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EXOTIC ASPHERICAL 4-MANIFOLDS

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Abstract. We prove that there exist closed, aspherical, smooth 4-manifolds that are homeomorphic but not diffeomorphic. These provide counterexamples to a smooth analog of the Borel conjecture in dimension four.

1. Introduction

The Borel conjecture predicts that closed aspherical manifolds are topologically rigid, with their homeomorphism type determined by their homotopy type. This conjecture seeks to generalize the rigidity exhibited by hyperbolic manifolds and other (non-positively curved) locally symmetric spaces, a setting in which every homotopy equivalence is realized by an isometry [Mos68, Mos73].

This paper concerns the smooth version[1] of the Borel conjecture, which asks whether closed, aspherical n-manifolds that are homotopy equivalent are in fact diffeomorphic. This is classical in dimensions $n \leq 2$ and is known to hold for orientable 3-manifolds, using Perelman's results [Per02, Per03b, Per03a, MT14, BBM$^+$10]. On the other hand, it is known [Wal99, Chapter 15] that there exist exotic aspherical manifolds in dimensions at least 5, including exotic smooth structures on T^n for $n \geq 5$. In dimensions at least 7, such examples can be obtained by connected sum of certain aspherical manifolds (such as tori or stably parallelizable hyperbolic manifolds) with an exotic sphere [BT23, FJ89]. We resolve the last remaining case, in dimension 4:

Theorem 1.1. *There exist pairs of smooth, closed, aspherical 4-manifolds that are homeomorphic but not diffeomorphic.*

In fact, we will find infinitely many such pairs; see Remark 5.4. The key adjective in the theorem is the word *closed*. It was previously known that there are exotic smooth structures on $\mathbb{R}^4$ (cf. [GS99]) and on compact aspherical manifolds with boundary, in fact on compact [AR16] contractible manifolds. (If one is working relative to a fixed identification of the boundary, as in some formulations of the Borel conjecture, then exotic contractible manifolds go back to [Akb91].)

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[1]See [Wei23] for some interesting historical comments on the formulation of the Borel conjecture as a statement about homeomorphism, rather than diffeomorphism, of homotopy equivalent aspherical manifolds.

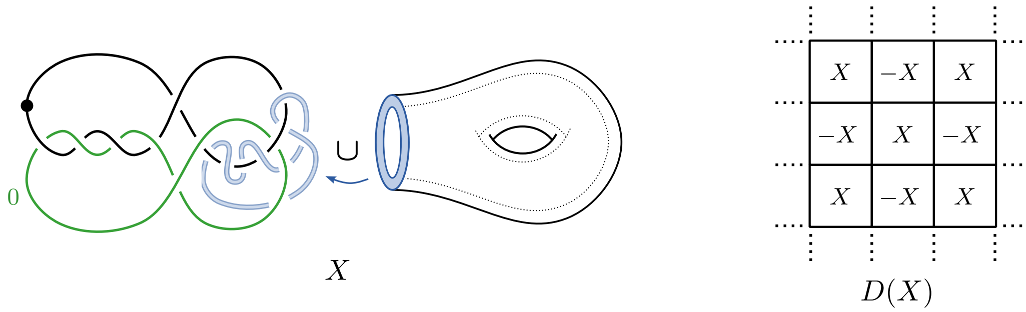

FIGURE 1. (LEFT) The 4-manifold X is the union of a contractible 4-manifold and a thickened, once-punctured torus. (RIGHT) The 4-manifold $D(X)$ is a union of countably many copies of X and $-X$, glued along 3-cells in their boundaries.

Among closed manifolds, most previous constructions of exotic 4-manifolds appear ill-suited to the aspherical setting. For example, one obstacle is that closed, aspherical 4-manifolds must have infinite fundamental groups, and these are typically not known to be "good" groups that allow [FQ90] topological surgery theory and the s-cobordism theorem to function in the 4-dimensional setting. Therefore one must take a more local approach to establishing homeomorphisms; our exotic pairs Q, Q' are related by explicit cork twisting, which is known to preserve homeomorphism type by work of Freedman [Fre82]. Since cork twisting takes place in a contractible region, it has more in common with counterexamples obtained by sums with exotic spheres than with other constructions from surgery theory.

The construction underlying Theorem 1.1 begins with a pair of exotic aspherical 4-manifolds X, X' with boundary (based on [HP19]) and then produces closed 4-manifolds by applying the reflection group trick developed by the first author [Dav83]. As depicted schematically in Figure 1, each of X and X' is obtained from a contractible 4-manifold C (namely the Akbulut cork [Akb91]) by attaching a thickened punctured torus, hence has the homotopy type of T^2, which is aspherical. Necessary data for the reflection group trick includes a triangulation $\mathcal{T}$ of ∂X as a flag complex. Then $\mathcal{T}$ defines a right-angled Coxeter group $W(\mathcal{T})$. The reflection group trick proceeds by constructing an associated noncompact space $D(X)$ (resp., $D(X')$) built from infinitely many copies of X and $-X$ (resp., X' and $-X'$), as depicted schematically on the right-hand side of Figure 1. The closed aspherical 4-manifolds $Q(X)$ and $Q(X')$ claimed in Theorem 1.1 are then obtained as certain quotients of $D(X)$ and $D(X')$.

