Hyperbolic Coxeter Groups

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(This is a transcription of G. Moussong's dissertation into LaTeX for my own personal use - J. McCammond. Not for distribution.)

Contents

0. Introduction

An important chapter of the theory of infinite groups is a collection of applications of various geometric or topological concepts, methods,or analogies to abstract groups. The main link between geometry, topology and group thoery is the theory of discrete group actions. Generally, the stronger regularity properties an action has, the closer the relationship is between geometric properties of the space acted on, and algebraic properties of the acting group. The requirements that an action of a finitely generated discrete group on a locally compact space be proper and have compact quotient are not too restrictive but sufficiently strong to provide a rich theory and an interesting interplay between geometry and algebra.

The most straightforward examples of such group actions are the actions of finitely generated discrete groups on their Cayley groups (cf. Section 16). With suitable metrics on the Cayley graphs these actions are isometric actions. Many geometric or graph theoretic properties of the Cayley graph directly correspond to certain algegbraic properties of the group. A disadvantage of this approach, however, is the dependence of the Cayley graph on the finite system of generators chosen for the group. It seems desirable to consider different Cayley graphs of the same group, in some sense, equivalent.

This question leads to the definition of quasi-isometry of metric spaces (cf. Section 16). Quasi-isomtry is a considerable weakening of the isometry relation among metric spaces. Quasi-isometric spaces may have very different local structures, only similarity of distances "in large" is required. The first obvious advantage of quasi-isometry is that it eliminates the ambiguity in defining Cayley graphs: all choices of finite systems of generators yield quasi-isometric graphs. Thus we have a geometric object associated to finitely generated groups that depends intrinsically on the group structure.

The most important and most attractive feature of the concept of quasi-isometry is that discrete actions of finitely generated groups with compact quotient induce quasi-isometries between themselves and the spaces acted on. This phenomenon naturally leads to the question of finding quasi-isometry invariant geometric properties of spaces, and quasi-isometry invariant algebraic properties of groups.

These properties, apart from the trivial boundedness, finiteness and compactness properties, usually are fairly complex. For example, in case of groups, a quasi-isometry invariant property must be a "virtual" property, since passing to an extension or a subgroup of finite index is always a quasi-isometry. Interesting examples of such algebraic properties are: non-existence of "straight" (cf. [**9**]) abelian subgroups of rank greater than 1, and virtual nilpotency. Similar properties naturally occur in various geometric contexts, usually in connection with negative curvature phenomena (Preismann's theorem and Margulis' lemma for mainifolds of negative curvature).

M. Gromov in [**10**] developed the theory of hyperbolic metric spaces and hyperbolic groups, and introduced far-reaching generalizations of the classical aspects of hyperbolic space. Hyperbolicity is one of the deepest and most interesting quasi-isometry invariant property of metric spaces and finitely generated groups.

An important class of discrete transformation groups is the class of reflection groups, or, in an abstract situation, Coxeter gruops. They occur naturally in a wide range of questions in geometry, the theory of Coxeter groups has its share both in the classical development of mathematics and in modern research.

A construction used in a recent result by M. Davis (cf. [**6**]) and Gromov's fundmental paper [**10**] both lead to the question of hyperbolicity among Coxeter groups. This work is aimed to answer this question.

The central result of the dissertation is a construction that proves the following theorem:

THEOREM A. *For any Coxeter group W there exists a complete, contractible, piecewise euclidean space U of non-positive curvature, on which W acts properly as a discrete group of isometries with compact quotient.*

The space U is a euclidean convex polyhedral compllex with all cells combinatorially equivalent to a cube.

For example, if W is the dihedral group of order $2m$, then U is a regular euclidean $2m$-gon and W acts on U as the subgroup of the full symmetry group of U generated by the reflections across the axes that do not pass through vertices. A 2-cell in U (and a fundamental chamber for the action of W on U) is a quadrilateral with angles π/m, $\pi/2$, $\pi - (\pi/m)$ and $\pi/2$. The vertex with angle $\pi - (\pi/m)$ is called the inside vertex of this quadrilateral. If W is a (p, q, r)-triangle group with $(1/p) + (1/q) + (1/r) \leq 1$ (that is, W is infinite), then a fundamental chamber in U is obtained by pasting together the three quadrilaterals coreesponding to the three dihedral subgroups along the sides starting out of the common inside vertex. The inequality

$$(\pi - (\pi/p)) + (\pi - (\pi/q)) + (\pi - (\pi/r)) \geq 2\pi$$

shows that the geometry on U has a cone singularity with non-positive curvature concentrated at the inside vertex.

The construction of U for arbitrary Coxeter groups is a natural generalization of these examples. Verification of a condition on links of cells in U, which is analogous to the above inequality and ensures that U has non-positive curvature, constritutes the bulk of the dissertation.

A modified version of the construction yields strictly negative curvature on U for a certain class of Coxeter groups, and this class is precisely the class of hyperbolic Coxeter groups. This leads to the following characterization of hyperbolicity among Coxeter groups:

THEOREM B. *For a Coxeter group W the following conditions are equivalent:*

(1) *W is hyperbolic,*
(2) *W does not contain abelian subgroups ofrank greater than* 1*.*
(3) *W has no affine standard subgroups of rank greater than* 2*, and has no pairs of disjoint commuting infinite standard subgroups.*

The dissertation is divided into four chapters. Chapter 1 is an introduction to the geometry of spaces of piecewise constant curvature, together with some of the key technical lemmas. Chapter 2 develops the theory of spherical complexes associated to certain type of matrices, the main steps of the proofs of Theorems A and B are made here. Chapter 3 describes the construction and proves Theorem A. Chapter 4 briefly introduces Gromov's concept of hyperbolicity, proves Theorem B, and discusses some naturally arising open questions.

CHAPTER 1

Geometry Complexes

First we reivew the definitions and basic properties of convex polyhedral cell complexes, then we show (Corollary 4.6) that they provide examples of so-called geodesic metric spaces. We develop some of the main technical tools (Lemma 4.5, 5.4, and 5.11) used in later chapters.

1. Convex polyhedral cells

Let $\mathbb{E}^n$, $\mathbb{H}^n$ and $\mathbb{S}^n$ denote the standard euclidean, hyperbolic and spherical space, respectively. That is, $\mathbb{E}^n$, $\mathbb{H}^n$ (for $n \geq 0$) and $\mathbb{S}^n$ (for $n \geq 2$) are the (unique) n-dimensional simply connected complete riemannian manifolds of constant sectional curvature 0, -1 and 1 respectively, $\mathbb{S}^1$ is the circle of length 2π with the arc metric, $\mathbb{S}^0$ is the two-point space with the discrete metric of diameter π, and $\mathbb{S}^{-1}$ is the empty space. For $n \geq 0$, we identify $\mathbb{E}^n$ with the n-dimensional coordinate spce $\mathbb{R}^n$, and $\mathbb{S}^{n-1}$ with the set of unit vectors in $\mathbb{R}^n$.

A *cell*, or more precisely, a *convex polyhedral cell* in $\mathbb{E}^n$ or $\mathbb{H}^n$ is the convex hull of a finite number of points. A *simplex* in $\mathbb{E}^n$ or $\mathbb{H}^n$ is the convex hull of at most $n+1$ points in general position. A *convex polyhedral cone* in a vector space is the set of nonnegative linear combinations of a finite number of vectors. A *simplicial cone* in a vector space is the set of all nonnegative linear combinations of a linearly independent set of vectors. In $\mathbb{S}^n$, a cell (simplex) B is the intersection of a convex polyhedral cone (simplicial cone) C in $\mathbb{R}^{n+1}$ with $\mathbb{S}^n$. This cone C is uniquely determined by B and is called the *euclidean cone* assoicated to the spherical cell B. A cell is called *proper* if it contains no pair of antipodal points; for example, all cells in $\mathbb{E}^n$ or $\mathbb{H}^n$, and all simplices in $\mathbb{S}^n$ are proper. A spherical cell is proper if and only if the associated euclidean cone contains no linear subspaces of positive dimension. In a proper cell any two points can be connected by a unique geodesic segement.

The *dimension* $\dim B$ of a cell B is the dimension of the smallest plane containing B. A *face* of B is either B itself or the intersection of B with the boundary of a closed half-space in $\mathbb{E}^n$ or $\mathbb{H}^n$, or closed hemisphere in $\mathbb{S}^n$, containing B. The structure of faces is preserved under isometries between cells. The faces of a cell B are cells contained in B, they form a finite partially ordered set with respect to inclusion. Every cell B has a unique smallest face; if B is proper, then this is the empty face,otherwise the smallest face is a sphere of nonnegative dimension. 0-dimensional faces are either singletons, where we call them vertices and identify them with their unique element, or doubletons in some non-proper spherical cases. Among all faces containing a given $x \in B$ there is a smallest one, called the *support* of x. The *interior* of B is the set of points whose support is B, the *boundary* of B is the union of faces different from B. B is the disjoint union of its interior and its boundary.

