Vanishing results for the cohomology of complex toric hyperplane complements

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Abstract

Suppose $\mathcal{R}$ is the complement of an essential arrangement of toric hyperlanes in the complex torus $(\mathbb{C}^*)^n$ and $\pi = \pi_1(\mathcal{R})$. We show that $H^*(\mathcal{R}; A)$ vanishes except in the top degree $n$ when $A$ is one of the following systems of local coefficients: (a) a system of nonresonant coefficients in a complex line bundle, (b) the von Neumann algebra $\mathcal{N}\pi$, or (c) the group ring $\mathbb{Z}\pi$. In case (a) the dimension of $H^n$ is $|e(\mathcal{R})|$ where $e(\mathcal{R})$ denotes the Euler characteristic, and in case (b) the $n^{th}$ $\ell^2$ Betti number is also $|e(\mathcal{R})|$.


Keywords: hyperplane arrangements, toric arrangements, local systems, $L^2$-cohomology.

1 Introduction

A complex toric arrangement is a family of complex subtori of a complex torus $(\mathbb{C}^*)^n$. The study of such objects is a relatively recent topic. Different versions of these arrangements, also known as toral arrangements, have been introduced and studied in works of Lehrer [16, 17], Dimca-Lehrer [11], Douglass [12], Looijenga [18] and Macmeikan [20, 21].

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The foundation of the topic can be traced to the paper [10] by De Concini and Procesi. There the main objects are defined and the cohomology of the complement of a toric arrangement is studied. An explicit goal of [10] is to generalize the theory of hyperplane arrangements. (For an extensive account of the work of De Concini and Procesi see [9].)

The next step is the work of Moci, in particular his papers [22, 23, 24], developing the theory with a special focus on combinatorics. In [25] Moci and Delucchi generalize results in [25] to a wider class of toric arrangements which they call thick. In [3] D’Antonio and Delucchi develop a special class of toric arrangements which they call complexified because of structural affinity with the case of hyperplane arrangements. They also prove that complements of complexified toric arrangements are minimal (see [4]).

In this paper we generalize to toric arrangements a well known result for affine arrangements: vanishing conditions for the cohomology of the complement $M(A)$ of an arrangement $A$ with coefficients in a complex local system $A$. Necessary conditions for $H^k(M(A); A) = 0$ if $k \neq n$, i.e., for the cohomology to be concentrated in top dimension, have been determined by a number of authors, including Kohno [15], Esnault, Schechtman, and Viehweg [13], Davis, Januszkiewicz, and Leary [5], Schechtman, Terao, and Varchenko [28] and Cohen and Orlik [2]. In particular, in [28] (see also [2]) it is proved that the cohomology of the complement $M(A)$ of an arrangement with coefficients in a complex local system is concentrated in top dimension provided certain nonresonance conditions for monodromies are fulfilled for a certain subset of edges (i.e., intersections of hyperplanes) that are called denses.

In order to generalize the above results we use techniques developed by the first author in a joint work with Januszkiewicz, Leary, and Okun, [5, 6, 7, 8]. One considers an open cover of the complement $M$ by “small” open sets each homeomorphic to the complement of a central arrangement. In the cases of nonresonant rank one local coefficients or $\ell^2$ coefficients, the $E_1$ page of the resulting Mayer-Vietoris spectral sequence is nonzero only along the bottom row, where it can be identified with the simplicial cochains with constant coefficients on a pair $(N(U), N(U_{\text{sing}}))$, which is homotopy equivalent to $(\mathbb{C}^n, \Sigma)$ where $\Sigma$ is the union of all hyperplanes in the arrangement. (The simplicial complex $N(U)$ is the nerve of an open cover of $\mathbb{C}^n$ and $N(U_{\text{sing}})$ is a subcomplex.)

It follows that the $E_2$ page can be nonzero only in position $(l, 0)$. One also can prove that for an affine hyperplane arrangement of rank $l$ only the
\( l \)-th \( \ell^2 \)-Betti number of the complement \( M \) can be nonzero and that it is equal to the rank of the reduced \((l-1)\)-homology of \( \Sigma \) (cf. [5]). Similarly, with coefficients in the group ring, \( \mathbb{Z}_\pi \), for \( \pi = \pi_1(M) \), \( H^*(M; \mathbb{Z}_\pi) \) is nonzero only in degree \( l \) (cf. [6]). We generalize all of these vanishing results to the toric case in Theorems 5.1, 5.2 and 5.3.