The main claim in the theorem is then proved in two steps. In §4, we construct a homeomorphism between $Q(X)$ and $Q(X')$, and argue that it is not homotopic to a diffeomorphism.[2] In fact, we show that there is no diffeomorphism between $Q(X)$

[2]In many treatments, e.g. [Far02, Wei23], the Borel conjecture is stated as saying that any homotopy equivalence is homotopic to a homeomorphism, so that this step would already give a counterexample to that version of the smooth Borel conjecture.

and $Q(X')$ that lifts to the covering spaces $D(X)$ and $D(X')$. (These covers are distinguished by comparing the genera of smoothly embedded surfaces; here some care with obstructions is required because these covers are built from copies of X with both orientations.) This reduces Theorem 1.1 to a lifting problem that we solve in §5; the key is an algebraic argument showing that the fundamental group $\pi_1(D(X))$ (resp., $\pi_1(D(X')))$ is characteristic in $\pi_1(Q(X))$ (resp. $\pi_1(Q(X')))$. In order to make this algebraic argument work we need to make a special choice of $\mathcal{T}$: it must satisfy the "flag-no-square condition." This implies that the resulting Coxeter group is word hyperbolic, a fact which we need in our algebraic argument. The fact that $\pi_1(D(X))$ and $\pi_1(D(X')$ are characteristic subgroups implies that $Q(X)$ and $Q(X')$ are not diffeomorphic.

We close this discussion by noting that the reflection group trick has had many applications to the construction of "exotic" spaces and groups, including closed aspherical manifolds that are not covered by Euclidean space [Dav83], Poincaré duality groups that are not finitely presented [Dav98] (c.f., [BB97]), 4-dimensional locally CAT(0)-manifolds that do not admit a Riemannian metric of nonpositive curvature [DJL12], and — closer to our purposes here — aspherical topological manifolds that admit no smooth structure [DH89]. However, to our knowledge, Theorem 1.1 is the first application of the reflection group trick to the study of exotic smooth structures on manifolds. Given the trick's capacity for promoting exotic phenomena from the compact to the closed setting, we expect further applications in this direction.

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2. The input manifolds

Let X be the 4-manifold shown in Figure 2, which arises from a very slight modification to the examples underlying [HP19, Theorem 4.1]. The manifold X is obtained from the contractible Akbulut cork C [Akb91] by attaching a "genus-1 handle" (i.e., a copy of $F \times D^2$ where F is a genus-1 surface with one boundary component) along a knot K in ∂C. The embedded copy $C \subset X$ can be seen as the union of the 0-handle, the 1-handle represented by the topmost dotted curve, and the green 2-handle.

Our proof of Theorem 1.1 will leverage the following properties of X.

Proposition 2.1. *The 4-manifold X satisfies the following:*

(a) *X is homotopy equivalent to the torus,*
(b) *X embeds smoothly in B^4,*
(c) *every homologically essential, smoothly embedded surface in X has genus ≥ 2 (hence the same is true of $-X$), and*
(d) *X is homeomorphic to a smooth 4-manifold X' such that $H_2(X')$ is generated by a smoothly embedded torus.*

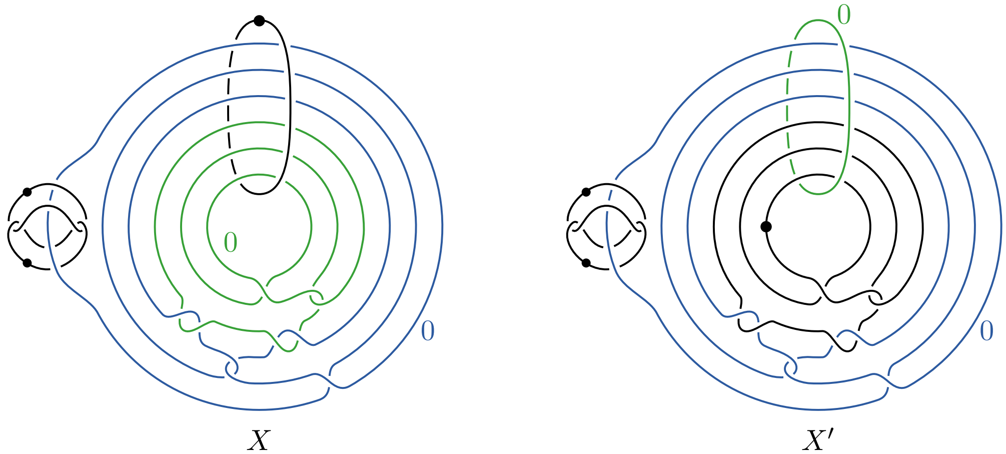

FIGURE 2. Kirby diagrams for the 4-manifolds X and X'.

Most of these properties follow verbatim from the proof of [HP19, Theorem 4.1]; for the reader's convenience, we sketch the arguments, adding detail only where necessary.

