof of this result makes use of the algebraic one. The argument works as follows. Look at the Levi decomposition of a minimal Γ-stable parabolic subgroup P of G. By minimality, the Levi factor is an algebraic $\mathbf{F}_q$-group which cannot contain a proper parabolic subgroup defined over $\mathbf{F}_q$. Indeed, it would otherwise lead to a Γ-stable parabolic subgroup smaller than P. Consequently, the Levi factor – which is reductive – has no proper parabolic subgroup defined over $\mathbf{F}_q$, but admits an $\mathbf{F}_q$-Borel subgroup: it must be a torus. $\square$

B. Down-to-earth constructions of forms. Restriction to quasi-split forms enables to define concretely twisted Kac-Moody groups. In fact, groups of this kind were already considered by J.-Y. Hée and J. Ramagge, who obtained structure theorems for «rational points» without reference to or use of buildings [H1 and H3], [Ra1 and Ra2]. These groups were studied in the wider context of generalized *Steinberg torsions*, which allowed J.-Y. Hée to define analogues of Ree-Suzuki groups. If we stick to Galois torsion as above, we have the following constructive result [Ré2, 13.2.3].

Proposition.— *Consider a generalized Cartan matrix A with Dynkin diagram D. Suppose we are given a Galois extension $\mathbf{E}/\mathbf{K}$ and a morphism $* : \mathrm{Gal}(\mathbf{E}/\mathbf{K}) \to \mathrm{Aut}(D)$. Then the simply connected (resp. adjoint) split Kac-Moody group defined over $\mathbf{E}$ admits a quasisplit form over $\mathbf{K}$ with $*$-action the morphism above.* □

Remark.— The Dynkin diagram of a generalized Cartan matrix is defined in [K]. The result is also valid with a suitable notion of automorphism for a Kac-Moody root datum. As we will see it as a procedure to construct twisted geometries which only depends on A, we will not be interested in these refinements.

3.4. Relative apartments and relative links

We precise now the determination of the shape of the apartments and of the relative links.

A. Geometry of relative apartments. Our aim here is to derive some more information about the geometry of $\mathbf{K}$-apartments from the $*$-action of the almost split form. This is to be combined with the constructive procedure of quasi-split forms 3.3.B.

We are still working with the datum of a pair of $\mathbf{K}$-facets $\pm F^\natural$ of opposite signs and of a pair $\pm c$ of opposite chambers s.t. $F^\natural$ and $-F^\natural$ define a $\mathbf{K}$-apartment $\mathbf{A}_\mathbf{K}$ by $\mathbf{A}_\mathbf{K} := \mathrm{conv}(F^\natural \cup -F^\natural)$; the Γ-stable facet $\pm F$ with $\pm F^\natural = (\pm F)^\Gamma$ is in the closure $\pm\overline{c}$ and the pair of opposite apartments $\mathbf{A}$ defined by $\pm c$ is Γ-stable. Recall that $\mathbf{A}$ is the union of the Tits cone and its opposite in the real vector space V^* and that $L^\natural$ is the linear span of $\mathbf{A}_\mathbf{K}$ in V^*. By transitivity of $G(\mathbf{K})$, all $\mathbf{K}$-chambers are fixed points in Γ-stable facets of fixed type S_0.

Lemma.— *Denote by Γ^* the automorphism group of the standard chamber c arising from the $*$-action. Then $L^\natural$ admits the following definition by linear equations.*

$$L^\natural = \{x \in V^* \mid a_s(x) = 0 \ \forall s \in S_0 \text{ and } a_s(x) = a_t(x) \text{ for } \Gamma^* s = \Gamma^* t\},$$

and the $\mathbf{K}$-chamber is a simplicial cone of dimension $d \geq S^\natural$ defined by the same equations for x in c_+. □

This result [Ré2, 12.6.1] enables to determine the shape of a $\mathbf{K}$-chamber or a $\mathbf{K}$-apartment from the datum of the $*$-action. In other words, this is a concrete procedure to determine a part of the geometry of the twisted twin building. To get more precise information, we have to use finer theoretical results concerning infinite Kac-Moody type root systems. Following the abstract results in [Ba], the datum of S_0 and of the $*$-action determines a *relative Kac-Moody matrix*, a relative version of the matrix A. Then, there is a similar rule to deduce a Coxeter from it, and it happens to be naturally isomorphic to the relative Weyl group $W^\natural$. So another piece of the twisted combinatorics is theoretically computable.