A *subdivision* of a cell B is a finite collection $\{C_1, \dots, C_k\}$ of cells with $B = C_1 \cup \dots \cup C_k$, such that for any i and j $C_i \cap C_j$ is a common face of C_i and C_j. A subdivision of B induces subdivisions of faces of B. Every cell has a subdivision that consists of simplices.

Let B be a cell in M, where M is $\mathbb{E}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n$, and let $x \in B$. The *tangent cone* T_xB of B at x is the set of vectors v in T_xM such that $\exp_x(\epsilon v) \in B$ for some $\epsilon > 0$. T_xB is a convex polyhedral cone in T_xM. We identify $\mathbb{S}^{n-1}$ with the unit sphere in T_xM, then the *geometric link* $\mathrm{LK}(x, B)$ of x in B is defined as the intersection $T_xB \cap \mathbb{S}^{n-1}$. $\mathrm{LK}(x, B)$ is a spherical cell of one less dimension than B. The isometry class of $\mathrm{LK}(x, B)$ is independent of x within the interior of the support of x, thus the geometric link $\mathrm{LK}(F, B)$ of a nonempty face F in B is well-defined up to isometry as $\mathrm{LK}(x, B)$ for some point x in the interior of F. For a proper spherical cell B we define the geometric link of the empty face as B itself. The *link* $\textsc{lk}(x, B)$ of $x \in B$ is defined as the set of vectors in $\mathrm{LK}(x, B)$ orthogonal to the subspace $T_xF \leq T_xM$, where F is the support of x. $\textsc{lk}(x, B)$ is a proper spherical cell of dimension $\dim B - \dim F - 1$, its isometry class only depends on F and B, and, as above, for any nonempty face F of B, the link $\textsc{lk}(F, B)$ of F in B is well-defined up to isometry. Again, for a proper spherical cell B, we define $\textsc{lk}(\emptyset, B) = B$.

Let $x \in F \subset G \subset B$, where F and G are faces of a cell B, and assume that x is in the interior of F. Then the inclusion $T_x \subset T_xB$ identifies $\textsc{lk}(F, G)$ with a face of $\textsc{lk}(F, B)$, and this identification is independent of the choice of x. The link $\textsc{lk}(\textsc{lk}(F, G), \textsc{lk}(F, B))$ is isometric to $\textsc{lk}(G, B)$.

Let B and C be two spherical cells, say $B \subset \mathbb{S}^{m-1}$ and $C \subset \mathbb{S}^{n-1}$. The *join* $B * C$ of B and C is defined as the intersection of $\mathbb{S}^{n+m-1}$ with the convex cone generated by B and C in the orthogonal sum $\mathbb{R}^m \oplus \mathbb{R}^n = \mathbb{R}^{m+n}$. $B*C$ is a spherical cell of dimension $\dim B + \dim C + 1$, the euclidean cone assoicated to $B * C$ is the orthogonal sum of those associated to B and C, and the faces of $B*C$ are precisely the cells of the form $F * G$, where F is a cell of B and G is a cell of C. If $\emptyset$ is a face of C or B, then we identify the faces $B * \emptyset$ and $\emptyset * C$ with B and C respectively. The join operation is commutative and associative up to isometry.

For a spherical cell B, the *suspension* $\mathbf{S}B$ of B is defined as the join $\mathbb{S}^0 * B$, and the *cone* $\mathbf{C}B$ of B is defined as $\mathbf{1} * B$, where the cone point $\mathbf{1}$ is the single-point cell at $1 \in \mathbb{R}$. If B is proper, then the links $\mathrm{LK}(\mathbb{S}^0, \mathbf{S}B)$, $\textsc{lk}(\mathbb{S}^0, \mathbf{S}B)$, $\mathrm{LK}(\mathbf{1}, \mathbf{C}B)$, $\textsc{lk}(\mathbf{1}, \mathbf{C}B)$ are naturally identified with B, and for $x \in B$, $\mathrm{LK}(x, \mathbf{C}B)$ and $\textsc{lk}(x, \mathbf{C}B)$ are naturally identified with the cones $\mathbf{C}\mathrm{LK}(x, B)$ and $\mathbf{C}\textsc{lk}(x, B)$ respectively. For any point $x \in B$ with support F of dimension $k+1$, the geometric link $\mathrm{LK}(x, B)$ is naturally identified with the join $\mathbb{S}^k * \textsc{lk}(x, B)$, where $\mathbb{S}^k$ is the unit sphere in T_xF. $\mathbb{S}^n$ is the $(n + 1)$-fold iterated suspension of $\emptyset$, therefore joins with spheres are naturally identified with iterated suspensions.

2. Complexes

We use a slightly generalized version of CW-complexes: we require that they have a sphere as a unique cell of smallest dimension, and that all higher dimensional cells be attached in the usual fashion. When the smallest cell is the empty set, we have a CW-complex in the usual sense. A suitable subdivision always makes generalized CW-complexes into usual ones.

Let M be one of the symbols $\mathbb{E}$, $\mathbb{H}$ or $\mathbb{S}$. An *M-complex* is a CW-complex K in the above sense, together with a collection of maps f_B, called *characteristic maps*, for each closed cell B of K, that satisfy the following requirements:

(1) f_B is a homeomorphism from B onto a convex polyhedral cell in M^n for some n. Inverse images under f_B of the faces of this cell are called the *faces* of B;
(2) If B and C are closed cells of K, then $B \cap C$ is a face of both B and C, and $f_C {f_B}^{-1}$ restricts to an isometry from $f_B(B \cap C)$ to $f_C(B \cap C)$.

Closed cells and their faces (which are, by virtue of (2), closed cells themselves) are called *cells* of K. By (1), the interiors of cells form a partition of K. For $x \in K$, the unique cell containing x in its interior is called the *support* of x. The *dimension* $\dim K$ or K is the supremum of dimensions of its cells.

The characteristic maps induce metrics on cells of K, these metrics agree on intersections. With a slight abuse of language, we make no distinction between cells of K and their images under characteristic maps, or between the M-complex K and its underlying topological space.

Two M-complexes are called *isometrically isomorphic*, if there is a homeomorphism between them that takes cells onto cells isometrically. Such a map is called an *isometric isomorphism*.

A *subcomplex* L of K is a closed subset that is a union of a family of cells, together with $\{f_B | B \subset L\}$ as a system of characteristic maps. If $x \in K$, then the *star* $\mathrm{S{\scriptsize T}}(x, K)$ of x in K is the subcomplex of K defined as the union of all cells containing x, and the *open star* $\mathrm{O{\scriptsize ST}}(x, K)$ of x in K is the union of interiors of all cells containing x. $\mathrm{O{\scriptsize ST}}(x, K)$ is an open neighborhood of x in K, contained in $\mathrm{S{\scriptsize T}}(x, K)$. If $y \in \mathrm{O{\scriptsize ST}}(x, K)$, then $\mathrm{O{\scriptsize ST}}(y, K) \subset \mathrm{O{\scriptsize ST}}(x, K)$.

Obvious examples of M-complexes are the convex polyhedral cells in M^n with their face structures, boundaries of cells, subdivisions of cells, etc., taking identity maps as characteristic maps.

An M-complex K is called *simplicial* (*proper*), if all cells of K are simplices (proper cells); *finite*, if the number of cells is finite; *locally finite*, if every point of K or (equivalently every cell of K) is only contained in a finite number of cells of K. Finiteness is equivalent to compactness, local finiteness is equivalent to local compactness of K.

If K and L are proper M-complexes, then the *disjoint union* of K and L has a natural proper M-complex structure with the disjoint union of the sets of characteristic maps for K and L.

A *subdivision* of K is a simultaneous subdivision of all faces of K in a compatible fashion, that is, for any cell B and any face F of B, the subdivision of F is the one induced by the subdivision of B. Characteristic maps for cells in the subdivision of K are restrictions of original characteristic maps. Any subdivision of the boundary of a cell B can be extended to a subdivision of B, simplicially, if the subdivision of the boundary is simplicial. Thus, subdivisions of M-complexes can be defined by the usual skeletal induction. It follows, for example, that all M-complexes have simplicial subdivisions.