Proof. For (a), recall from above that X is obtained from the contractible Akbulut cork C by attaching a genus-1 handle $F \times D^2$. Collapsing $F \times D^2$ to $F \times 0$ and C to a point (hence ∂F to a point) yields a map to T^2 that is easily seen to be a weak homotopy equivalence, hence a homotopy equivalence.

For (b), first attach 0-framed 2-handles along meridians to all of the 1-handle curves of X in Figure 2. This has the diagrammatic effect of erasing the 1-handles of X, leaving only the green and blue 0-framed 2-handle curves. These can be seen to form a 2-component unlink, so attaching two 3-handles yields B^4.

For (c), consider the handle diagram for X shown in Figure 3, which is in Gompf's standard form [Gom98]. It can be checked that Thurston-Bennequin numbers tb and rotation numbers r of the (oriented) 2-handle curves G and B satisfy

$$tb(G) = 1, \quad tb(B) = 1 \qquad\qquad r(G) = 1, \quad r(B) = 3$$

The 2-handle framings are both $0 = tb - 1$, hence X admits a Stein structure [Gom98]. Observe that $H_2(X)$ is generated by a class α corresponding to the difference of the 2-handles attached along the oriented curves B and G. Gompf's formula [Gom98, Proposition 2.1] for the Chern class $c_1(X)$ of the Stein structure on X yields

$$\langle c_1(X), \alpha \rangle = r(B) - r(G) = 3 - 1 = 2.$$

Now suppose that S is a smoothly embedded surface in X satisfying $[S] = k\alpha$ for $k \neq 0$. The adjunction inequality for homologically essential, smoothly embedded surfaces of non-negative self-intersection in X [LM98] gives us

$$2g(S) - 2 \geq |\langle c_1(X), [S] \rangle| + [S] \cdot [S] = |\langle c_1(X), k\alpha \rangle| + k^2(\alpha \cdot \alpha) = 2|k| + k^2 \cdot 0,$$

hence $2g(S) \geq 2|k| + 2$ and thus $g(S) \geq |k| + 1 \geq 2$.

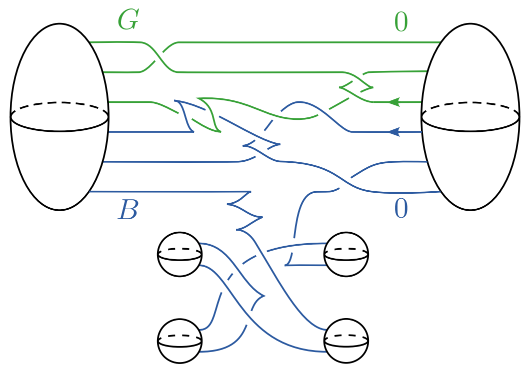

FIGURE 3. A Stein handle diagram for X.

For (d), let X' be the 4-manifold shown on the right-hand side of Figure 2. It is obtained from X by twisting along the Akbulut cork $C \subset X$, which can be described diagrammatically by exchanging the roles of the green 0-framed 2-handle and the top-most dotted 1-handle curve. This corresponds to removing $\mathring{C} \subset X$ and regluing by an involution of ∂C that extends to a homeomorphism of C (but not a diffeomorphism); see [Akb91, Fre82].

To see the torus that generates $H_2(X')$, note that X' is also obtained by attaching a genus-1 handle to the Akbulut cork C, where the attaching curve corresponds to the underlying blue curve B in Figure 2. At the cost of dragging the green 2-handle curve, one can check that this blue curve is an unknot that can be isotoped away from the black dotted 1-handle curve in C, hence bounds a smooth disk in C. Capping off this disk with the core genus-1 surface in the genus-1 handle yields the desired torus. □

Remark 2.2. We can extend this to an infinite family of such exotic pairs X_m, X'_m for $m \geq 0$ (all homotopy equivalent to T^2) such that $H_2(X'_m)$ is represented by a smoothly embedded torus but a nonzero element of $H_2(X_m)$ cannot be represented by a smoothly embedded surface S of genus $\leq 2 + m$. This can be achieved through modifying the attaching curve B by adding m positive clasps across the 1-handle (and m positive stabilizations) as in [HP19, Figure 8], which has the effect of increasing the evaluation of the Chern class $c_1(X_m)$ on the generator of $H_2(X_m)$ by m.

3. The Reflection Group Trick

We now recall some components of Davis' construction [Dav83, Dav08].

3.1. **The Coxeter group.** Fix a flag triangulation $\mathcal{T}$ of ∂X. (Recall that a simplicial complex is *flag* if any finite subset of its vertices that are pairwise connected by edges spans a simplex. For example, the barycentric subdivision of any simplicial complex is flag.) Letting V denote the set of vertices in $\mathcal{T}$, there is an associated Coxeter system (W, V) where W is the right-angled Coxeter group with a generator v of order two for each vertex $v \in V$ and a relation of the form $(vw)^2 = 1$ for each pair of vertices v, w joined by an edge in $\mathcal{T}$.

More generally, for any flag simplicial complex $\mathcal{T}$ (which is not necessarily a manifold), we can defined an associated right-angled Coxeter group, denoted by $W(\mathcal{T})$, in a similar way.