B. Geometry of relative residues. Once we have determined the geometry of the slices of the twisted twin building, one may want to know more about the local geometry of it. This we will also accomplish thanks to the knowledge of S_0 and of the $*$-action. In the split case, according to 2.2.B, the fixator of a pair of opposite spherical facets $F_\pm$ is abstractly isomorphic to a split reductive group. Besides, a realization of its twin building is given by the union of the residues of F_+ and F_-. The relative version of this is also valid. More precisely, let us consider a pair of opposite spherical $\mathbf{K}$-facets $\pm F^\natural$ whose union will be denoted by $\Omega^\natural$. It can be shown then that the Galois actions on the Γ-stable residues yields a natural $\mathbf{K}$-structure on the reductive algebraic groups to which the small subgroup $\mathrm{Fix}_G(\Omega^\natural)$ is isomorphic. For instance, if we assume that $\pm F^\natural$ are spherical $\mathbf{K}$-panels, this provides information about rational root groups. In fact, we have [Ré2, 12.5.2 and 12.5.4]:

Lemma.— (i) *The union of the sets of* $\mathbf{K}$*-chambers whose closure contains* $\pm F^\natural$ *is a geometric realization of the twinning of an algebraic reductive* $\mathbf{K}$*-group.*
(ii) *Rational root groups are isomorphic to root groups in algebraic reductive* $\mathbf{K}$*-groups. In particular, the only possible proportionality factor* > 1 *between* $\mathbf{K}$*-roots is* 2 *and rational root groups are metabelian.* $\square$

Combined with the geometric consequence of the Moufang property, the second point enables to compute thicknesses.

3.5. A classical twin tree

A. The split situation. Recall that $\mathbf{SL}_3(\mathbf{K}[t,t^{-1}])$ is a split Kac-Moody group defined over the field $\mathbf{K}$ and of affine type $\widetilde{A}_2$. The Dynkin diagram is the simply-laced triangle. The simple roots of the $\widetilde{A}_2$-affine root system are denoted by a_0, a_1 and a_2, for which we choose respectively the parametrizations $x_0 : r \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -rt & 0 & 1 \end{pmatrix}$, $x_1 : r \mapsto \begin{pmatrix} 1 & r & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $x_2 : r \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$. The standard Cartan subgroup is $T = \{D_{u,v} := \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1}v & 0 \\ 0 & 0 & v^{-1} \end{pmatrix} \mid u, v \in \mathbf{E}^\times\}$. At last, we work for this example with the Bruhat-Tits realization of the buildings.

B. The involution. Let us consider a separable quadratic extension $\mathbf{E}/\mathbf{K}$. We want to define a quasi-split $\mathbf{K}$-form of $\mathrm{SL}_3(\mathbf{E}[t,t^{-1}])$ so as to produce a semi-homogeneous twin tree by quasi-split Galois descent. Fom a theoretical point of view (3.3.B), it is enough to find an involution of the Dynkin diagram. We choose the inversion of types 1 and 2. We describe now concretely to which Galois action this choice leads. Let ι (resp. τ) be the inversion (resp. the «antidiagonal» symmetry) of matrices.

$$\tau : \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} i & f & c \\ h & e & b \\ g & d & a \end{pmatrix}.$$