Let K be an M-complex and $x \in K$. Define the *geometric link* $\mathrm{LK}(x, K)$, the *link* $\mathrm{{\scriptsize LK}}(x, K)$ and the *tangent space* $T_x K$ of x in K as the disjoint union of cells $\mathrm{LK}(x, B)$, $\mathrm{{\scriptsize LK}}(x, B)$ and cones $T_x B$ respectively, for all cells B of K containing x, with the natural identifications: a vector u in $T_x B$ is identified with the vector

v in T_xC if and only if the differential of the map $f_B {f_C}^{-1}$ at $f_C(x)$ maps v to u. $\mathrm{LK}(x,K)$ and $\textsc{lk}(x,K)$ have a natural $\mathbb{S}$-complex structure, using the identity maps as characteristic maps. The system of exponential maps at x for various cells of K containing x is compatible with the identifications, so $\exp_x$ is well-defined on a certain subset of T_xK; for any $v \in \mathrm{LK}(x,K)$ there is an $\epsilon > 0$, such that $\exp_x(\delta v)$ is defined for all $0 \leq \delta \leq \epsilon$. If $\mathrm{LK}(x,K)$ is finite, then ϵ can be chosen independently of v. If K is locally finite, then $\mathrm{LK}(x,K)$ and $\textsc{lk}(x,K)$ are finite for all $x \in K$. The isometric isomorphism class of $\mathrm{LK}(x,K)$ and $\textsc{lk}(x,K)$ depends only on the support of x, so we can define the geometric link and the link of a nonempty cell B of K up to isometric isomorphism by $\mathrm{LK}(B,K) = \mathrm{LK}(x,K)$ and $\textsc{lk}(B,K) = \textsc{lk}(x,K)$ with some point x in the interior of B. For a proper $\mathbb{S}$-complex K we define $\mathrm{LK}(\emptyset,K) = \textsc{lk}(\emptyset,K) = K$.

If F and B are cells of an M-complex K with $F \subset B$, then $\textsc{lk}(F,B)$ is a cell in the $\mathbb{S}$-complex $\textsc{lk}(F,K)$, and $\textsc{lk}(\textsc{lk}(F,B),\textsc{lk}(F,K))$ is isometrically isomorphic to $\textsc{lk}(B,K)$.

Let K and L be two $\mathbb{S}$-complexes. The *join* $K * L$ of K and L is defined as the disjoint union of all cells of the form $B * C$, where B is a cell in K and C is a cell in L with the natural identifications given by inclusions of faces. $K * L$ has a natural $\mathbb{S}$-complex structure with identities as characteristic maps. The join operation is commutative and associative up to isometric isomorphism. The special cases $\mathbf{S}K = \mathbb{S}^0 * K$ and $\mathbf{C}K = \mathbf{1} * K$ are called the *suspension* and the *cone* of K, respectively. The complex $\mathbf{2} * K$, where $\mathbf{2}$ is the disjoint union of two copies of $\mathbf{1}$, is called the *double cone* of K. It is isometrically isomorphic to the "equatorial" subdivision of $\mathbf{S}K$. If K is proper, then the complexes $\mathrm{LK}(\mathbb{S}^0,\mathbf{S}K)$, $\textsc{lk}(\mathbb{S}^0,\mathbf{S}K)$, $\mathrm{LK}(\mathbf{1},\mathbf{C}K)$ and $\textsc{lk}(\mathbf{1},\mathbf{C}K)$ are all naturally isometrically isomorphic to K, and for $x \in K$, $\mathrm{LK}(x,\mathbf{C}K)$ and $\textsc{lk}(x,\mathbf{C}K)$ are naturally isometrically isomorphic to $\mathbf{C}\mathrm{LK}(x,K)$ and $\mathbf{C}\textsc{lk}(x,K)$ respectively.

3. The intrinsic metric

Let X be a set. A family $\{d_i | i \in I\}$ is called a *compatible family of partial metrics* on X if

(1) for all $i \in I$, d_i is a metric on some subset X_i of X,
(2) $X = \cup\{X_i | i \in I\}$, and
(3) $d_i(x,y) = d_j(x,y)$ whenever $x,y \in X_i \cap X_j$ and $i,j \in I$.

An *allowable m-chain* is a sequence $C = (x_0,\ldots,x_m)$ of points i X such that for each $k = 1,\ldots,m$ there exists an index $i(k) \in I$ with $x_{k-1}, x_k \in X_{i(k)}$. The *length* $\lambda(C)$ of C is defined as the sum $d_{i(1)}(x_0,x_1) + \cdots + d_{i(m)}(x_{m-1},x_m)$. Due to the compatibility condition 3, $\lambda(C)$ is indenpendent of the choices of indices $i(k)$. We say that C is *from x_0 to x_m*. An *allowable chain* is a sequence which is an allowable m-chain for some $m \geq 0$.

For $x,y \in X$ we define

$$d(x,y) = \inf\{\lambda(C) | C \text{ is an allowable chain from } x \text{ to } y\}$$

Then d is a pseudometric on X (in a slightly generalized sense: d takes the value ∞ on pairs that cannot be connected by an allowable chain), called the *intrinsic pseudometric* defined by the family $\{d_i | i \in I\}$. Clearly d is maximal among all pseudometrics on X satisfying $d(x,y) \leq d_i(x,y)$ whenever $x,y \in X_i$.

For an M-complex K and for a cel B of K let d_B denote the metric on the cell B, then $\{d_B | B \text{ is a cell of } K\}$ is a compatible pseudometric defined by this family. If B is a cell and L is a subdivision of B, then obviously $d_L = d_B$. Therefore we have $d_{K'} = d_K$ for any subdivision K' of K. Isometric isomorphisms between M-complexes are isometries with respect to the intrinsic pseudometrics.

LEMMA 3.1. *Let K be a locally finite M-complex, $x \in K$, and put*

$$\epsilon(x) = \min\{d_B(x, B - \mathrm{OST}(x, K)) | B \textit{ is a cell in } K \textit{ containing } x\}$$

Then $\epsilon(x) > 0$, and if $y \in K$ with $d(x, y) < \epsilon(x)$, then x and y are contained in some cell B of K, and $d_B(x, y) = d(x, y)$.

PROOF. $\epsilon(x) > 0$ since K is locally finite, and since $\mathrm{OST}(x, K) \cap B = \mathrm{OST}(x, B)$ is an open neighborhood of $x \in B$ in B.

Let $C = (x_0, \ldots, x_m)$ be an allowable chain from x to y with $\lambda(C) < \epsilon(x)$, and put $C_k = (x_0, \ldots, x_k)$ for $k \leq m$. Let m' be the greatest $k \leq m$ with the property that $x_i \in \mathrm{ST}(x, K)$ for all $i \in \{0, \ldots, k\}$. Then either $m' = m$, or $x_{m'} \in \mathrm{ST}(x, K) - \mathrm{OST}(x, K)$. For all $1 \leq k \leq m'$ there is a cell $\mathrm{B}(k)$ containing the points x, x_{k-1} and x_k. We prove by induction on k that $d_{B(k)}(x, x_k) \leq \lambda(C_k)$. For $k = 1$ this is obvious, and for $1 < k \leq m'$ we have

$$\begin{aligned}
d_{B(k)}(x, x_k) &\leq d_{B(k)}(x, x_{k-1}) + d_{B(k)}(x_{k-1}, x_k) \quad \text{(triangle inequality in B}(k)\text{)} \\
&= d_{B(k-1)}(x, x_{k-1}) + d_{B(k)}(x_{k-1}, x_k) \\
&\leq \lambda(C_{k-1}) + d_{B(k)}(x_{k-1}, x_k) \quad \text{(induction hypothesis)} \\
&= \lambda(C_k).
\end{aligned}$$

Thus $d_{B(k)}(x, x_{m'}) \leq \lambda(C_{m'}) \leq \lambda(C_m) < \epsilon(x)$ and $x_{m'} \in \mathrm{OST}(x, K)$, therefore $m' = m$, and with $B = B(m)$ we have $d_B(x, y) \leq \lambda(C)$. This proves the lemma, since $d_B(x, y)$ is independent of the choice of C. □

From this lemma it immediately follows that:

COROLLARY 3.2. *If K is a locally finite, connected M-complex, then its intrinsic pseudometric is a metric compatible with the topology of K.*

For non-connected M complexes the intrinsic pseudometric is a c(compatible) metric in a generalized sense: the distance between two points is finite if and only if they are in the same connected component of K.

4. Geodesics

Let (X, d) be a metric space. A *geodesic segment* in X is an isometric map $p : I \to X$, where I is a metric space isometric to (and usually identified with) some interval $[a, b] \subset \mathbb{R}$, $a \leq b$, equipped with its usual metric. We say that p *connects* its*endpoints* $p(a)$ and $p(b)$. The *length* of p is $b - a$. A *closed geodesic* in X is an isometric map $q : S(\lambda) \to X$, where $S(\lambda)$ is a circle of length λ equipped with the arc metric. The *length* of q is λ. Closed geodesics in X can be represented by isometric maps $[a, b] \to X$ where $a < b$ and $[a, b]$ is equipped with the pseudometric $\rho(s, t) = min(|s - t|, b - a - |s - t|)$. A map $S(\lambda) \to X$ is a closed geodesic in X if and only if its restriction to any arc of $S(\lambda)$ of length $\leq \lambda/2$ is a geodesic segment in X.

(X, d) is called a *geodesic metric space*, if every pair of points in X can be connected with a geodesic segment. Important examples of geodesic metric spaces are the complete connected riemannian manifolds, and their geodesically convex subspaces.