In recent work [27], Papadima and Suciu generalize the result in [2] to arbitrary minimal CW-complex, i.e., a complex having as many \( k \)-cells as the \( k \)-th Betti number. It would be very interesting to decide if the complement of toric arrangement also could be minimal. In this case Theorem 5.1 would be a consequence of minimality.

Our paper begins with a review of some background about toric and affine arrangements. Then, in Section 3, we give a brief account of open covers by “small” convex sets. In Section 4 we recall basic definitions on systems of local coefficients. Finally in Section 5 we prove that the cohomology of the complement of a toric arrangement with coefficient in a local system \( A \) vanishes except in the top degree when \( A \) is a nonresonant local system, the von Neumann algebra \( N_\pi \) or the group ring \( \mathbb{Z}_\pi \).

2 Affine and toric hyperplane arrangements

\textbf{Affine hyperplanes arrangements} A hyperplane arrangement \( \mathcal{A} \) is a finite collection of affine hyperplanes in \( \mathbb{C}^n \). A subspace of \( \mathcal{A} \) is a nonempty intersection of hyperplanes in \( \mathcal{A} \). Denote by \( L(\mathcal{A}) \) the poset of subspaces, partially ordered by inclusion, and let \( \overline{L}(\mathcal{A}) := L(\mathcal{A}) \cup \{\mathbb{C}^n\} \). An arrangement is central if \( L(\mathcal{A}) \) has a minimum element. Given \( G \in L(\mathcal{A}) \), its rank, \( \text{rk}(G) \), is the codimension of \( G \) in \( \mathbb{C}^n \). The minimal elements of \( L(\mathcal{A}) \) form a family of parallel subspaces and they all have the same rank. The rank of an arrangement \( \mathcal{A} \) is the rank of a minimal element in \( L(\mathcal{A}) \). \( \mathcal{A} \) is essential if \( \text{rk}(\mathcal{A}) = n \).

The singular set \( \Sigma(\mathcal{A}) \) of \( \mathcal{A} \) is the union of hyperplanes in \( \mathcal{A} \). The complement of \( \Sigma(\mathcal{A}) \) in \( \mathbb{C}^n \) is denoted \( M(\mathcal{A}) \).

\textbf{Toric arrangements} Let \( T = (\mathbb{C}^*)^n \) be a complex torus and let \( \Lambda = \text{Hom}(T, \mathbb{C}^*) \) denote the group of characters of \( T \). Then \( \Lambda \cong \mathbb{Z}^n \). A character is primitive if it is a primitive vector in \( \Lambda \). Given a primitive character \( \chi \) and an element \( a \in \mathbb{C}^* \) put
\[
H_{\chi,a} = \{ t \in T \mid \chi(t) = a \}.
\]
The subtorus $H_{\chi,a}$ is a toric hyperplane. A finite subset $X \subset \Lambda \times \mathbb{C}^*$ defines a toric arrangement,

$$\mathcal{T}_X := \{H_{\chi,a}\}_{(\chi,a) \in X}$$

The projection of $X$ onto the first factor is denoted $p(X)$ and is called the character set of $\mathcal{T}_X$. (Thus, $p(X) := \{\chi \mid (\chi,a) \in X\}$.) The singular set, $\Sigma_X$, is the union of toric hyperplanes in the arrangement. Its complement, $T - \Sigma_X$, is denoted $\mathcal{R}_X$. The intersection poset $L_X$ is the set of nonempty intersections of toric hyperplanes and $\mathcal{T}_X = L_X \cup \{T\}$. $\mathcal{T}_X$ is partially ordered by inclusion. The rank of the arrangement is the dimension of the linear subspace of $\Lambda \otimes \mathbb{Z} \mathbb{R}$ spanned by $p(X)$. The arrangement is essential if its rank is $n$.

Suppose $G \in L_X$. Choose a point $x \in G$. The tangential arrangement along $G$ is the arrangement $\mathcal{A}_G$ of linear hyperplanes which are tangent to the complex toric hyperplanes containing $G$ (i.e., all hyperplanes of the form $T_x(H_{\chi,a})$ where $T_x(G) \subset T_x(H_{\chi,a})$). It is a central hyperplane arrangement of rank equal to $n - \dim G$.