Let $\cdot^\sigma$ be at the same time the nontrivial element of $\mathrm{Gal}(\mathbf{E}/\mathbf{K})$ and this operation on all matrix coefficients. Then the involution $\cdot^* := \tau \circ \iota \circ \cdot^\sigma$ of $\mathbf{SL}_3(\mathbf{E}[t,t^{-1}])$ is the one given by the quasi-split $\mathbf{K}$-form. It is completely determined by the formulas $x_1(r)^* = x_2(r^\sigma)$, $x_2(r)^* = x_1(r^\sigma)$ and $x_0(r)^* = x_0(r^\sigma)$, the analogues for negative simple roots, and $D^*_{u,v} = D_{v^\sigma,u^\sigma}$. The group of monomial matrices lifting modulo T the affine Weyl group $W = \mathcal{S}_3 \ltimes \mathbf{Z}^2$ is also $\cdot^*$-stable. This implies the $\cdot^*$-stability of the standard apartments $\mathbf{A}_+ = \{wB_+\}_{w\in W}$ and $\mathbf{A}_- = \{wB_-\}_{w\in W}$. The set of *-fixed points inside is a $\mathbf{K}$-apartment, since according to the permutation of types, there cannot be a two-dimensional **K**-chamber. This set $\mathbf{A_K}$ is the median of the standard equilateral triangle (intersecting the edge of type 0) w.r.t. to which $\cdot^*$ operates by symmetry in **A**. Consequently, each building is a tree and the relative Weyl group $W^\natural$ is the infinite dihedral group D_∞. Translations are lifted by the matrices $\begin{pmatrix} at^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (at^n)^{-1} \end{pmatrix}$; reflexions by the matrices $\begin{pmatrix} 0 & 0 & (at^n)^{-1} \\ 0 & 1 & 0 \\ -at^n & 0 & 0 \end{pmatrix}$, $(n \in \mathbf{Z},\ a \in \mathbf{K}^\times)$. The reason why we call the twin building so obtained a classical twin tree is that, up to sign conventions, the Steinberg torsion we defined is that of the unitary group SU_3 considered in [BrT2, 4.1.9].

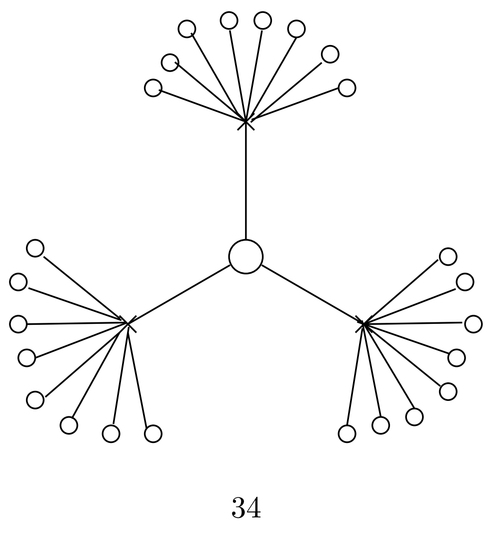
C. Semi-homogeneity. What is left to do is to compute valencies. We will get semi-homogeneous trees since there are two shapes of intersections of the **K**-apartment with walls. We sketch arguments for positive standard vertices and edges, but the situation is completely symmetric w.r.t. signs and can be deduced for every facet by transitivity of the rational points.

The first (simplest) situation is given by the intersection of the standard edge of type 0 with $\mathbf{A_K}$. In this case, only one wall passes through the intersection point and each of the two opposite **K**-roots is the trace on $\mathbf{A_K}$ of a single root. The corresponding root groups are stable and their groups of rational points are isomorphic to the additive group of **K**. Hence a valency at all transforms of this **K**-vertex equal to $1 + \#\mathbf{K}$.

The other situation corresponds to the fact that $\mathbf{A_K}$ contains the standard vertex of type $\{1;2\}$ for which 3 roots leave the same **K**-root as trace on $\mathbf{A_K}$. The small subgroup fixing this vertex and its opposite in **A** is stable and abstractly isomorphic to SL_3 over **E**. According to the action around the vertex, the three positive root groups passing through it are permuted and the small subgroup is abstractly isomorphic to a quasi-split reductive **K**-group of rank one. The root groups of such groups are described in detail in [BrT2, 4.1.9]; they are rational points of the unipotent radical of a Borel subgroup of SU_3, in bijection with $\mathbf{K}^3$. This can also be seen directly on the explicit description of the Galois action above. Hence a valency at all transforms of this **K**-vertex equal to $1 + \#\mathbf{K}^3$.

This shows that we built a semi-homogeneous twin tree of valencies $1 + \#\mathbf{K}$ and $1 + \#\mathbf{K}^3$. So in the case of a finite groundfield $\mathbf{K} = \mathbf{F}_q$, one gets a locally finite semi-homogeneous twin tree of valencies $1 + q$ and $1 + q^3$.

Figure.— [Here $q = 2$]



4. HYPERBOLIC EXAMPLES. LATTICES

So far, we just considered examples of Chevalley groups over Laurent polynomial rings or twisted versions of them. Section 4.1 deals with split and twisted Kac-Moody groups whose buildings are hyperbolic. In §4.2, we show that Kac-Moody theory provides analogues of arithmetic groups over function fields.

4.1. Hyperbolic examples

At last, some more exotic Kac-Moody geometries will be described.