Let K be a proper M-space with intrinsic metric d. For an allowable chain $C = (x_0, \dots, x_m)$ we define the *path* $p_C : [0, \lambda(C)] \to K$ *associated to* C as the concatenation of the unique geodesic segments $p_C^{(k)}$ from x_{k-1} to x_k in a cell containing both x_{k-1} and x_k ($k = 1, \dots, m$). (Note that these segments need not be geodesic segments in K.) We say that p_C is *represented* by C. If $x_{k-1} \neq x_k$, then let $v_{k-1}^{\text{out}} \in \mathrm{LK}(x_{k-1}, K)$ denote the unit tangent vector to $p_C^{(k)}$ at x_{k-1}, and let $v_k^{\text{in}} \in \mathrm{LK}(x_k, K)$ denote the backward unit tangent vector to $p_C^{(k)}$ at x_k ($k = 1, \dots, m$). If $x_0 \neq x_1$, then the *initial vector* $i(C)$ of C is defined as $v_0^{\text{out}} \in \mathrm{LK}(x_0, K)$. A *local geodesic segment* in K is an allowable chain $C = (x_0, \dots, x_m)$, such that $x_{k-1} \neq x_k$ ($k = 1, \dots, m$) and $d(v_k^{\text{in}}, v_k^{\text{out}}) \geq \pi$, where d denotes the initrinsic metric of $\mathrm{LK}(x_k, K)$ ($k = 1, \dots, m-1$). If, additionally, $x_0 = x_m$ and $d(v_m^{\text{in}}, v_0^{\text{out}}) \geq \pi$ in $\mathrm{LK}(x_0, K)$, then C is called a *closed local geodesic* in K.

Lemma 4.1. *Let C be an allowable chain in a proper M-complex K. If p_C is a geodesic segment, then (after removing repetitions, if necessary) C is a local geodesic segment in K.*

Proof. We can assume that $C = (x_0, x_1, x_2)$ and $x_0 \neq x_1 \neq x_2$, where we only have to check that $d(u, v) \geq \pi$ in $\mathrm{LK}(x, K)$ where $u = v_1^{\text{in}}$, $v = v_1^{\text{out}}$ and $x = x_1$. Suppose that $d(u, v) < \pi$, then the euclidean cone in $T_x K$ associated to the image of p_D for an allowable chain D of length $\alpha < \pi$ from u to v in $\mathrm{LK}(x, K)$ is the image of a euclidean plane sector $S \subset \mathbb{R}^2$ with angle α under a map $f : S \to T_x K$ that is isometric on the cones $f^{-1}(T_x B)$ for cells B containing x. The unit vectors $a = f^{-1}(u)$ and $b = f^{-1}(v)$ in $\mathbb{R}^2$ generate the boundary half lines of S. Since p_D is contained in a finite subcomplex of K, with a suitable choice of the points $a_0 = a, a_1, \dots, a_m = b$ along the segment connecting a and b in S, the sequence $E_\epsilon = (\exp_x(f(\epsilon a_0)), \dots, \exp_x(f(\epsilon a_m)))$ for sufficiently small $\epsilon > 0$ is an allowable chain in K. The limit of $\lambda(E_\epsilon)/\epsilon$ as $\epsilon \to 0$ is the distance (< 2) of a and b in $\mathbb{R}^2$, since $\exp_x$ is a near isometry near the origin. So, with a sufficiently small $\epsilon > 0$, $\lambda(E_\epsilon) < 2\epsilon$, and $C' = (x_0, \exp_x(f(\epsilon a_0)), \dots, \exp_x(f(\epsilon a_m)), x_2)$ is an allowable chain in K with $\lambda(C') < \lambda(C)$, a contradiction. □

Lemma 4.2. *In a proper M-complex every geodesic segment can be represented by a local geodesic segment, and every closed geodesic can be represented by aclosed local geodesic.*

Proof. By Lemma 4.1, it suffices to prove that every geodesic segment can be represented by an allowable chain. Let $p : [0, \lambda] \to K$ be a geodesic segment in a proper M-complex K. We can assume that K is finite, since the image f p is contained in a finite subcomplex. For $t \leq \lambda$, let p_t denote the restriction of p to the subinterval $[0, t]$. Define

$$H = \{t \in [0, \lambda] | p_t \text{ can be represented by an allowable chain}\}.$$

Then H is a subinterval of $[0, \lambda]$ and $0 \in H$. We show that H is open and closed in $[0, \lambda]$, which proves the lemma. Let $t \in H$, $t < \lambda$. Choose an allowable chain $C = (x_0, \dots, x_m)$ with $p_C = p_t$, and $\tau \in (t, t + \epsilon(x_m)$, $\tau < \lambda$, where $\epsilon(x_m)$ is given in Lemma 3.1. Then $C' = (x_0, \dots, x_m, p(\tau))$ is an allowable chain with $p_{C'} = p_\tau$, showing $\tau \in H$ and that H is open.

Let $t = \sup H$. Choose $\tau \in (t - \epsilon(p(t)), t)$ $\tau \geq 0$, and an allowable chain $C = (x_0, \ldots, x_m)$ with $p_C = p_\tau$, then $C' = (x_0, \ldots, x_m, p(t))$ is an allowable chain with $p_{C'} = p_t$, showing that $t \in H$ and that H is closed. □

Let K be an M-complex with intrinsic metric d, and $x, y \in K$. For $m \in \mathbb{N}$ define

$$d_m(x, y) = \inf\{\lambda(C) | C \text{ is an allowable } m\text{-chain from } x \text{ to } y\}$$

Clearly $d_m(x, y) \leq d_n(x, y)$ for $m \geq n$ and $d_m(x, y) \to d(x, y)$ as $m \to \infty$.

LEMMA 4.3. *If K is finite and connected, then for any $x, y \in K$ and $m \in \mathbb{N}$ there is an allowable m-chain C from x to y with $\lambda(C) = d_m(x, y)$.*

PROOF. If $z, w \in K$, then $\mathrm{OST}(z, K) \cap \mathrm{OST}(w, K) = \emptyset$ if and only if no cell contains both z and w. This shows that the set of allowable m-chains in the topological product K^{m+1} is closed, therefore it is compact. The length function is continuous, so it assumes its infimum. □

LEMMA 4.4. *Let K be a finite, connected M-complex with intrinsic metric d, and let $x, y \in K$, $x \neq y$. Then there exists a point $z \in \mathrm{ST}(x, K)$, such that $z \neq x$ and $d(x, z) + d(z, y) = d(x, y)$.*

PROOF. For $m \in \mathbb{N}$ choose an allowable m-chain C_m from x to y with $\lambda(C_m) = d_m(x, y)$. This is possible by Lemma 4.3. For $n \in \mathbb{N}$ choose an $m(n) \in \mathbb{N}$, such that $d_{m(n)}(x, y) < d(x, y) + 1/n$. Let $\epsilon = \epsilon(x)$ be given by Lemma 3.1. For $n \in \mathbb{N}$ let v_n be the initial vector $i(C_{m(n)}) \in \mathrm{LK}(x, K)$ and $z_n = \exp_x(\epsilon v_n)$. By compactness of K, we can assume that $z_n \to z$ as $n \to \infty$ for some $z \in \mathrm{ST}(x, K)$. Then

$$\begin{aligned} d(x, z) + d(z, y) &\leq \epsilon + d(z, z_n) + d(z_n, y) \\ &\leq \epsilon + d(z, z_n) + \lambda(C_{m(n)}) - \epsilon \\ &\leq d(z, z_n) + d(x, y) - 1/n \\ &\to d(x, y) \text{ as } n \to \infty \end{aligned}$$

□

LEMMA 4.5. *Let K be a finite M-complex with intrinsic metric d, and let $y \in K$. Suppose that a continuous function $f : U \to \mathbb{R}$ defined on an open neighborhood U of y satisfies the following conditions:*

(1) *$f(y) = 0$*
(2) *$f(z) > 0$ $(z \in U, z \neq y)$*
(3) *for every $z \in U$, the set $\{z' \in U | f(z') \leq f(z)\}$ is closed in K, and*
(4) *for every $x \in U$ there exists a point $r(x) \in \mathrm{ST}(x, K) \cap U$, such that $d(x, r(x)) + f(r(x)) \leq f(x)$ and $r(x) \neq x$, if $x \neq y$*

Then for every $x \in U$ there exists an allowable chain C from y to x with $\lambda(C) \leq f(x)$.