3.2. **The Davis complex.** The dual decomposition $\mathcal{T}'$ to $\mathcal{T}$ gives a cell structure on ∂X; we view the top-dimensional cells of $\mathcal{T}'$ as equipping X with the structure of a *manifold with faces*. Davis' construction yields a noncompact manifold $D(X)$ formed from gluing a countable collection of copies of X via reflections across these faces in a prescribed way as follows. For each vertex v of $\mathcal{T}$, let X_v be the closed cell in $\mathcal{T}'$ dual to v. For each point $x \in \partial X$, let W_x be the subgroup of W generated by all $v \in V$ such that $x \in X_v$. Define an equivalence relation $\sim$ on $W \times X$ by $(h, x) \sim (g, y)$ if and only if $x = y$ and $h^{-1}g \in W_x$. Give $G \times X$ the product topology and define $D(X)$ to be the quotient space $(G \times X)/\sim$.

The Coxeter group W acts smoothly and properly on $D(X)$ with closed fundamental domain X so that we can write $D(X)$ as a union

$$D(X) = \bigcup_{g \in W} gX.$$

We note that the fact that X is a smooth manifold ensures that $D(X)$ and the action by W are smooth [Dav08, Remark 10.1.11]

The copies of X can be ordered in a convenient way. To set this up, note that the generating set V of reflections gives us a length function ℓ on W based on the word length in terms of this generating set. Next choose any ordering on W such that $g < g'$ implies $\ell(g) \leq \ell(g')$. This in turn gives us an ordering of the tiles $X_i := g_i X$.

Using the above ordering, we can write $D(X)$ as an increasing union of subspaces $P_n = \cup_{i=1}^n X_i$, each of which is a codimension-zero submanifold of $D(X)$. (While we may view these subspaces P_n as smooth manifolds with corners, it will suffice to consider them in the PL category.) Moreover, each P_n is a boundary sum of the tiles X_i for $i \leq n$:

$$P_n \cong X_1 \natural \cdots \natural X_n. \tag{1}$$

This decomposition is key to our arguments, so we sketch its proof in Lemma 3.2 below. Its main input is the following:

Lemma 3.1 ([Dav83, Remark 10.6])**.** *The intersection of P_n and X_{n+1} is a PL codimension-zero disk $\Delta_n \subset \partial P_n$.*

This description introduces two important subtleties. First, when the union $P_n = \cup_{i=1}^n X_i$ is expressed as a boundary sum, we must allow each summand X_i to be diffeomorphic to either X or $-X$. (To see why both orientations on X arise, note that copies of X whose boundaries share a common face must have opposite orientations, since the gluing is achieved by reflection.)

The second subtlety is that the summing region Δ_n in $P_n = P_{n-1} \natural X_n$ is not confined to the boundary of $X_{n-1} \subset P_{n-1}$, as is the case in the usual construction of $X_1 \natural \cdots \natural X_n$. Therefore the inductive identification of P_n with $X_1 \natural \cdots \natural X_n$ is slightly unnatural. (This is evidenced by the fact that any given tile X_i eventually lies strictly in the interior of each P_n for $n \gg i$, yet the i^{th} summand of $X_1 \natural \cdots \natural X_n$ does not lie in the interior of $X_1 \natural \cdots \natural X_n$.) For completeness, we sketch a proof of (1):

Lemma 3.2. *For each n, there exists a PL homeomorphism $P_n \to X_1 \natural \cdots \natural X_n$.*

Proof. We argue inductively, with the claim being trivially true for $n = 1$. Next, suppose that for a given n there exists a PL homeomorphism

$$f : P_n \to X_1 \natural \cdots \natural X_n.$$

By construction, we have $P_{n+1} = P_n \cup_{\Delta_n} X_{n+1}$, where $\Delta_n \subset \partial P_n$ is the disk $P_n \cap X_{n+1}$ from Lemma 3.1. It follows that f induces a PL homeomorphism

$$P_{n+1} \to (X_1 \natural \cdots \natural X_n) \cup_{f(\Delta_n)} X_{n+1},$$

where $f(\Delta_n) \subset \partial(X_1 \natural \cdots \natural X_n)$ is identified with the disk in X_{n+1} that was previously identified with $\Delta_n \subset \partial P_n$.

Now choose a small codimension-zero disk $\Delta'_n \subset \partial X_n$ that lies away from the boundary-summing region in $X_1 \natural \cdots \natural X_n$. By the PL version of Palais' disk theorem [RS72, Theorem 3.34], there is an isotopy of $\partial(X_1 \natural \cdots \natural X_n)$ carrying $f(\Delta_n)$ to Δ'_n, and such an isotopy extends to an isotopy of $X_1 \natural \cdots \natural X_n$ supported near a collar neighborhood of its boundary (cf. [RS72, Theorem 3.22]). This defines a PL homeomorphism

$$g : (X_1 \natural \cdots \natural X_n) \cup_{f(\Delta_n)} X_{n+1} \longrightarrow (X_1 \natural \cdots \natural X_n) \cup_{\Delta'_n} X_{n+1},$$

and the latter space is naturally identified with the boundary sum $X_1 \natural \cdots \natural X_{n+1}$. Composing f and g yields the desired PL homeomorphism from P_{n+1} to $X_1 \natural \cdots \natural X_{n+1}$, completing the inductive argument. □

We record one more simple consequence of Lemma 3.1.