A. Hyperbolic buildings. The definition of such spaces [GP] is the metric version of that of buildings from the apartment system point of view.

Definition.— *Let R be a hyperbolic polyhedron providing a Poincaré tiling of the hyperbolic space $\mathbb{H}^n$. A* hyperbolic building of type R *is a piecewise polyhedral cell complex, covered by a family of subcomplexes – the* apartments*, all isomorphic to the tiling above and satisfying the following incidence axioms.*

(i) *Two points are always contained in an apartment.*
(ii) *Two apartments are isomorphic by a polyhedral arrow fixing their intersection.*

A building whose apartments are tilings of the hyperbolic plane $\mathbb{H}^2$ will be called Fuchsian.

The main point about hyperbolic buildings is that we get this way CAT(-1)-spaces i.e., metric spaces where geodesic triangles are at least as thin as in $\mathbb{H}^2$. In particular, they are hyperbolic in the sense of Gromov. The additional incidence axioms make these spaces nice geometries where to refine results known for hyperbolic metric spaces, or to generalize results about symmetric spaces with (strict) negative curvature.

Remark.— Note that in contrast with the affine case, for hyperbolic buildings there is usually no more relation between the dimension of the apartments and the rank of the Weyl group of the building. The two-dimensional case with a regular right-angled r-gon is particularly striking: $r \geq 5$ may be arbitrary large. Nevertheless, we will sometimes be interested in the case called *compact hyperbolic* by Bourbaki. It corresponds to a tesselation of $\mathbb{H}^n$ by a hyperbolic simplex. In this particular case, the relation alluded to above is valid; and the Weyl group shares another property with the affine case: every proper subset of canonical generators gives rise to a finite Coxeter subgroup.

B. Split twin hyperbolic buildings. Our purpose here is to show which kind of hyperbolic buildings can be produced via Kac-Moody machinery. Let us make preliminary remarks, valid in both split and twisted cases. First, the rule of point (i) in the second definition of 2.1.A shows that the Coxeter coefficients involved in Kac-Moody theory are very specific: 2, 3, 4, 6 or ∞. So by far not any Coxeter group appears as the Weyl group of a generalized Cartan matrix. Besides, the thicknesses will always be of the form «1 + prime power» , and will be constant in the split case. These are strong conditions, but not complete obstructions.

Hyperbolic realizations.— It is clear that standard gluing techniques [D2, §10] apply in the hyperbolic case once affine tilings replaced by hyperbolic ones. Given P a hyperbolic Poincaré polyhedron with dihedral angles $\neq 0$ are of the form π/m with $m = 2$, 3, 4 or 6, and which may have vertices at infinity; and given p^k a prime power, there exists a Kac-Moody building of constant thickness $1 + p^k$ admitting a metric realization whose apartments are Poincaré tilings defined by P [Ré1 §2]. This building is a complete geodesic proper CAT(-1) metric space. This is just an existence result, it is expected to be combined with a uniqueness assertion, but the following result due to D. Gaboriau and F. Paulin ([GP], théorème 3.5) is quite disheartening.

Theorem.— *Let P be a regular hyperbolic Poincaré polygon with an even number of edges and angles equal to $\pi/m, m \geq 3$. Let L be a fixed generalized m-gon of classical type and sufficiently large characteristic. Then, there exist uncountably many hyperbolic buildings with apartments the tiling provided by P and links all isomorphic to L.* □

The latter theorem is in contrast with the following characterization proved by M. Bourdon [Bou1, 2.2.1], which shows that in the two-dimensional case, there is some kind of dichotomy according to whether $m \geq 3$ or $m = 2$. The latter kind of buildings will be referred to as the class of *right-angled Fuchsian buildings.*

Proposition.— *Consider the Poincaré tiling of the hyperbolic plane $\mathbb{H}^2$ by a right-angled r-gon ($r \geq 5$). Then, there exists a unique building $I_{r,q+1}$ satisfying the following conditions:*

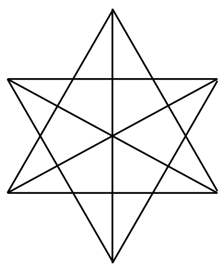
– *apartments are all isomorphic to this tesselation;*
– *the link at each vertex is the complete bipartite graph of parameters $(q+1, q+1)$.* □

Consequently, we know by uniqueness that the building $I_{r,q+1}$ – $r \geq 5$, q a prime power – comes from a Kac-Moody group.