PROOF. By transfinite recursion on countable ordinals we define the points $x_\alpha inU$ for $\alpha < \omega_1$, such that the following condition holds:

$$(*)\ t_\alpha + f(x_\alpha) \leq f(x), \text{ where } t_\alpha = \sum_{\xi < \alpha} d(x_\xi, x_{\xi+1}).$$

Put $x_0 = x$. Suppose $0 < \beta < \omega_1$, and for $\alpha < \beta$ we have defined x_α satisfying $(*)$. If β is a non-limit ordinal, say, $\beta = \alpha + 1$, put $x_\beta = r(x_\alpha)$. Then

$$\begin{aligned} t_\beta + f(x_\beta) &= t_\alpha + d(x_\alpha, x_\beta) + f(x_\beta) \\ &\leq t_\alpha + f(x_\alpha) \qquad \text{(by 4)} \\ &\leq f(x) \qquad \text{(by the induction hypothesis)} \end{aligned}$$

If β is a limit ordinal, then $\beta = \lim \alpha(n)(n \to \infty)$ for some sequence of ordinals $\alpha(n) < \beta$, and the sequence $x_{\alpha(n)}$ converges to some $x_\beta \in K$. By 3, $x_\beta \in U$, and

$$\begin{aligned} t_\beta + f(x_\beta) &= \lim t_{\alpha(n)} + f(\lim x_{\alpha(n)}) \\ &= \lim(t_{\alpha(n)} + f(x_{\alpha(n)})) \qquad \text{(by continuity of } f\text{)} \\ &\leq f(x) \qquad \text{(by induction hypothesis)} \end{aligned}$$

The increasing sequence t_α $(\alpha < \omega_1)$, and therefore the sequence x_α $(\alpha < \omega_1)$, must be eventually constant. Since $r(x) \neq x$ for $x \neq y$, x_α must stabilize at y, say $y = x_{\alpha(0)}$. Define the sequence of ordinals $(\alpha(n)|n \in \mathbb{N})$ by recursion as follows: If $\alpha(n)$ is a non-limit ordinal, then $\alpha(n+1)$ is defined by $\alpha(n) = \alpha(n+1) + 1$, unless $\alpha(n) = 0$ when the recursion stops. If $\alpha(n)$ is a limit ordinal, then choose $\alpha(n+1) < \alpha(n)$ so that $d(x_{\alpha(n)}, x_{\alpha(n+1)}) < \epsilon(x_{\alpha(n)})$, where $\epsilon(\alpha(n))$ is given by Lemma 3.1. A strictly decreasing sequence of ordinals must be finite, therefore $\alpha(n) = 0$ for some n. Then $C(x_{\alpha(0)}, \ldots, x_{\alpha(n)})$ is an allowable chain from y to x with

$$\lambda(C) = \sum_{i=1}^{n} d(x_{\alpha(i-1)}, x_{\alpha(i)}) \leq \sum_{\xi < \alpha(0)} d(x_\xi, x_{\xi+1}) \leq f(x) \qquad \text{(by } (*)\text{)}$$

□

Corollary 4.6. *Let K be a finite, connected M-complex and d be the intrinsic metric on K. Then (K, d) is a geodesic metric space.*

Proof. By passing to a subdivision if necessary, we can assume that K is proper. Given $y \in K$, the open set $U = K$ and the function $f : K \to \mathbb{R}$, $f(x) = d(x, y)$ satisfy the conditions 1, 2 and 3 in Lemma 4.5 automatically, 4 follows from Lemma 4.4. Given $x \in K$, p_C is a geodesic segment from y to x, where C is the allowable chain given by Lemma 4.5. □

Corollary 4.7. *Let K be a locally finite, connected M-complex, and assume that there exists an $\epsilon > 0$, such that all closed ϵ-balls with respect to the intrinsic metric d of K are compact. Then (K, d) is a geodesic metric space.*

Proof. The condition on ϵ-balls implies that all closed metric balls of finite radius are compact. Given $x, y \in K$, the closed $d(x, y)$-ball around x is contained in a finite subcomplex L, then a geodesic segment in L from x to y is a geodesic segment in K. □

5. The girth of finite $\mathbb{S}$-complexes

Throughout this section, let K denote a finite $\mathbb{S}$-complex, and d its intrinsic metric. Define the *girth* $\mathbf{g}(K)$ of K as the infinum of lengths of closed geodesics in K.

Let (Y, σ) be a metric space. A *closed ϵ-geodesic of length λ* in Y is a map $q : S(\lambda) \to Y$ such that for any pair of points $s, t \in S(\lambda)$

$$(1-\epsilon)\rho_\lambda(s,t) \leq \sigma(q(s), q(t)) \leq (1+\epsilon)\rho_\lambda(s,t)$$

where ρ_λ denotes the arc metric on the circle $S(\lambda)$ of length λ.

LEMMA 5.1. *Let (X, ρ) and (Y, σ) be compact metric spaces, and $S = (f_n : X \to Y | n \in \mathbb{N})$ be a sequence of maps, such that for any pair of points $a, b \in X$ $\sigma(f_n(a), f_n(b)) \to \rho(a, b)$ as $n \to \infty$. Then there exists an isometric embedding $f : X \to Y$.*

PROOF. Given any subsequence S' of the sequence S, and any finite subset F of X, there is a subsequence $F(S')$ of the sequence S' that is convergent on F. Choose a countable dense subset $A = \{a_k | k \in \mathbb{N}\}$ in X, put $F_k = \{a_1, \ldots, a_k\}$ ($k \in \mathbb{N}$), and define by recursion $S_0 = S$ and $S_k = F_k(S_{k-1})$ ($k \in \mathbb{N}$). Define $f(a_k)$ as the limit of S_k at a_k. Then for any $k \leq m$, we have

$$\begin{aligned}
\sigma(f(a_k), f(a_m)) &= \sigma(\lim_{n\to\infty} S_k(a_k), \lim_{n\to\infty} S_m(a_m)) \\
&= \sigma(\lim_{n\to\infty} S_m(a_k), \lim_{n\to\infty} S_m(a_m)) \\
&= \lim_{n\to\infty} \sigma(S_m(a_k), S_m(a_m)) \\
&= \lim_{n\to\infty} \sigma(f_n(a_k), f_n(a_m)) \\
&= \rho(a_k, a_m).
\end{aligned}$$

Since f maps a dense subset A of X isometrically into Y, and X and Y are compact, f extends (uniquely) to an isometric map from X, the metric completion of A, into Y. $\square$

COROLLARY 5.2. *If for every $\epsilon > 0$ there exists a closed ϵ-geodesic of length λ in K, then there exists a closed geodesic of length λ in K.*

PROOF. Apply Lemma 5.1 to a sequence of closed $1/n$-geodesics of length λ ($n \in \mathbb{N}$). $\square$

COROLLARY 5.3. *If $0 < \mathbf{g}(K) < \infty$, then there exists a closed geodesic of length $\mathbf{g}(K)$ in K.*

PROOF. Given $\epsilon > 0$, choose a closed geodesic $q : S(\lambda) \to K$, where $\mathbf{g}(K) \leq \lambda < (1+\epsilon)\mathbf{g}(K)$. Then the composition $q \circ s$, where $s : S(\mathbf{g}(K)) \to S(\lambda)$ is a uniform stretch, is a closed ϵ-geodesic of length $g(K)$ in K, and Corollary 5.2 applies. $\square$

LEMMA 5.4. *Suppose that $x, y \in K$ and $d(x, y) < \pi$. Then every geodesic segment i the cone $\mathbf{C}K$ of K from x to y is contained in K.*

PROOF. The following argument is due to M. Gromov, cf. [**10**] p.122. Suppose that $p : [0, \lambda] \to \mathbf{C}K$ is a geodesic segment from x to y, where, without loss of generality, we can assume that $p(t) \notin K$ for $t \neq 0, \lambda$. Assume that K is proper, then the cone point $\mathbf{1}$ is a vertex of $\mathbf{C}K$. We can assume that the image of p is contained in $U = \mathbf{C}K - \{\mathbf{1}\}$, since any geodesic segment in $\mathbf{C}K$ connecting two points of K through $\mathbf{1}$ must consist of two segments of length $\pi/2$. Let $pr : U \to K$ denote the radial projection, then pr is injective on the image P of p. The projection of an allowable chain representing p gives $pr(P)$ a one-dimensional, and the cone

$\mathbf{C}pr(P)$ a two-dimensional $\mathbb{S}$-complex structure. p is a geodesic segment in $\mathbf{C}pr(P)$, therefore the map $f : \mathbf{C}pr(P) \to \mathbb{S}^2$ defined by wrapping $pr(P)$ around the equator and mapping each cell of $\mathbf{C}pr(P)$ isometrically into $\mathbb{S}^2$ must take P isometrically onto a great circle arc, and x and y into a pair of antipodal points. So, $\lambda = \pi$, which proves the lemma. □