Lemma 3.3. *Each inclusion $X_n \hookrightarrow D(X)$ induces an injective map on homology.*

Proof. First consider the inclusions $X_n \hookrightarrow P_n$ and $P_{n-1} \hookrightarrow P_n$. For all n, observe that these induce injections on homology by applying a Mayer-Vietoris argument to the decomposition $P_n = P_{n-1} \cup X_n$ and using the fact that the intersection $P_{n-1} \cap X_n$ is a disk. It follows that, for all $m > n$, the composition of inclusions

$$X_n \hookrightarrow P_n \hookrightarrow P_m$$

induces an injection on homology. It follows that the inclusion $X_n \hookrightarrow D(X)$ induces an injection on homology, as a class $\alpha \in H_*(X_n)$ that becomes null-homologous in $H_*(D(X))$ must become null-homologous in $H_*(P_m)$ for some finite $m > n$. □

3.3. **A compact quotient.** Finally, we produce a closed manifold $Q(X)$: The Coxeter group W must contain a finite index, torsion-free subgroup W_0 [Dav08, Corollary 6.12.12]. Set $Q(X) = D(X)/W_0$. (Note that this quotient $Q(X)$ is smooth [Dav08, Remark 10.1.11].) For example, given that the Coxeter group W is right-angled, one can take W_0 to be the commutator subgroup of W (cf. [Dav08, p. 213]).

When X is aspherical (as it is in our case), it is easy to see that $D(X)$ and $Q(X)$ are also aspherical using (1). For $D(X)$, consider a map $f : S^k \to D(X)$ with $k \geq 2$. Its image is compact, hence lies in some P_n. Applying (1), we see that P_n is homotopy equivalent to a wedge sum of n copies of X. A wedge sum of aspherical spaces is aspherical, hence $f(S^k)$ is nullhomotopic in P_n and thus in $D(X)$. We conclude $D(X)$ is aspherical, hence so is $Q(X)$, because the homotopy groups π_k of $Q(X)$ and its cover $D(X)$ are isomorphic for $k \geq 2$.

4. An exotic homeomorphism

Let X and X' denote the exotic 4-manifolds from §2. Applying the construction from §3, we obtain an associated pair of closed, aspherical 4-manifolds $Q(X)$ and $Q(X')$ with covers $D(X)$ and $D(X')$. As a step in the direction of Theorem 1.1, we prove the following:

Theorem 4.1. *There is a homeomorphism $Q(X) \to Q(X')$ that is not homotopic to any diffeomorphism.*

Proof. Recall that X' is obtained from X by removing the interior of the Akbulut cork $C \subset \mathring{X}$ and regluing C with a twist. The homeomorphism $X \to X'$ constructed in the proof of Proposition 2.1(d) can be viewed as the identity away from $\mathring{C}$, where the definitions of X and X' agree. This induces a W-equivariant homeomorphism $\tilde{f} : D(X) \to D(X')$, hence descends to a homeomorphism $f : Q(X) \to Q(X')$.

We claim that there is no diffeomorphism $D(X) \to D(X')$, which will imply that $f : Q(X) \to Q(X')$ is not homotopic to any diffeomorphism. (By the homotopy lifting property, such a homotopy would lift to one from $\tilde{f} : D(X) \to D(X')$ to a diffeomorphism $D(X) \to D(X')$.) To prove this, recall from Proposition 2.1 that X' contains a smoothly embedded torus generating $H_2(X')$. In particular, since each inclusion-induced map $H_2(X'_i) \to H_2(D(X'))$ is injective by Lemma 3.3, we see that $D(X')$ contains smoothly embedded, homologically essential tori.

In contrast, we claim that any smoothly embedded, homologically essential surface S in $D(X)$ has genus at least two. Since S is compact, it must be contained in one of the compact subspaces P_n in the exhaustion of $D(X)$. By Lemma 3.2, there is a PL homeomorphism $\varphi : P_n \to X_1 \natural \cdots \natural X_n$. The surface $\varphi(S)$ is a locally flat, PL codimension-2 submanifold of the smooth manifold $X_1 \natural \cdots \natural X_n$, hence is isotopic to a smoothly embedded surface S' by Wall [Wal67]; also see the proof of [HLL22, Lemma A.3].

Since $[S]$ is nonzero in $H_2(D(X))$, it is nonzero in $H_2(P_n)$, hence its image $\varphi_*[S] = [S']$ is nonzero in $H_2(X_1 \natural \cdots \natural X_n)$. Note that $H_2(X_1 \natural \cdots \natural X_n)$ splits as a direct sum of $H_2(X_i)$, hence gives canonical projection $H_2(X_1 \natural \cdots \natural X_n) \to H_2(X_i)$ for each i.