Given a toric arrangement $\mathcal{T}_X$ of rank $l$, let $K_X$ denote the identity component of the intersection of all kernels in $p(X)$, i.e., $K_X$ is the identity component of

$$\bigcap_{\chi \in p(X)} \ker \chi = \{t \in T \mid \chi(t) = 1, \forall \chi \in p(X)\}.$$ 

Put $\mathcal{T}_X := T/K_X$. Thus, $K_X$ and $\mathcal{T}_X$ are tori of dimensions $n - l$ and $l$, respectively. ($K_X \cong (\mathbb{C}^*)^{n-l}$ and $\mathcal{T}_X \cong (\mathbb{C}^*)^l$.) Let $\overline{\Sigma}_X$ denote the image of $\Sigma_X$ in $\mathcal{T}_X$. Since $T \to T/K_X$ is a trivial $K_X$-bundle, we have a homeomorphism of pairs,

$$(T, \Sigma_X) \cong (\mathcal{T}_X, \overline{\Sigma}_X).$$ (1)

In other words, the arrangement in $T$ is just the product of the arrangement in $\mathcal{T}_X$ with the torus $K_X$. We call $\mathcal{T}_X$ the essentialization of $\mathcal{T}_X$. So, it is not restrictive to consider essential toric arrangements.

**Lemma 2.1.** (cf. [5, Prop. 2.1]). Suppose $\mathcal{T}_X$ is an essential toric arrangement on $T$ and $\Sigma = \Sigma_X$. Then $H_*(T, \Sigma)$ is free abelian and concentrated in degree $n$.

**Proof.** We follow the “deletion-restriction” argument in [5, Prop. 2.1]) using induction on $\text{Card}(\mathcal{T}_X)$. Choose a toric hyperplane $H \in \mathcal{T}_X$. Let $\mathcal{T}' = \mathcal{T}_X - H$.
\{H\}$ and let $\mathcal{T}''$ be the restriction of $\mathcal{T}_{\chi}$ to $H$, i.e., $\mathcal{T}'' = \{H \cap H' \mid H' \in \mathcal{T}_{\chi}\}$. Let $\Sigma'$ and $\Sigma''$ denote the singular sets of $\mathcal{T}'$ and $\mathcal{T}''$, respectively. Consider the exact sequence of the triple $(T, \Sigma, \Sigma')$,

$$\to H_*(T, \Sigma') \to H_*(T, \Sigma) \to H_{*-1}(\Sigma, \Sigma') \to$$

(2)

There is an excision, $H_{*-1}(\Sigma, \Sigma') \cong H_{*-1}(H, \Sigma')$. The rank of $\mathcal{T}'$ is either $n$ or $n - 1$, while the rank of $\mathcal{T}''$ is always $n - 1$. The argument breaks into two cases depending on the rank of $\mathcal{T}'$.

**Case 1:** the rank of $\mathcal{T}'$ is $n$. By induction, $H_*(T, \Sigma')$ and $H_*(H, H \cap \Sigma)$ are free abelian and concentrated in degrees $n$ and $n - 1$, respectively. So, (2) becomes,

$$0 \to H_n(T, \Sigma') \to H_n(T, \Sigma) \to H_{n-1}(H, H \cap \Sigma') \to 0$$

and all other terms are 0. Therefore, $H_*(T, \Sigma)$ is concentrated in degree $n$ and $H_n(T, \Sigma)$ is free abelian.

**Case 2:** the rank of $\mathcal{T}'$ is $n - 1$. Then the projection $T \to \overline{T}$ takes $H$ isomorphically onto $\overline{T}$ and the arrangement $\mathcal{T}''$ on $H$ maps isomorphically to the arrangement $\overline{T}_{\chi}$ on $\overline{T}$. So, $(H, H \cap \Sigma') \cong (\overline{T}, \overline{\Sigma})$. By (1), $(T, \Sigma') \cong K_{\chi} \times (H, H \cap \Sigma') \cong \mathbb{C}^* \times (H, H \cap \Sigma')$. By the Künneth Formula, $H_*(T, \Sigma') \cong H_*(\mathbb{C}^*) \otimes H_*(H, H \cap \Sigma')$. So,

$$H_{n-1}(T, \Sigma') \cong H_0(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma')$$

and

$$H_n(T, \Sigma') \cong H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma');$$

moreover, the first isomorphism is induced by the inclusion $(H, H \cap \Sigma') \to (T, \Sigma')$. So, (2) becomes,

$$0 \to H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma') \to H_n(T, \Sigma) \to H_{n-1}(H, H \cap \Sigma')$$

$$\to H_0(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma')$$

where the last map is an isomorphism. It follows that $H_{n-1}(T, \Sigma) = 0$ and that $H_n(T, \Sigma) \cong H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma')$, which, by inductive hypothesis, is free abelian. This proves the lemma.