LEMMA 5.5. *Every closed geodesic in $\mathbf{C}K$ is contained in K.*

PROOF. The image of Q of a closed geodesic q in $\mathbf{C}K$ obviously cannot contain $\mathbf{1}$. If $Q \subset \mathbf{C}K - (\{\mathbf{1}\} \cup K)$, then the image of the wrapping map described in the proof of Lemmma 5.4 starting with any point on Q is a great circle arc, which, sooner or later, must intersect the equator of $\mathbb{S}^2$, a contradiction. Finally, if $Q \cap K \neq \emptyset$, then let the allowable chain C represent q. Then, since $\mathbf{C}B - B$ is convex in B for any spherical cell B, some element x_k of C must be in K. The distance of v_k^{in} and v_k^{out} in $\mathrm{LK}(x_k, \mathbf{C}K) = \mathbf{C}\mathrm{LK}(x_k, K)$ can equal π only if both v_k^{in} and v_k^{out} are in $\mathrm{LK}(x_k, K)$, which, by induction around C, shows that $Q \subset K$. □

Combining Lemma 5.4 and Lemma 5.5, it immediately follows that:

COROLLARY 5.6. *If $\mathbf{g}(K) \geq 2\pi$, then $\mathbf{g}(\mathbf{S}K) \geq 2\pi$. If $\mathbf{g}(K) \geq 2\pi$, then every closed geodesic in $\mathbf{S}K$ passes through both suspension points and has length 2π.*

LEMMA 5.7. *For any $x \in K$, the ball of radius $\epsilon(x)/2$ around x in K is isometric to the ball of radius $\epsilon(x)/2$ around $\mathbf{1}$ in $\mathbf{C}L$, where L is a proper subdivision of $\mathrm{LK}(x, K)$, and $\epsilon(x)$ is defined in Lemma 3.1.*

PROOF. Identification of tangent spaces $T_x K = T_{\mathbf{1}}\mathbf{C}L$ defines an embedding f of the open $\epsilon(x)$-neighborhood of x in K into the suspension of $\mathbf{S}L$, f is isometric when restricted to each cell in $\mathrm{ST}(x, K)$. By Lemma 3.1, any geodesic segment between two points in the $\epsilon(x)/2$-ball is contained in the $\epsilon(x)/2$-ball (either around x in K, or around $\mathbf{1}$ in $\mathbf{C}L$), therefore f is an isometry between the $\epsilon(x)/2$-balls. □

Combining Lemmas 5.5 and 5.7, it immediately follows that:

COROLLARY 5.8. *For any $x \in K$, the $\epsilon(x)/2$-ball around x cannot contain a closed geodesic in K.*

COROLLARY 5.9. *$\mathbf{g}(K)$ is always positive.*

PROOF. Let λ denote the Lebesgue number of the covering of K by all open $\epsilon(x)/2$-balls for $x \in K$, then by Corollary 5.8, $\mathbf{g}(K) \geq 2\lambda$. □

In the remainder of this section, we show that $\mathbf{g}(K)$, in some sense, is a lower semicontinuous function of K.

Let B, B' be spherical simplices and $\delta > 1$. We say that B' is a *δ-change* of B, if there exists a map $f : B' \to B$, such that for any $x, y \in B'$ the following inequality holds:

$$(1/\delta)d'(x, y) \leq d(f(x), f(y)) \leq \delta d'(x, y) \tag{1}$$

where d and d' denote the metrics on B and B' respectively. Any map f with this property is called *δ-map* for B'. The inverse of f shows that then B is a δ-change of B'. If K and K' are finite simplicial $\mathbb{S}$-complexes, then K' is a δ-change of K, if there exists a homeomorphism $f : K' \to K$ that takes simplices onto simplices, and restricts to a δ-map on each simplex of K'. Any such map, called a δ-map for

K', obviously satisfies 1, where d and d' denote the intrinsic metrics on K and K' respectively. Again, the inverse of f shows that K is a δ-change of K'.

LEMMA 5.10. *Let K be a finite simplicial $\mathbb{S}$-complex. There exists a poisitive real number λ, such that if K' is a δ-change of K with $\delta \leq 2$, then $\mathbf{g}(K') \geq \lambda$.*

PROOF. Let λ denote the Lebesgue number of the covering of K with open $\epsilon(x)/8$-balls for all $x \in K$. If K' is a δ-change of K with δ-map f, then for any subset $A \subset K'$ of diameter $\leq \lambda/2$, there exists an $x \in K$, such that $f(A)$ is contained in the $\epsilon(x)/8$-ball around x, and therefore A is contained in the $\epsilon(x)/4$-ball around $f^{-1}(x)$, which is contained in the $\epsilon(f^{-1}(x))/2$-ball around $f^{-1}(x)$. Then by Corollary 5.8, A cannot be the image of a closed geodesic. $\square$

LEMMA 5.11. *Let K be a finite simplicial $\mathbb{S}$-complex. For any real number $\alpha < \mathbf{g}(K)$ there exists a $\delta > 1$, such that $\mathbf{g}(K') \geq \alpha$ hods for any δ-change K' of K.*

PROOF. Suppose that, to the contrary, for all $n \in \mathbb{N}$ there is an $(1 + 1/n)$-change K'_n of K and a closed geodesic $q_n : S(\lambda_n) \to K'_n$ with $\lambda_n \leq \alpha$. Then by Lemma 5.10, the sequence (λ_n) has a positive limit point λ, and the composition $q_n \circ s_n$ with uniform stretches $s_n : S(\lambda) \to S(\lambda_n)$ gives closed ϵ-geodesics of length $\lambda < \mathbf{g}(K)$ with arbitrary small $\epsilon > 0$, which contradicts to Corollary 5.2. $\square$

Generally, $\mathbf{g}(K)$ is not an upper semicontinuous function of K. Indeed, if K is a simplicial subdivision of a hemisphere in $\mathbb{S}^2$, then $\mathbf{g}(K) = 2\pi$, and there exist δ-changes of K with infintie girth for arbitrarily small $\delta > 1$.

6. Curvature

Motivated by [**10**], p. 120, we say that an M-complex K satisfies the *link axiom*, if

$$\mathbf{g}(\text{LK}(B, K)) \geq 2\pi \text{ for each non-empty cell } B \text{ of } K.$$

We say that an $\mathbb{E}$-complex ($\mathbb{H}$-complex, or $\mathbb{S}$-complex) K has *curvature* ≤ 0 (≤ -1, or ≤ 1 respectively), if K satisfies the link axiom.

There are a number of generalizations of the condition "curvature $\leq \kappa$" on riemannian manifolds to broader classes of metric spaces. Let us review a few of these.

In [**1**], A.D. Aleksandrov considered complete geodesic metric spaces X with a certain definition of angles between geodesic segments originating from a common endpoint. (For example, locally finite M-complexes K satisfying the condition of Corollary 4.7 are such spaces, the angle between two geodesic segments with common starting point x being the distance of their initial vectors in $\text{LK}(x, K)$.) Let κ be a real number. If Δ is a geodesic triangle in X (with perimeter less than $2\pi\kappa^{-1/2}$ if $\kappa > 0$), then the comparison triangle $\Delta(\kappa)$ in the simply connected complete riemannian 2-manifold $M(\kappa)$ fo constant sectional curvature κ is uniquely defined up to isometry by the requirement that $\Delta(\kappa)$ have the same sides as Δ. The defect of a geodesic triangle is defined as π minus the sum of the interior angles. Now we can formulate Aleksandrov's comparison axiom

> A(κ): For every geodesic triangle Δ in X, the defect of the comparison triangle $\Delta(\kappa)$ does not exceed the defect of Δ.

In [**1**], Aleksandrov proved that if a space satisfies the axiom $A(\kappa)$, then in any geodesic triangle Δ each angle is less than or equal to the corresponding angle of $\Delta(\kappa)$. Aleksandrov defined X to have curvature $\leq \kappa$, if X satisfies $A(\kappa)$ locally, that is, every point in X has a neighborhood satisfying $A(\kappa)$. He proved that if $\kappa \leq 0$ and X satisfies $A(0)$ globally and has curvature $\leq \kappa$, then X satisfies $A(\kappa)$ globally.

In [**10**], M. Gromov considered the comparison axiom

> $\mathrm{CAT}(\kappa)$: Let Δ be a geodesic triangle in X with vertices x, y and z, and let w be a point on the side with vertices y and z. Let x', y' and z' be the corresponding vertices of the comparison triangle $\Delta(\kappa)$, and let w' be a point on the side of $\Delta(\kappa)$ with vertices y' and z' uniquely determined by the requirement that the distance from y' to w' equal the distance from y to w. Then the distance from x to w does not exceed the distance from x' to w'.

Aleksandrov proved the equivalence of $A(\kappa)$ and $\mathrm{CAT}(\kappa)$ in [**1**].

In [**9**], Gromov introduced the following "convexity axiom":

> $C(0)$: The metric of X is a convex function on $X \times X$.

Here, a real function f defined on a geodesic space Y is called convex, if all compositions $f \circ p$ where p is a geodesic segment in Y are convex functions. A complete geodesic space X is called a convex (locally convex) space, if X satisfies $C(0)$ (locally).

In [**4**] H. Busemann introduced the axiom

> $B(0)$: If m_i is the midpoint of a geodesic segment from x to y_i in X $(i = 1, 2)$, then the distance from m_1 to m_2 does not exceed half the distance from y_1 to y_2.