It follows that $[S']$ must project to a nonzero element in the homology $H_2(X_i)$ of at least one summand X_i in $X_1\natural\cdots\natural X_n$. For notational convenience, let us assume $X_i = X_1$. By Proposition 2.1, we may attach 2- and 3-handles to all the other summands $X_2,\ldots,X_n$ in $X_1\natural\cdots\natural X_n$ to turn them into 4-balls, giving an embedding of $X_1\natural\cdots\natural X_n$ into $X_1\natural B^4\natural\cdots\natural B^4 \cong X_1$. This embedding of $X_1\natural\cdots\natural X_n$ into X_1 induces the projection from $H_2(X_1\natural\cdots\natural X_n)$ to $H_2(X_1)$, so it carries S' to a smoothly embedded surface in X_1 that is still homologically essential. By Proposition 2.1, it follows that S' has genus at least two, hence so does the original surface S. □

It is important to note that the argument above proves an *a priori* stronger statement than that of Theorem 4.1, which we record here for use in the final argument for Theorem 1.1.

Theorem 4.2. *There is no diffeomorphism $Q(X) \to Q(X')$ that lifts to a diffeomorphism $D(X) \to D(X')$.*

Remark 4.3. Let $\widetilde{X}$ (resp. $\widetilde{X}'$) denote the universal cover of X (resp. X'). The triangulation of ∂X lifts to a triangulation $\tilde{\mathcal{T}}$ of $\partial\widetilde{X}$. Use this to define a right-angled Coxeter group $\tilde{W}$ and a corresponding Davis complex $D(\widetilde{X})$. Then $D(\widetilde{X})$ is the universal cover of $D(X)$. Similarly, we get $\widetilde{X}'$ and its universal cover $D(\widetilde{X}')$. We conjecture that $D(\widetilde{X})$ and $D(\widetilde{X}')$ are not simply connected at infinity and hence, that neither is homeomorphic to $\mathbb{R}^4$. (In particular, neither is an exotic $\mathbb{R}^4$.) An interesting open question is whether the open contractible manifolds $D(\widetilde{X}')$ and $D(\widetilde{X})$ are diffeomorphic.

5. Characteristic subgroups

To complete the proof of Theorem 1.1 (in light of Theorems 4.1 and 4.2), it suffices to show that *any* potential diffeomorphism $Q(X) \to Q(X')$ would lift to a diffeomorphism $D(X) \to D(X')$. We will see that this lifting problem is easily recast in terms of the behavior of the subgroup $\pi_1(D(X)) \le \pi_1(Q(X))$ under automorphisms of $\pi_1(Q(X))$.

5.1. Flag-no-square complexes and characteristic subgroups.

A *cycle* in a simplicial complex $\mathcal{T}$ is a subcomplex homeomorphic to the circle $\mathbb{S}^1$, and its *length* is the number of edges in the cycle. A *diagonal* of a cycle is an edge connecting any two non-consecutive vertices in this cycle. A simplicial complex $\mathcal{T}$ is said to satisfy the *flag-no-square* condition if $\mathcal{T}$ is a flag complex and any cycle of length 4 in $\mathcal{T}$ has a diagonal.

Proposition 5.1. [PŚ09, Proposition 2.13] *Let $\mathcal{T}$ be a 3-dimensional simplicial complex. Then it admits a subdivision which is flag-no-square.*

The following is a consequence of [Mou88, Theorem 17.1], see also [Dav08, Corollary 12.6.3].

Proposition 5.2. *Let $\mathcal{T}$ be a simplicial complex which is flag-no-square. Let $W(\mathcal{T})$ be the right-angled Coxeter group associated with $\mathcal{T}$ (as defined in §3.1). Then $W(\mathcal{T})$ is word hyperbolic and so does not contain any subgroup which is isomorphic to $\mathbb{Z}^2$.*

In turn, the above results provide a degree of control over $\mathbb{Z}^2$-subgroups of $\pi_1(Q(X))$, enabling us to prove that $\pi_1(D(X))$, which is an infinite free product of copies of $\pi_1(X) \cong \mathbb{Z}^2$, is a characteristic subgroup of $\pi_1(Q(X))$.

Lemma 5.3. *Let X be a compact 4-manifold equipped with a flag triangulation $\mathcal{T}$ of ∂X. Let $W = W(\mathcal{T})$ be the right-angled Coxeter group associated with $\mathcal{T}$. Let $D(X)$ be the associated Davis complex as in §3, with the natural action $W \curvearrowright D(X)$, and let $Q(X)$ be the quotient of $D(X)$ by a torsion-free finite index subgroup W_0 of W. If $\pi_1(X)$ is isomorphic to $\mathbb{Z}^2$ and the triangulation $\mathcal{T}$ of ∂X is flag-no-square, then $\pi_1(D(X))$ is a characteristic subgroup of $\pi_1(Q(X))$.*

Proof. By construction, $D(X)$ is a normal covering of $Q(X)$ with the deck group W_0. This gives an exact sequence:

$$1 \to \pi_1(D(X)) \to \pi_1(Q(X)) \to W_0 \to 1. \tag{2}$$

We claim if $H \leq \pi_1(Q(X))$ is isomorphic to $\mathbb{Z}^2$, then $H \leq \pi_1(D(X))$. Recall that $\pi_1(D(X))$ is a free product of copies of $\pi_1(X)$, with one copy of $\pi_1(X)$ for each element in W (c.f., [Dav83, Remark 15.9]). As $\pi_1(X) \cong \mathbb{Z}^2$, this claim implies for any automorphism f of $\pi_1(Q(X))$, we have $f(\pi_1(D(X))) \leq \pi_1(D(X))$; hence, $\pi_1(D(X))$ is a characteristic subgroup of $\pi_1(Q(X))$.