**Complexified toric arrangements** In [3] D’Antonio-Delucchi consider the case of “complexified toric arrangements.” This means that for each $(\chi, a) \in X$, the complex number $a$ has modulus 1 (where $X \subset \Lambda \times \mathbb{C}^*$ is a
set defining a toric arrangement $\mathcal{T}_X$. Let $T^{\text{cpt}} = (S^1)^n \subset \mathbb{C}^n$ be the compact torus. Then for each $H \in \mathcal{T}_X$, $H \cap T^{\text{cpt}}$ is a compact subtorus of $T^{\text{cpt}}$. The set of subtori, $\mathcal{T}_X^{\text{cpt}} := \{H \cap S \mid H \in \mathcal{T}\}$, is called the associated compact arrangement.

Let $\Sigma^{\text{cpt}} := \Sigma_X \cap T^{\text{cpt}}$. We note that $(T, \Sigma_X)$ deformation retracts onto $(T^{\text{cpt}}, \Sigma^{\text{cpt}})$. Here are a few observations.

(i) The universal cover of $T^{\text{cpt}}$ is $\mathbb{R}^n$ (actually the subspace $i\mathbb{R}^n \subset \mathbb{C}^n$). Let $\pi: \mathbb{R}^n \to T^{\text{cpt}}$ be the covering projection. Then for each $H^{\text{cpt}} \in \mathcal{T}_X^{\text{cpt}}$, each component of $\pi^{-1}(H^{\text{cpt}})$ is an affine hyperplane and the collection of these hyperplanes is a periodic affine hyperplane arrangement in $\mathbb{R}^n$.

(ii) If $\mathcal{T}_X$ is essential, then $\Sigma^{\text{cpt}}$ cuts $T^{\text{cpt}}$ into a disjoint union of convex polytopes, called chambers (see [25]). The inverse images of these polytopes under $\pi$ give a tiling of $\mathbb{R}^n$.

(iii) When $\mathcal{T}_X$ is essential, it follows from (ii) that for $n \geq 2$, $\Sigma^{\text{cpt}}$ is connected and that for $n \geq 3$, $\pi_1(\Sigma^{\text{cpt}}) = \pi_1(T^{\text{cpt}})$.

(iv) It is easy to prove Lemma 2.1 in the case of a compact arrangement. We have an excision $H_*(T^{\text{cpt}}, \Sigma^{\text{cpt}}) \cong H_*(\bigsqcup (P_i, \partial P_i))$ where each chamber $P_i$ is an $n$-dimensional convex polytope. Hence, $H_*(T^{\text{cpt}}, \Sigma^{\text{cpt}})$ is concentrated in degree $n$ and is free abelian. Moreover, the rank of $H_*(T^{\text{cpt}}, \Sigma^{\text{cpt}})$ is the number of chambers.

(v) Let $\bar{\Sigma}^{\text{cpt}}$ denote the inverse image of $\Sigma^{\text{cpt}}$ in $\mathbb{R}^n$ and let $\bar{\Sigma}_X$ be the induced cover of $\Sigma_X$. Suppose $\mathcal{T}_X$ is essential. Then $\Sigma^{\text{cpt}}$ cuts $\mathbb{R}^n$ into compact chambers. It follows that $\bar{\Sigma}^{\text{cpt}}$ (and hence, $\bar{\Sigma}$) is homotopy equivalent to a wedge of $(n-1)$-spheres.

3 Certain covers and their nerves

Equip the torus $T = (\mathbb{C}^*)^n$ with an invariant metric. This lifts to a Euclidean metric on $\mathbb{C}^n$ induced from an inner product. Hence, geodesics in $T$ lift to straight lines in $\mathbb{C}^n$ and each component of the inverse image of a subtorus of $T$ is an affine subspace of $\mathbb{C}^n$. A convex subset of $T$ means a geodesically convex subset. Thus, each component of the inverse image of a convex subset of $T$ is a convex subset of $\mathbb{C}^n$. 
The intersection of an open convex subset of $T$ with the toric hyperplanes in $\mathcal{T}_X$ is equivalent to an affine arrangement. An open convex subset $U \subset T$ is *small* (with respect to $\mathcal{T}_X$) if this affine arrangement is central. In other words, $U$ is *small* if the following two conditions hold (cf. [5, 6]):

(i) $\{G \in \overline{L}(\mathcal{T}_X) \mid G \cap U \neq \emptyset\}$ has a unique minimum element, $\text{Min}(U)$.