Busemann defined a complete geodesic metric space X to have curvature ≤ 0, if X satisfies $B(0)$ locally. He proved that simply connected spaces of curvature ≤ 0 are contractible and satisfy $B(0)$ globally.

It is easy to see the equivalence of the axioms $C(0)$ and $B(0)$, and that $A(\kappa)$ and $\mathrm{CAT}(\kappa)$ with $\kappa \leq 0$ imply $C(0)$ and $B(0)$. In the case of locally finite M-complexes, it is easy to see that $A(0)$, $\mathrm{CAT}(0)$, $C(0)$ and $B(0)$ are equivalent.

Gromov's argument in [**10**], p.120 implies that locally finite $\mathbb{H}$-complexes ($\mathbb{E}$-complexes or $\mathbb{S}$-complexes) satisfy the axiom $\mathrm{CAT}(-1)$ ($\mathrm{CAT}(0)$ or $\mathrm{CAT}(1)$ respectively) locally. Thus, our definition of "curvature ≤ -1 (0, or 1 respectively)" coincides with the classical ones. Combining theorems of Aleksandrov and Busemann, it follows that simply connected locally finite $\mathbb{H}$-complexes ($\mathbb{E}$-complexes) satisfying the condition of Corollary 4.7 and the link axiom are contractible and satisfy the axioms $\mathrm{CAT}(-1)$ ($\mathrm{CAT}(0)$ respectively) globally. In particular, they are convex spaces.

CHAPTER 2

Almost Negative Matrices

We use a classical theorem in linear algebra to prove our main technical tools (Lemmas 9.5 and 9.7) in investigating the geometry of spherical simplicial complexes with prescribed edge lengths. The chief result of this chapter (Proposition 10.1) will imply that the geometric complexes we construct in Chapter 3 satisfy the link condition.

7. The nerve of a symmetric matrix

Let $A = (a_{ij})$ denote a real symmetric matrix of order n. The set of indices of rows and columns usually is the set $\{1, \ldots, n\}$, when we call A an $n \times n$-matrix, but we often use different index sets, for example a subset I of $\{1, \ldots, n\}$, when we say that A is an $I \times I$-matrix. We define a finite $\mathbb{S}$-complex $\mathbf{N}(A)$, called the *nerve* of A, as follows:

Let $u_1, \ldots, u_n$ be a basis in a real vector space V, and let $\langle, \rangle$ denote the bilinear form on V whose matrix with respect to the basis $u_1, \ldots, u_n$ is A; that is, $a_{ij} = \langle u_i, u_j \rangle$. For $I \subset \{1, \ldots, n\}$, let A_I denote the principal submatrix of A corresponding to the indices in I, and let V_I denote the linear subspace of V generated by $\{u_i | i \in I\}$. If A_I is positive definite, then let B_I denote the spherical simplex associated to the simplicial cone generated by $\{u_i | i \in I\}$ in the unit sphere of the euclidean vector space V_I. The nerve $\mathbf{N}(A)$ is defined as

$$\mathbf{N}(A) = \cup \{B_I | I \subset \{1, \ldots, n\} \text{ and } A_I \text{ is positive definite}\}$$

together with identity maps as characteristic maps. $\mathbf{N}(A)$ clearly is a finite $\mathbb{S}$-complex. Let J denote the index set $\{i \in \{1, \ldots, n\} | \langle u_i, u_i \rangle > 0\}$. Then the vertex set of $\mathbf{N}(A)$ is $\{v_i | i \in J\}$, where $v_i = \langle u_i, u_i \rangle^{-1/2} u_i$ $(i \in J)$. Every $x \in \mathbf{N}(A)$ can be uniquely be written as a linear combination, with positive coefficients, of the vertices of the support of x in $\mathbf{N}(A)$.

We say that a matrix is *normalized*, if all diagonal entries equal 1. Let A' denote the $J \times J$-matrix with entries $a'_{ij} = \langle v_i, v_j \rangle (i, j \in J)$. (Here, if $-1 < \langle v_i, v_j \rangle < 1$, then $\langle v_i, v_j \rangle$ equals the cosine of the length of the edge in $\mathbf{N}(A)$ with vertices v_i and v_j.) Then $A' = DA''D$, where A'' is the principal submatrix of A corresponding to the index set J and D is a diagonal matrix with entries $d_{ii} = \langle u_i, u_i \rangle^{-1/2} (i \in J)$. Thus A' is a normalized symmetric matrix with entries $\mathbf{N}(A') = \mathbf{N}(A)$. We say that A' is the normalized matrix associated to A (and to A'').

If V is a vector space equipped with a bilinear form $\langle, \rangle$, then for any subset $U \subset V$ let $U^\perp$ denote the subspace $\{v \in V | \langle u, v \rangle = 0 \text{ for all } u \in U\}$. For singletons $U = \{u\}$ we write $u^\perp$ instead of $\{u\}^\perp$. Suppose that the restriction of $\langle, \rangle$ to the linear span L_U of U is positive definite. Then V is the orthogonal direct sum of the subspaces L_U and $U^\perp$. Let ϕ_U denote the orthogonal projection of V onto $U^\perp$. If

U' is any subset of U, then

$$\phi_U = \phi_{\phi_{U'}}(U) \circ \phi_{U'} \tag{2}$$

If S is the unit sphere in a euclidean vector space $(E, \langle , \rangle)$ and $x \in S$, then T_xS is identified with the subspace $x^\perp \leq E$ with the bilinear form inherited from E. Unit speed geodesic ranys originating from x are the maps $p_v(t) = x\cos t + v \sin t (t \geq 0)$, where $v \in x^\perp \cap S$. Thus, if $\mathbf{N}(A)$ is the nerve of A and $x \in \mathbf{N}(A)$, then the complexes $T_x\mathbf{N}(A)$, $\mathrm{LK}(x, \mathbf{N}(A))$ and $\textsc{lk}(x, \mathbf{N}(A))$ are identified with subsets of the subspace $x^\perp$ of V, and the exponential map at x takes the form $\exp_x(tv) = x\cos t + v\sin t$, where $v \in \mathrm{LK}(x, \mathbf{N}(A))$ and $0 \leq t \leq t_0$ for some $t_0 > 0$ depending on v.

If B is a simplex in $\mathbf{N}(A)$ with vertex set $\{v_i | i \in I\}$, $I \subset \{1, \dots, n\}$, then define the $I^* \times I^*$-matrix $\textsc{lk}(I, A) = (a^*_{ij})$, where $I^* = \{1, \dots, n\} - I$, by

$$a^*_{ij} = \langle \phi_U(u_i), \phi_U(u_j) \rangle \ (i, j \in I^*)$$

where $U = \{u_i | i \in I\}$. Then, with the above identification, $\mathbf{N}(\textsc{lk}(I, A)) = \textsc{lk}(B, \mathbf{N}(A))$. Forula 2 implies that for any $I' \subset I$ we have $\textsc{lk}(I', \textsc{lk}(I - I', A)) = \textsc{lk}(I, A)$.

If A' and A'' are real symmetric matrices, then the nerve of the direct sum

$$A' \oplus A'' = \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix} \tag{3}$$

is isometrically isomorphic to (and identified with) the join of the nerves of A' and A'': $\mathbf{N}(A' \oplus A'') = \mathbf{N}(A') * \mathbf{N}(A'')$. A real symmetric matrix A of order n is called *reducible*, if some permutation of the index set $\{1, \dots, n\}$ brings A into the form 3, where the orders of A' and A'' are less than n. In this case we say that the principal submatrices A' and A'' are orthogonal. Otherwise A is called *irreducible*. The nerves of the reducible matrices

$$\mathbf{C}A = (\mathbf{1}) \oplus A \text{ and } \mathbf{D}A = \mathbf{D} \oplus A \text{ where } \mathbf{D} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

are the cone and the double cone of $\mathbf{N}(A)$ respectively. If $\{w_1, w_2\}$ is a basis in a real vector space W equipped with the bilinear form with $\mathbf{D}$ in this basis, and as earlier $\mathbf{1}$ denotes the standard unit basis vector in $\mathbb{R}$, then the linear map $f : W \to \mathbb{R}$ defined by $f(w_1) = \mathbf{1}$ and $f(w_2) = -\mathbf{1}$ preserves scalar products. Thus, $f \oplus \textsc{id}_V$ restricts to a scalar product-preserving isometric isomorphism from the double cone of $\mathbf{N}(A)$ onto the equatorial subdivision of the suspension $\mathbf{SN}(A)$. Identification of the double cone of $\mathbf{N}(A)$ with $\mathbf{SN}(A)$ via $f \oplus \textsc{id}_V$ enables us to work with the $\mathbb{S}$-complex $\mathbf{SN}(A)$ as if it were the nerve of $\mathbf{D}A$.