It remains to prove the claim. As the triangulation of ∂X is flag-no-square, Proposition 5.2 implies that W_0 does not contain a subgroup isomorphic to $\mathbb{Z}^2$. Let L be the image of H under $\pi_1(Q(X)) \to W_0$. As W_0 is torsion-free, if L is nontrivial, the only possibility for L is $L \cong \mathbb{Z}$. Next we show $L \cong \mathbb{Z}$ leads to a contradiction, which justifies the claim. Our strategy will be to consider the action of $L \subset W_0$ on the complementary subgroup $N = H \cap \pi_1(D(X)) \cong \mathbb{Z}$ in the restricted short exact sequence

$$1 \to N \to H \to L \to 1.$$

that parallels (2); the fact that H is abelian will constrain this action and lead to a contradiction.

To that end, note that each element of W_0 lifts to an element in $\pi_1(Q(X))$, which gives an automorphism of $\pi_1(D(X))$; moreover, different lifts give rise to the same automorphism up to an inner automorphism of $\pi_1(D(X))$. This gives a well-defined homomorphism $\varphi : W_0 \to \mathrm{Out}(\pi_1(D(X)))$, where $\mathrm{Out}(\pi_1(D(X)))$ denotes the outer automorphism group of $\pi_1(D(X))$. Later, we will need the following topological description of φ. Take a base point $p \in D(X)$. Then we can identify $\pi_1(D(X), p)$ with $\pi_1(D(X), q)$ for any $q \neq p$, by choosing a path from p to q. This identification is well-defined up to an inner automorphism of $\pi_1(D(X), p)$. Given an element $\bar{g} \in W_0$, the action $W_0 \curvearrowright D(X)$ by deck transformations gives an isomorphism $\bar{g}_* : \pi_1(D(X), p) \to \pi_1(D(X), \bar{g}(p))$. As we can identify $\pi_1(D(X), \bar{g}(p))$ with $\pi_1(D(X), p)$, the isomorphism $\bar{g}_*$ gives an element in the outer automorphism group of $\pi_1(D(X), p)$, which is exactly $\varphi(\bar{g})$ for the map φ defined above.

Now suppose $L \cong \mathbb{Z}$, and let $\bar{g} \in L$ be a generator of L. Let $N = \pi_1(D(X)) \cap H \cong \mathbb{Z}$ be defined as before. Let $g \in H \leq \pi_1(Q(X))$ be a lift of $\bar{g}$. As H is abelian, we know

$ghg^{-1} = h$ for all $h \in N \leq H$. Thus conjugating by g gives an automorphism $\beta : \pi_1(D(X)) \to \pi_1(D(X))$ that restricts to the identity on N. The element in the outer automorphism group of $\pi_1(D(X))$ represented by β is exactly $\varphi(\bar{g})$, where $\varphi : W_0 \to \mathrm{Out}(\pi_1(D(X)))$ is defined in the previous paragraph. In particular, $\varphi(\bar{g})$ fixes the $\pi_1(D(X))$-conjugacy class of each element of N. (Note that the action of $\mathrm{Out}(\pi_1(D(X)))$ is only well-defined on the *conjugacy* classes of elements in $\pi_1(D(X))$. We have a particular lift of $\varphi(\bar{g})$ to $\mathrm{Aut}(\pi_1(D(X)))$, namely conjugation by g, that acts as the identity on each element of N, hence the associated outer automorphism $\varphi(\bar{g})$ acts as the identity on the $\pi_1(D(X))$-conjugacy class of N.)

Now let α be a loop based at p which represents a generator of N in $\pi_1(D(X), p)$. Then the previous paragraphs imply that for any path β from p to $\bar{g}(p)$, the loop $\beta\bar{g}(\alpha)\beta^{-1}$ gives an element in $\pi_1(D(x), p)$ which is conjugate to $[\alpha]$ by an element of $\pi_1(D(X), p)$. Here β^{-1} denotes the inverse path of β. In particular, this implies that α and $\bar{g}(\alpha)$ are freely homotopic in $D(X)$. Thus α and $\bar{g}^m(\alpha)$ are freely homotopic in $D(X)$ for any $m \geq 1$. We will show below that this leads to a contradiction.