C. Quasi-split Kac-Moody groups and hyperbolic buildings. On the one hand, according to subsection 2.2.B, there is no possibility to construct buildings with non-constant thickness via split Kac-Moody groups. On the other hand, we know by 3.5 that we can obtain semi-homogeneous twin trees from classical groups. Here is an example of what can be done in the context of hyperbolic buildings.

Semi-homogeneous trees in Fuchsian buildings. [Ré2, 13.3]— Definition of $\mathbf{F}_q$-forms relies on the existence of symmetries in the Dynkin diagram. Let us sketch the application of method 3.3.B to the buildings $I_{r,q+1}$. Denote by D the Coxeter diagram of the reflection group associated to the regular right-angled r-gon $R \subset \mathbb{H}^2$. To get it, draw the r-th roots of 1 on the unit circle and connect each pair of non consecutive points by an edge $\overset{\infty}{\text{———}}$. To keep all symmetries, lift this Coxeter diagram into a Dynkin diagram replacing the edges $\overset{\infty}{\text{———}}$ by $\Longleftrightarrow$. The corresponding generalized Cartan matrix A is indexed by $\mathbf{Z}/r$; it is defined by $A_{\bar{i},\bar{i}} = 2$, $A_{\bar{i},\bar{i}+1} = 0$ and -2's elsewhere.

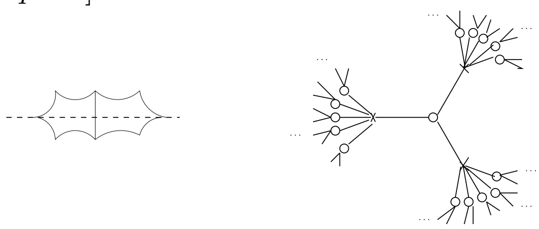
Figure.— [Here $r = 6$]



We suppose now that the integer r is odd, so that D is symmetric w.r.t. each axis passing through the middle of two neighbors and the opposite vertex. We fix such a symmetry, and consider the quadratic extension $\mathbf{F}_{q^2}/\mathbf{F}_q$: this defines a quasi-split form as in 3.5.B. Moreover, the arguments to determine the shape of an $\mathbf{F}_q$-chamber or an $\mathbf{F}_q$-apartment are the same.

We get a straight line cut into segments and acted upon by the infinite dihedral group D_∞. The twisted buildings are trees.

Figure.— [Here $r = 5$ and $q = 2$]



For the valencies, we must once again distinguish two kinds of intersection of the boundary of R with the $\mathbf{F}_q$-apartment, which leads to two different computations. When the $\mathbf{F}_q$-apartment cuts an edge, the rational points of the corresponding root group are isomorphic to $\mathbf{F}_q$, hence a valency equal to $1 + q$. The other case is when the $\mathbf{F}_q$-apartment passes through a vertex contained in two orthogonal walls, with two roots leaving the same trace on the Galois-fixed line. The corresponding Galois-stable group is then a direct product of two root groups (each isomorphic to $\mathbf{F}_{q^2}$) acted upon by interversion and Frobenius: the valency at these points is then $1 + q^2$. Consequently, we obtain this way a semi-homogeneous twin tree with valencies $1 + q$ and $1 + q^2$.

Twisted Fuchsian buildings in higher-dimensional hyperbolic buildings. [Ré1, §4.5]— Galois descent makes the dimension of apartments decrease, so if we want Fuchsian buildings with non constant thickness, we must consider higher dimensional hyperbolic buildings. Let us consider a specific example obtained from a polyhedron in the hyperbolic ball $\mathbf{B}^3$ and admitting many symmetries: it is the truncated regular hyperideal octahedron with dihedral angles $\pi/2$ and $\pi/3$. This polyhedron P is obtained by moving at same speed from the origin eight points on the axis of the standard orthogonal basis of $\mathbf{R}^3$. The condition on the angles impose to move the points outside of $\mathbf{B}^3$. What is left to do then is to cut the cusps by totally geodesic hyperplanes centered at the intersections of $\mathbf{S}^2 = \partial\mathbf{B}^3$ with the axis. A figure of this polyhedron is available in [GP]. The polyhedron P admits a symmetry w.r.t. hyperplanes containing edges and cutting P along an octogon. The residue at such an edge – where the dihedral angle is $\pi/3$ – is the building of SL_3. This enables to define a quasi-split $\mathbf{F}_q$-form whose building is Fuchsian. The $\mathbf{F}_q$-chambers are octogons and thicknesses at $\mathbf{F}_q$-edges are alternatively $1 + q$ and $1 + q^3$.