(ii) A toric hyperplane $H \in \mathcal{T}_X$ has nonempty intersection with $U$ if and only if $\text{Min}(U) \subset H$.

If (i) and (ii) hold, then the arrangement in $U$ is equivalent to the tangential arrangement along $\text{Min}(U)$, which we denote by $\mathcal{A}_{\text{Min}(U)}$. The intersection of two small convex open sets is also a small convex set; hence, the same is true for any finite intersection of such sets.

Let $U = \{U_i\}_{i \in I}$ be an open cover of $T$ by small convex sets, put

$$U_{\text{sing}} := \{U \in U \mid U \cap \Sigma_X \neq \emptyset\}.$$ 

Given a nonempty subset $\sigma \subset I$, put $U_\sigma := \bigcap_{i \in \sigma} U_i$. The *nerve* $N(U)$ of $U$ is the simplicial complex defined as follows. Its vertex set is $I$ and a finite, nonempty subset $\sigma \subset I$ spans a simplex of $N(U)$ if and only if $U_\sigma$ is nonempty. We have the following lemma.

**Lemma 3.1.** Suppose $\mathcal{T}_X$ is essential. $N(U)$ is homotopy equivalent to $T$ and $N(U_{\text{sing}})$ is a subcomplex homotopy equivalent to $\Sigma_X$. Moreover, $H_n(N(U), N(U_{\text{sing}}))$ is concentrated in degree $n$ and $H_n(N(U), N(U_{\text{sing}}))$ is free abelian.

**Proof.** $U_{\text{sing}}$ is an open cover of a neighborhood of $\Sigma_X$ which deformation retracts onto $\Sigma_X$. For each simplex $\sigma$ of $N(U)$, $U_\sigma$ is contractible (in fact, it is a small convex open set). By a well-known result (see [14, Cor. 4G.3 and Ex. 4G(4)]) $N(U)$ is homotopy equivalent to $T$ and $N(U_{\text{sing}})$ is homotopy equivalent to $\Sigma_X$. The last sentence of the lemma follows from Lemma 2.1. 

**Definition 3.2.** $\beta(\mathcal{T}_X)$ is the rank of $H_n(N(U), N(U_{\text{sing}}))$.

Equivalently, $\beta(\mathcal{T}_X)$ is the rank of $H_n(T, \Sigma_X)$. It is not difficult to see that, for essential arrangements, $(-1)^n \beta(\mathcal{T}_X) = e(T, \Sigma_X) = -e(\Sigma_X) = e(\mathcal{R}_X)$, where $e(\cdot)$ denotes Euler characteristic.
4 Local coefficients

Generic and nonresonant coefficients  Consider an affine arrangement \( \mathcal{A} \) The fundamental group \( \pi \) of its complement, \( M(\mathcal{A}) \), is generated by loops \( a_H \) for \( H \in \mathcal{A} \), where the loop \( a_H \) goes once around the hyperplane \( H \) in the “positive” direction. Let \( \alpha_H \) denote the image of \( a_H \) in \( H_1(M(\mathcal{A})) \). Then \( H_1(M(\mathcal{A})) \) is free abelian with basis \( \{\alpha_H\}_{H \in \mathcal{A}} \). So, a homomorphism \( H_1(M(\mathcal{A})) \to \mathbb{C}^* \) is determined by an \( \mathcal{A} \)-tuple \( \Lambda \in (\mathbb{C}^*)^{\mathcal{A}} \), where \( \Lambda = (\lambda_H)_{H \in \mathcal{A}} \) corresponds to the homomorphism sending \( \alpha_H \) to \( \lambda_H \). Let \( \psi_{\Lambda} : \pi \to \mathbb{C}^* \) be the composition of this homomorphism with the abelianization map \( \pi \to H_1(M(\mathcal{A})) \). The resulting rank one local coefficient system on \( M(\mathcal{A}) \) is denoted \( \mathcal{A}_\Lambda \).