Take a fundamental domain $X_1 \cong X$ for the action of W on $D(X)$. As in §3.2, we choose an enumeration $g_1 = \mathrm{id}, g_2, g_3, \ldots$ of elements of W such that $\ell(g_i) \leq \ell(g_j)$ whenever $i \leq j$. Let $X_1 = P_1 \subset P_2 \subset P_3 \subset \cdots$ be the exhaustion of $D(X)$ as defined in §3.2 with $P_n = \bigcup_{1 \leq i \leq n} g_i X_1$. As α is compact, there exists n_0 such that $\alpha \subset P_{n_0}$. As the deck group action $W \curvearrowright D(X)$ is properly discontinuous and g has infinite order in W, we know there exists $m_0 > 0$ such that

$$P_{n_0} \cap \bar{g}^{m_0}(P_{n_0}) = \emptyset. \tag{3}$$

Let $Y \subset X$ be a subset obtained by removing a collar neighborhood of ∂X in X (homeomorphic to $\partial X \times [0, 1)$) from X. We collapse $Y \subset X$ to a point and obtain the topological space $\bar{X}$ which is homeomorphic to a cone over ∂X. Let $Y_1 \subset X_1$ be the subspace of X_1 arising from $Y \subset X$. For each $i > n_0$, we collapse $g_i Y_1$ in $D(X)$ to a point. This gives a new topological space $\bar{D}(X)$, with $\pi : D(X) \to \bar{D}(X)$ being the natural continuous map. As α and $\bar{g}^{m_0}(\alpha)$ are freely homotopic in $D(X)$, we know $\pi(\alpha)$ and $\pi(\bar{g}^{m_0}(\alpha))$ are freely homotopic in $\bar{D}(X)$. In what remains, we will show $\pi(\bar{g}^{m_0}(\alpha))$ is null-homotopic in $\bar{D}(X)$, but $\pi(\alpha)$ is not null-homotopic in $\bar{D}(X)$, which gives the desired contradiction.

Let $\bar{P}_n = \pi(P_n)$. As the procedure of obtaining $\bar{D}(X)$ from $D(X)$ does not change the boundary of each chamber, we know from Lemma 3.1 that $\bar{P}_n \cap \pi(g_{n+1}X_1)$ is a codimension-zero disk in ∂P_n. Thus the van Kampen theorem implies that $\bar{P}_n \to \bar{P}_{n+1}$ is π_1-injective for each n, which further implies that $\bar{P}_n \to \bar{D}(X)$ is π_1-injective for each n. As α is not null-homotopic in $D(X)$, we know it is not null-homotopic in P_{n_0}. As π restricted to P_{n_0} is a homeomorphism onto $\bar{P}_{n_0}$, we know $\pi(\alpha)$ is not null-homotopic in $\bar{P}_{n_0}$. Hence $\pi(\alpha)$ is not null-homotopic in $\bar{D}(X)$.

Now we look at $\pi(\bar{g}^{m_0}(\alpha))$. By (3), each chamber in $\bar{g}^{m_0}(P_{n_0})$ is collapsed under π. We apply the Davis construction to $\bar{X}$ to obtain $D(\bar{X})$. Note that $D(\bar{X})$ has a similar filtration, denoted by $R_1 \subset R_2 \subset \cdots$. As $\bar{X}$ (that is, the cone on ∂X) is simply-connected, we know R_n is simply-connected for each n by the same argument as the previous paragraph. Moreover, by construction, $\pi(\bar{g}^{m_0}(P_{n_0}))$ is homeomorphic to R_{n_0}, hence is simply-connected. It follows that $\pi(\bar{g}^{m_0}(\alpha))$ is null-homotopic in $\pi(\bar{g}^{m_0}(P_{n_0}))$, and hence, is null-homotopic in $\bar{D}(X)$, as desired. □

5.2. **Conclusion.** We now complete the proof of our main result.

Proof of Theorem 1.1. By Theorem 4.1, there is a homeomorphism $f : Q(X) \to Q(X')$; that by construction is such that f_* carries $\pi_1(D(X)) \leq \pi_1(Q(X))$ isomorphically to $\pi_1(D(X')) \leq \pi_1(Q(X'))$.

Next suppose there is a diffeomorphism $g : Q(X) \to Q(X')$. To obtain a contradiction, by Theorem 4.2, it suffices to show that this lifts to a diffeomorphism $\tilde{g} : D(X) \to D(X')$. Such a lift exists if and only if $g_*(\pi_1(D(X)))$ lies inside $\pi_1(D(X')) \leq \pi_1(Q(X'))$. To show that this condition is met, note that $f_*^{-1} \circ g_*$ is an automorphism of $\pi_1(Q(X))$, hence preserves the characteristic subgroup $\pi_1(D(X))$ by Lemma 5.3. Since f_*^{-1} restricts to an isomorphism between $\pi_1(D(X'))$ and $\pi_1(D(X))$, it follows that g_* must carry $\pi_1(D(X))$ to $\pi_1(D(X'))$, as desired. □

Remark 5.4. To obtain infinitely many such examples of exotic aspherical pairs, we can apply these arguments to the 4-manifolds X_m, X'_m described in Remark 2.2.

Alternatively, for fixed X and a fixed flag-no-square triangulation $\mathcal{T}$ of ∂X, taking different finite index torsion free subgroups of the associated right-angled Coxeter group W (there are plenty of such subgroups as any right-angled Coxeter group is residually finite) in the construction of Section 3.3 also gives infinitely many exotic aspherical pairs.

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