D. Hyperbolic buildings from a geometrical point of view. Here is now a small digression concerning single hyperbolic buildings, without the assumption that they arise from Kac-Moody theory. This often implies to restrict oneself to specific classes of hyperbolic buildings, but provides stimulating results.

Existence, uniqueness and homogeneity.— We saw in the previous subsection that unicity results were useful to know which buildings come from Kac-Moody groups. Nevertheless, the «disheartening theorem» above shows that the shape of the chambers and the links are not sufficient in this direction. F. Haglund [Ha] determined conditions to be added so as to obtain characterization results for two-dimensional hyperbolic buildings. His study enables him to know when the (type-preserving) full automorphism group is transitive on chambers.

Automorphism groups.— The first step towards group theory then is to consider groups acting on a given hyperbolic building. Two kinds of groups are particularly interesting: full automorphism groups and lattices. Concerning full automorphism groups, F. Haglund and F. Paulin [HP] proved abstract simplicity and non-linearity results for two-dimensional buildings. Besides, É. Lebeau [L] proved that some others of these groups are strongly transitive on their buildings, hence admit a natural structure of BN-pair – see [Ron], §5. Concerning lattices, the first point is the existence problem. In fact, the main construction technique is a generalization in higher dimension of Serre's theory of graph of groups [Se]. D. Gaboriau and F. Paulin [GP] determined the precise local conditions on a *complex of groups* in the sense of A. Haefliger [BH] to have a hyperbolic building as universal covering. This method has the advantage to produce at the same time a uniform lattice with the building. The lattices we will obtain by Kac-Moody theory are *not* uniform.

Boundaries: quasiconformal geometry and analysis.— The reason why boundaries of metric hyperbolic metric spaces are interesting is that they can be endowed with a quasiconformal structure – see [GH], §7. In the case of hyperbolic buildings, the structure can be expected to be even richer or better known, in view of the CAT(-1)-property and the incidence axioms. At least for Fuchsian buildings, this is indeed the case as shown by works due to M. Bourdon. In this situation, he relates the precise conformal dimension of the boundary and the growth rate of the Weyl group [Bou2]. In a joint paper with H. Pajot [BP1], the existence of Poincaré inequalities and the structure of Loewner space is proved for the boundaries of the $I_{r,q+1}$'s.

Rigidity.— Rigidity is a vast topic in geometric group theory. In this direction, results were proved by M. Bourdon first [Bou1] and then by M. Bourdon and H. Pajot [BP2]. The first paper deals with Mostow type rigidity of uniform lattices. It concerns right-angled Fuchsian buildings. A basic tool there is a combinatorial crossratio constructed thanks to the building structure involved. It enables to generalize arguments of hyperbolic geometry by D. Sullivan to the singular case of (boundaries of) buildings. The second paper uses techniques from analysis on singular spaces to prove a stronger result implying Mostow rigidity, and in the spirit of results by P. Pansu [P1], B. Kleiner and B. Leeb [KL] for Riemannian symmetric spaces (minus the real and complex hyperbolic spaces) and higher rank Euclidean buildings. More precisely, M. Bourdon and H. Pajot proved [BP2]:

Theorem (Rigidity of quasi-isometries).— *Every quasi-isometry of a right-angled Fuchsian building lies within bounded distance from an isometry.* □

For the notion of a *quasi-isometry*, see for instance [GH, §5]. Roughly speaking, it is an isometry up to an additive and a multiplicative constant, surjective up to finite distance.

4.2. Analogy with arithmetic lattices

The purpose of this section is to present several arguments to see Kac-Moody groups over finite fields as generalizations of arithmetic groups over function fields.

A. Kac-Moody topological groups and lattices. Let G be a Kac-Moody group, almost split – hence, quasi-split (3.3.A) – over the finite field $\mathbf{F}_q$. We denote by $\Delta_\pm$ the locally finite metric realizations of the buildings associated to G. Recall that all points there are of spherical type, hence the fixator of any point is a spherical parabolic subgroup. By local finiteness, the (isomorphic) corresponding full automorphism groups are locally compact. We use the notations $\mathrm{Aut}_\pm := \mathrm{Aut}(\Delta_\pm)$. The groups Aut_+ and Aut_- are besides unimodular and totally disconnected. We choose a left invariant Haar measure $\mathrm{Vol}_\pm$ and normalize it by $\mathrm{Vol}_\pm\big(\mathrm{Fix}_{\mathrm{Aut}_\pm}(c_\pm)\big) = 1$, where $c_\pm$ is the standard chamber.