Returning to the case where \( \mathcal{T}_X \) is a toric arrangement, for each simplex \( \sigma \) in \( N(\hat{U}) \), let \( \mathcal{A}_\sigma := \mathcal{A}_{\text{Min}(U_\sigma)} \) be the corresponding central arrangement (so that \( \hat{U}_\sigma \cong M(\mathcal{A}_\sigma) \)). Given \( \Lambda_\sigma \in (\mathbb{C}^*)^{\mathcal{A}_\sigma} \), put

\[
\lambda_\sigma := \prod_{H \in \mathcal{A}_\sigma} \lambda_H.
\]

Let \( \Lambda_T \in \text{Hom}(H_1(\mathcal{R}_X), \mathbb{C}^*) \) be a local coefficient system on \( \mathcal{R}_X \). The localization of \( \Lambda_T \) on the open set \( \hat{U}_\sigma \) has the form \( A_{\Lambda_\sigma} \), where \( \Lambda_\sigma \) is an \( \mathcal{A}_\sigma \)-tuple in \( \mathbb{C}^* \). We call \( \Lambda_T \) generic if \( \lambda_\sigma \neq 1 \) for all \( \sigma \in N(U_{\text{sing}}) \). We call \( \Lambda_T \) nonresonant if \( \Lambda_\sigma \) is nonresonant in the sense of [2] for all \( \sigma \in N(U_{\text{sing}}) \). i.e., if the Betti numbers of \( M(\mathcal{A}_\sigma) \) with coefficients in \( \mathcal{A}_{\Lambda_\sigma} \) are minimal.

\( \ell^2 \)-cohomology and coefficients in a group von Neumann algebra

For a discrete group \( \pi \), \( \ell^2 \pi \) denotes the Hilbert space of complex-valued, square integrable functions on \( \pi \). There are unitary \( \pi \)-actions on \( \ell^2 \pi \) by either left or right multiplication; hence, \( \mathbb{C} \pi \) acts either from the left or right as an algebra of operators. The associated von Neumann algebra \( \mathcal{N} \pi \) is the commutant of \( \mathbb{C} \pi \) (acting from, say, the right on \( \ell^2 \pi \)).

Given a finite CW complex \( Y \) with fundamental group \( \pi \), the space of \( \ell^2 \)-cochains on the universal cover \( \tilde{Y} \) is equal to \( C^*(\tilde{Y}; \ell^2 \pi) \), the cochains with local coefficients in \( \ell^2 \pi \). The image of the coboundary map need not be closed; hence, \( H^*(\tilde{Y}; \ell^2 \pi) \) need not be a Hilbert space. To remedy this, one defines the reduced \( \ell^2 \)-cohomology \( H^\text{red}_*(\tilde{Y}; \ell^2 \pi) \) to be the quotient of the space of cocycles by the closure of the space of coboundaries. The von Neumann algebra admits a trace. Using this, one can attach a “dimension,”
dim\(N_\pi\) \(V\), to any closed, \(\pi\)-stable subspace \(V\) of a finite direct sum of copies of \(\ell^2\pi\) (it is the trace of orthogonal projection onto \(V\)). The nonnegative real number \(\dim_{N_\pi}(H^p_{\text{red}}(Y; \ell^2\pi))\) is the \(p\)th \(\ell^2\)-Betti number of \(Y\).

A technical advance of Lück [19, Ch. 6] is the use local coefficients in \(N_\pi\) in place of the previous version of \(\ell^2\)-cohomology. He shows there is a well-defined dimension function on \(N_\pi\)-modules, \(A \to \dim_{N_\pi} A\), which gives the same answer for \(\ell^2\)-Betti numbers, i.e., for each \(p\) one has that \(\dim_{N_\pi} H^p(Y; N_\pi) = \dim_{N_\pi} H^p_{\text{red}}(Y; \ell^2\pi)\).

**Group ring coefficients**  Let \(Y\) be a connected CW complex, \(\pi = \pi_1(Y)\) and \(r : \tilde{Y} \to Y\) the universal cover. There is a well-defined action of \(\pi\) on \(\tilde{Y}\) and hence, on the cellular chain complex of \(\tilde{Y}\). Given the left \(\pi\)-module \(\mathbb{Z}_\pi\), define the cochain complex with group ring coefficients

\[
C^*(Y; \mathbb{Z}_\pi) := \text{Hom}_\pi(C_*(\tilde{Y}); \mathbb{Z}_\pi).
\]

Taking cohomology gives \(H^*(Y; \mathbb{Z}_\pi)\).

## 5 The Mayer-Vietoris spectral sequence

**Statements of the main theorems**  Suppose \(T_X\) is an essential toric arrangement in \(T\) and \(\pi = \pi_1(R_X)\).