Let us turn now to the analogy with arithmetic groups [Mar, p.61]. The affine case is indeed a guideline since Chevalley groups over Laurent polynomials are $\{0;\infty\}$-arithmetic groups over the global field $\mathbf{F}_q(t)$. This is the example of an arithmetic group for two places. If we consider now the fixator of a facet, we get a group commensurable with the value of the latter Chevalley group over the polynomials $\mathbf{F}_q[t]$, so that each fixator of facet is an arithmetic group for one place. Besides, the normalization we chose for the invariant measure consists in the affine case in assigning volume 1 to an Iwahori subgroup. We just replaced here a p-adic Lie group by the full automorphism group of a building.

If we consider now the general case, here is a result supporting the analogy, independently proved by L. Carbone and H. Garland in the split case [CG, Ré3].

Theorem.— *Let G be an almost split Kac-Moody group over a finite field $\mathbf{F}_q$. Denote by $\sum_{n\geq 0} d_n t^n$ the growth series of the Weyl group W of G. Assume $\sum_{n\geq 0} \frac{d_n}{q^n} < \infty$. Then:*

(i) *The group G is a lattice of (the full automorphism group of) $\Delta_+ \times \Delta_-$.*
(ii) *For any point in $x_- \in \Delta_-$, $\mathrm{Fix}_G(x_-)$ is a lattice of Δ_+.*
(iii) *These lattices are not uniform.* □

Remarks.— 1. Non-uniformness is due to J. Tits, who determined fundamental domains for actions of parabolic subgroups in a combinatorial setting more general than the (TRD) axioms [T3, A1 §3].
2. Of course, the situation is completely symmetric w.r.t. signs: point (ii) is still true after inversion of $+$ and $-$.
3. There is a straightforward generalization of this result to the case of locally finite Moufang twin buildings. The cardinal of the groundfield has to be replaced by the maximal cardinal q of root groups. According to the Moufang property (1.3.B), this integer is geometrically computed as follows. Consider any chamber, then $q+1$ is the maximal thickness among all the panels in its closure.

B. *Sketch of proof.*— We just consider the case of the fixator of a negative point – spherical negative parabolic subgroup. As we work over a finite field, we can make use of the following facts due to Levi decompositions, 1.5.B and 1.5.A respectively.

(i) The fixator of a pair of (spherical) points of opposite signs is finite.
(ii) Borel subgroups are of finite index in the spherical parabolic subgroups containing them.

Discreteness follows from point (i), and according to point (ii), it is enough to show that the fixator B_- of a negative chamber is of finite covolume. The idea is to adapt Serre's criterion on trees to compute a volume. This requires to look at the series $\sum_{d_+\in\mathcal{C}} \frac{1}{\#\mathrm{Stab}_{B_-}(d_+)}$, where $\mathcal{C}$ is a complete set of representatives of chambers modulo the discrete group considered. According to Tits' result alluded to above, one has $\mathcal{C} = \mathbf{A}_+$. Thanks to a special case of the Levi decompositions (1.5.B), the series is $\leq \frac{1}{\#T} \sum_{w\in W} \frac{1}{q^{\ell(w)}}$. Rearranging elements of W w.r.t. length in the dominating series, one gets the series appearing in the statement. □

Remark.— When G is split over $\mathbf{F}_q$, the assumption on the numerical series is sharp.

C. Classical properties for Kac-Moody lattices. The result above should be seen as a starting point. Indeed, one may wonder whether some classical properties of arithmetic groups are relevant or true for this kind of lattices. Some answers are already known.

Moufang property.— The first remark in order for the analogy is to consider Moufang property. Recall that this property is already of interest in classical situations but is not satisfied by all p-adic Lie groups. For instance, it can be proved for many Bruhat-Tits buildings arising from reductive groups over local fields of equal (positive) characteristic, but it is known for no local field of characteristic zero [RT1, §2]. This remark suggests that the analogy with arithmetic groups in characteristic 0 will not be fruitful.

Kazhdan's property (T)*.—* This property has proved to be useful in many situations, and makes sense for arbitrary topological groups, endowed with the discrete topology for instance. In the classical case of Lie groups and their lattices, its validity is basically decided by the rank of the ambient Lie group [HV, 2.8]. In rank ≥ 2, all Lie groups are (T) and closed subgroups of finite covolume inherit the property [HV, 3.4]. Over non-archimidean local fields and in rank 1, no group is (T) since a necessary condition is to admit a fixed point for each isometric action on a tree [HV, 6.4]: contradiction with the natural action on the Bruhat-Tits tree. Another necessary condition is for instance compact generation, that is finite generation in the discrete case.

Property (T) is often characterized in terms of unitary representations, but the former necessary condition involving actions on trees suggests to look for geometrical criteria having the advantadge to make generalizations easier. An important example of this approach works as follows. On the one hand, H. Garland used the combinatorial Laplace operator on Bruhat-Tits buildings to obtain vanishing results for cohomology with coefficients in finite-dimensional representations [G]. On the other hand, property (T) is characterized by cohomology vanishing for every unitary representation. The idea to generalize Garland's method to infinite-dimensional representations is due to Pansu-Zuk [Z, P2], then many people elaborated on this. For instance, nice updatings and improvements as well as constructions of exotic discrete groups with property (T) are available in Ballmann-Swiatkowski's paper [BS]. That this concerns Kac-Moody groups with compact hyperbolic Weyl groups was seen independently by Carbone-Garland and Dymara-Januszkiewicz [CG, DJ]. According to the first cohomology vanishing criterion [HV, 4.7], the following result [DJ, Corollary 2] proves more than property (T).

Theorem.— *Let G be a Kac-Moody group over $\mathbf{F}_q$ with compact hyperbolic Weyl group. Then for q large enough and $1 \leq k \leq n-1$, the continuous cohomology groups $H^k_{ct}\big(\mathrm{Aut}(\Delta_\pm), \rho\big)$ with coefficients in any unitary representation ρ vanish.* $\square$

Remarks.— 1. Note that the case for which we have a positive answer is for a simplex as chamber. Actually, negative results are proved for Fuchsian buildings with non simplicial chambers.
2. Note also the condition on thickness, inside of which is hidden a spectral condition on the Laplace operator. It is actually necessary, in contrast with the classical situation as we will see in the forthcoming subsection.

Finiteness properties.— A way to define these finiteness properties – F_n or FP_n – is to use cohomology or actions on complexes, but for the first two degrees it is closely related to finite generation and finite presentability. A criterion defining the class of non-classical Kac-Moody groups for which positive results are known, involves finiteness of standard subgroups of the Weyl group. More precisely, we say that the Coxeter matrix M is *n-spherical* if $W_J = \langle J \rangle$ is finite for any subset J with $\# J \leq n$. For instance, if M is of irreducible affine or compact hyperbolic type, it is $(r-1)$-spherical but not r-spherical; and 2-spherical means that all coefficients are finite. When results are known, they look like those of arithmetic groups in characteristic p, for which much less is known than in the characteristic 0 case. The abstract framework of group actions on twin buildings in [A1] enables to formulate and prove results in both classical and Kac-Moody cases. The arguments there rely on homotopy considerations, in particular make use of a deep result by K. Brown [Br1]. If we stick to finite generation or finite presentability, techniques of ordered sets from [T3] can also be used [AM]. Here are more recent results by P. Abramenko [A2]:

Theorem.— (i) *If $\# S > 1$ and M is not 2-spherical, then $B_\pm$ is not finitely generated (for arbitrary q).*
(ii) *Suppose $q > 3$ and M 2-spherical but not 3-spherical. Then $B_\pm$ is finitely generated but not finitely presentable.*
(iii) *Suppose that $q > 13$ and M is 3-spherical. Then $B_\pm$ is finitely presentable.* □

Remarks.— 1. Non finite generation (resp. non finite presentability) is a way to contradict property (T) (resp. Gromov hyperbolicity).
2. The hypothesis «$\mathbf{F}_q$ large enough» once again comes into play. P. Abramenko noticed also that over tiny fields, some groups as in the theorem are lattices of their building but are *not* finitely generated. This shows that the condition on thickness is necessary for the previous theorem concerning property (T), while it is not so in the classical case (rank argument).

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