**Theorem 5.1.** Let \(\Lambda_T\) be a generic \(X\)-tuple with entries in \(k^*\). Then \(H^*(R_X; A_{\Lambda_T})\) is concentrated in degree \(n\) and

\[
\dim_k H^n(R_X; A_{\Lambda_T}) = \beta(T_X).
\]

**Theorem 5.2.** (cf. [7]). The \(\ell^2\)-Betti numbers of \(R_X\) are 0 except in degree \(n\) and \(\ell^2b_n(R_X) = \beta(T_X)\).

**Theorem 5.3.** (cf. [6, 8]). \(H^*(R_X; \mathbb{Z}_\pi)\) vanishes except in degree \(n\) and \(H^n(R_X; \mathbb{Z}_\pi)\) is free abelian.

**Remark 5.4.** Suppose \(W\) is a Euclidean reflection group acting on \(\mathbb{R}^n\) and that \(\mathbb{Z}^n \subset W\) is the subgroup of translations. The quotient \(W' := W/\mathbb{Z}^n\) is a finite Coxeter group. The reflection group \(W\) acts on the complexification \(\mathbb{C}^n\) and \(W'\) acts on the torus \(T = \mathbb{C}^n/\mathbb{Z}^n\). The image of the affine reflection arrangement in \(\mathbb{C}^n\) gives a toric arrangement \(T_X\) in \(T\). The fundamental
group of $\mathcal{R}_X$ is the Artin group $A$ associated to $W$ and $\mathcal{R}_X$ is the Salvetti complex associated to $A$. The quotient of the compact torus by $W'$ can be identified with the fundamental simplex $\Delta$ of $W$ on $\mathbb{R}^n$. (If $W$ is irreducible, then $\Delta$ is a simplex.) It follows that $\beta(T)$ is the order of $W'$ (i.e., the index of $\mathbb{Z}^n$ in $W$). So, in this case Theorem 5.2 is a special case of the main result of [7] and Theorem 5.3 is a special case of a result of [8, Thm. 4.1].

**Lemma 5.5.** Suppose $A$ is a finite, central arrangement of affine hyperplanes. Let $\pi' = \pi_1(M(A))$. Then

(i) (cf. [28, 2, 5]). For any generic system of local coefficients $A$, $H^*(M(A); A)$ vanishes in all degrees.

(ii) (cf. [5]). $H^*(M(A); N\pi')$ vanishes in all degrees. Hence, all $\ell^2$-Betti numbers are 0.

(iii) (cf. [6]). If the rank of $A$ is $l$, then $H^*(M(A); \mathbb{Z}\pi')$ vanishes except in the top degree, $l$.

**Proofs using the Mayer-Vietoris spectral sequence** The proofs of these three theorems closely follow the argument in [7], [5] and particularly, in [6]. For $\pi = \pi_1(\mathcal{R}_X)$, let $A$ denote one of the left $\pi$-modules in Section 4.

Let $\mathcal{U} = \{U_i\}$ be an open cover of $T$ by small convex sets. We may suppose that $\mathcal{U}$ is finite and that it is closed under taking intersections. For each $G \in L_X$, put

$$U_G := \{U \in \mathcal{U} \mid \text{Min}(U) \leq G\},$$

$$U_G^{\text{sing}} := \{U \in \mathcal{U} \mid \text{Min}(U) < G\} = \{U \in U_G \mid U \cap \Sigma_X \cap G \neq \emptyset\}.$$ 

The open cover $\mathcal{U}$ restricts to an open cover $\hat{\mathcal{U}} = \{U - \Sigma_X\}_{U \in \mathcal{U}}$ of $\mathcal{R}_X$. Any element $\hat{U} = U - \Sigma_X$ of the cover is homotopy equivalent to the complement of a central arrangement $M(A_{\text{Min}(U)})$.

Suppose $N(\mathcal{U})$ is the nerve of $\mathcal{U}$ and $N(\mathcal{U}_G)$ is the subcomplex defined by $U_G$. Since $N(\mathcal{U}_G)$ and $N(\mathcal{U}_G^{\text{sing}})$ are nerves of covers of $G$ and $\Sigma_X \cap G$, respectively, by contractible open subsets, we have that for each $G \in L(A)$,

$$H^*(N(\mathcal{U}_G), N(\mathcal{U}_G^{\text{sing}})) = H^*(G, \Sigma(T_X \cap G)).$$ \hspace{1cm} (3)
For each $k$-simplex $\sigma = \{i_0, \ldots, i_k\}$ in $N(U)$, let

$$U_\sigma := U_{i_0} \cap \cdots \cap U_{i_k}$$

denote the corresponding intersection.

Let $r : \tilde{R}_X \to R_X$ be the universal cover. The induced open cover \(\{r^{-1}(\tilde{U})\}\) of $R_X$ has the same nerve $N(\tilde{U})$ (= $N(U)$). We have the Mayer–Vietoris double complex,

$$C_{i,j} := \bigoplus_{\sigma \in N^{(i)}} C_j(r^{-1}(\tilde{U}_\sigma)),$$

where $N^{(i)}$ denotes the set of $i$-simplices in $N(U)$ (cf. [1, Ch. VII].) We get a corresponding double cochain complex,

$$E_0^{i,j} := \text{Hom}_\pi(C_{i,j}, A), \quad (4)$$

where $\pi = \pi_1(R_X)$. The filtration on the double complex gives a spectral sequence converging to the associated graded module for cohomology:

$$\text{Gr} H^m(R_X; A) = E_\infty := \bigoplus_{i+j=m} E_{i,j}^\infty.$$ 

By first using the horizontal differential, there is a spectral sequence with $E_1$ page

$$E_1^{i,j} = C^i(N(U); H^j(A))$$

where $H^j(A)$ is the coefficient system on $N(U)$ defined by

$$\sigma \mapsto H^j(\tilde{U}_\sigma; A),$$

where $\tilde{U}_\sigma \cong M(A_{\text{Min}(U_\sigma)})$. For $A = A_T$ or $A = N \pi$ these coefficients are 0 for $G \neq T$. For $A = \mathbb{Z} \pi$, they are 0 for $j \neq \text{dim}(G)$. Hence, in all cases, for any coface $\sigma'$ of $\sigma$, if $G' := \text{Min}(U_{\sigma'}) < G$, the coefficient homomorphism $H^j(M(A_{G}); A) \to H^j(M(A_{G'}); A)$ is the zero map. Moreover, the $E_1$ page of the spectral sequence decomposes as a direct sum (cf. [8, Lemma 2.2]). In fact, for a fixed $j$, by using Lemma 5.5, we see that the $E_1^{i,j}$ term decomposes as

$$E_1^{i,j} = \bigoplus_{G \in \mathcal{T}_X^{-j}} C^i(N(U_G), N(U_G^{\text{sing}}); H^j(M(A_G); A)).$$
where we have constant coefficients in each summand. Hence, at $E_2$ we have

$$E_2^{i,j} = \bigoplus_{G \in \mathcal{L}^n_{X^{-j}}} H^i(N(U_G), N(U_G^{\text{sing}}); H^j(M(A_G); A))$$

$$= \bigoplus_{G \in \mathcal{L}^n_{X^{-j}}} H^i(G, \Sigma_X \cap G; H^j(M(A_G); A)), \quad (5)$$

where the second equation follows from (3).

When $A = A_{\Lambda_T}$ or $A = N\pi$, all summands vanish for $G \neq T$ and $j \neq 0$. So, we are left with $E_2^{0,0} = H^n(T, \Sigma_X; A)$, which is isomorphic to the tensor product free abelian group of rank $\beta(T_X)$ with $A$. It follows that $H^*(\mathcal{R}_X; A)$ is concentrated in degree $n$ and that $\dim_{\mathbb{C}} H^n(\mathcal{R}_X; A_{\Lambda_T}) = \beta(T_X) = \dim_{N\pi} H^n(\mathcal{R}_X; N\pi)$. This proves Theorems 5.1 and 5.2.

Consider formula (5) for $A = \mathbb{Z}\pi$. By Lemma 2.1, $H^i(G, \Sigma_X \cap G)$ is concentrated in degree $\dim G = n - j$. Hence, $E_2^{i,j}$ is nonzero (and free abelian) only for $i + j = n$. It follows that the spectral sequence degenerates at $E_2$, i.e., $E_2 = E_\infty$. This proves Theorem 5.3.

**Remark 5.6.** Let us remark that the statement of Theorem 5.1 holds even if the local system $\Lambda_T$ is nonresonant or if it verifies the Schechtman, Terao and Varchenko nonresonance conditions in all small open convex sets, i.e. $\Lambda_{\sigma}$ verifies the nonresonance conditions in [28] for all $\sigma \in N(U_{\text{sing}})$. Indeed under these conditions Lemma 5.5 holds.

**References**


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