8. Nerves of almost negative matrices

We say that a real symmetric is *almost negative*, if all off-diagonal entries of A are non-positive. Throughout this section, let A be an almost negative matrix and $\mathbf{N} = \mathbf{N}(A)$ be the nerve of A. Since $DA'D$ is almost negative for any principal submatrix A' of A and any non-negative diagonal matrix D of the size of A', a normalized symmetric matrix with nerve N is also almost negative. Direct sums of almost negative matrices are almost negative, so joins, especially cones and double cones of nerves of almost negative matrices are nerves of almost negative matrices.

Lemma 8.1. *For any vertex v of $\mathbf{N}$ and any $x \in \mathbf{N}$ if $\langle v, x \rangle \geq 0$, then $x \in$ $\mathrm{St}(v, N)$.*

PROOF. Let $\{v_i | i \in I\}$ be the vertex set of the support of x, then

$$x = \sum_{i \in I} \alpha_i v_i$$

where $\alpha_i > 0 (i \in I)$. If $\langle v, x \rangle \leq 0$, then either $v = v_i$ for some $i \in I$, in which case $x \in \mathrm{OST}(v, \mathbf{N})$, or $\langle v, v_i \rangle = 0$ for all $i \in I$, in which case the set $\{v_i | i \in I\} \cup \{v\}$ generates positive definite subspace in V, and is the vertex set of some simplex (the cone of the support of x) in $\mathbf{N}$, showing that $x \in \mathrm{ST}(v, \mathbf{N})$. □

COROLLARY 8.2. *Let v be a vertex of $\mathbf{N}$. Then*

$$\mathrm{LK}(v, \mathbf{N}) = v^{\perp} \cap \mathbf{N}, \text{ and}$$
$$\mathbf{C}\mathrm{LK}(v, \mathbf{N}) \subset \mathrm{ST}(v, \mathbf{N}) \subset \mathbf{S}\mathrm{LK}(v, \mathbf{N}),$$

where the cone and the suspension of $\mathrm{LK}(v, \mathbf{N}) \subset v^{\perp}$ are in the orthogonal sum $v^{\perp} \oplus \mathbb{R}_v$.

PROOF. If in a spherical simplex B all edges at a vertex v have length $\geq \pi/2$, then $\mathrm{LK}(v, B) = \{x \in B | d_B(x, v) = \pi/2\} = v^{\perp} \cap B$. This shows that $\mathrm{LK}(v, \mathbf{N}) \subset v^{\perp} \cap \mathbf{N}$. On the other hand, if $x \in \mathbf{N}$ and $\langle v, x \rangle = 0$, then, by Lemma 8.1, $x \in B$ for some simplex B containing v, and so $x \in \mathrm{LK}(v, B) \subset \mathrm{LK}(v, \mathbf{N})$. The inclusions $\mathbf{C}\mathrm{LK}(v, \mathbf{N}) \subset \mathrm{ST}(v, \mathbf{N}) \subset \mathbf{S}\mathrm{LK}(v, \mathbf{N})$ are obvious. □

LEMMA 8.3. *If B is a simplex in $\mathbf{N}$ with vertex set $\{v_i | i \in I\}$, then the matrix $\mathrm{LK}(I, A)$ is almost negative.*

PROOF. For $B = \emptyset$ this is true since $\mathrm{LK}(\emptyset, A) = A$. For any $i \in I$ we have $\mathrm{LK}(I, A) = \mathrm{LK}(\{i\}, \mathrm{LK}(I - \{i\}, A))$, so it suffices to show that for any almost negative matrix A and vertex $v_i \in \mathbf{N}(A)$ the matrix $\mathrm{LK}(\{i\}, A)$ is almost negative, then the statement of the lemma follows by induction on the dimension of B. Using the notations of Section 7, we have to show that $a^*_{ij} \leq 0$ if $j \neq k (j, k \in \{i\}^*)$. Indeed,

$$\begin{aligned}
a^*_{jk} &= \langle \phi_{\{u_i\}}(u_j), \phi_{\{u_i\}}, u_k \rangle \\
&= \langle u_j - \frac{\langle u_i, u_j \rangle}{\langle u_i, u_i \rangle} u_i, u_k - \frac{\langle u_i, u_k \rangle}{\langle u_i, u_i \rangle} u_i \rangle \\
&= \langle u_j, u_k \rangle - \frac{\langle u_i, u_j \rangle \langle u_i, u_k \rangle}{\langle u_i, u_i \rangle} \\
&\leq 0
\end{aligned}$$

□

COROLLARY 8.4. *If B is a simplex of $\mathbf{N}$, then $\mathrm{LK}(B, \mathbf{N})$ is the nerve of an almost negative matrix.*

PROOF. Indeed, $\mathrm{LK}(B, \mathbf{N}) = \mathbf{N}(\mathrm{LK}(I, A))$, where the vertex set of B is $\{v_i | i \in I\}$, and Lemma 8.3 applies. □

COROLLARY 8.5. *If $x, y \in \mathbf{N}$, then $\langle x, y \rangle \leq 1$ with equality only if $x = y$.*

PROOF. We say that an $\mathbb{S}$-complex $\mathbf{N}$, which is the nerve of an almost negative matrix, has the property P, if the statement of Corollary 8.5 is true for $\mathbf{N}$. First we show that if P holds for $\mathbf{N}$, then P holds for the suspension of $\mathbf{N}$. Let $x, y \in \mathbf{SN}$, where we can assume that both x and y are different form $\mathbf{1}$ or $-\mathbf{1}$. Then $x = x' + \alpha \mathbf{1}$

and $y = y' + \beta\mathbf{1}$ where $x', y' \in V - \{0\}$, $\alpha, \beta \in \mathbb{R}$, and $\langle x, x'\rangle + \alpha^2 = \langle y, y'\rangle + \beta^2 = 1$. The unit vectors $x'' = \langle x, x'\rangle^{-1/2}x'$ and $y'' = \langle y, y'\rangle^{-1/2}y'$ are in $\mathbf{N}$, so we have

$$\begin{aligned}
\langle x, y\rangle &= \langle x', y'\rangle + \alpha\beta \\
&= \langle x', x'\rangle^{-1/2}\langle y', y'\rangle^{-1/2}\langle x'', y''\rangle + \alpha\beta \\
&\leq \langle x', x'\rangle^{-1/2} - \langle y', y'\rangle^{-1/2} + \alpha\beta \qquad \text{(since } P \text{ holds for } \mathbf{N}) \\
&\leq (\langle x', x'\rangle + \alpha^2)^{-1/2}(\langle y', y'\rangle + \beta^2)^{-1/2} \qquad \text{(Cauchy-Schwarz inequality)} \\
&= 1,
\end{aligned}$$

with equality only if $x' = y'$ and $\alpha = \beta$, that is $x = y$.

We prove that the nerve $\mathbf{N}$ of every almost negative matrix has property P by induction on $\dim \mathbf{N}$. If $\dim \mathbf{N} \leq 0$, then P obviously holds for $\mathbf{N}$. Assume that $\dim \mathbf{N} \geq 1$, and P holds in all dimensions $< \dim \mathbf{N}$. Let $x, y \in \mathbf{N}$. We can assume that the supports of x and y have a vertex v in common, since otherwise $\langle x, y\rangle \leq 0$. Then $x, y \in \mathrm{St}(v, \mathbf{N})$, and, by Corollary 8.2, $x, y \in \mathbf{S}\mathrm{LK}(v, \mathbf{N})$. By Corollary 8.4 the $\mathbb{S}$-complex $\mathrm{LK}(v, \mathbf{N})$ is the nerve of an almost negative matrix, so since $\dim \mathrm{LK}(v, \mathbf{N}) < \dim \mathbf{N}$, P holds for $\mathrm{LK}(v, \mathbf{N})$. Then, by the above argument, P holds for $\mathbf{S}\mathrm{LK}(v, \mathbf{N})$, so $\langle x, y\rangle \leq 1$ with equality only if $x = y$. □

9. The intrinsic metric on $\mathbf{N}(A)$

Let us recall the Frobenius-Perron Theorem (see, for instance, []) on the spectrum of nonnegative irreducible square matrices. We formulate the theorem for symmetric matrices.

THEOREM (Frobenius, Perron). *Let M be a nonnegative (that is, all entries of M be nonnegative) and irreducible real symmetric $n \times n$-matrix ($n \geq 1$). Let μ denote the largest eigenvalue of M, then*

(1) *$|\nu| \leq \mu$ for every eigenvalue ν of M,*
(2) *the multiplicity of μ is* 1,
(3) *the (unique up to multiplication by scalars) eigenvector corresponding to μ has all positive or all negative coordinates, and*
(4) *all eigenvalues of principal submatrices of order $\leq (n-1)$ of M are strictly less than μ.*

A normalized real symmetric matrix A is almost negative if and only if $M = \mathrm{ID} - A$ is a nonnegative matrix with zero diagonal entries, and M is irreducible if and only if A is irreducible. Thus if A is irreducible, then A has a smallest eigenvalue λ of multiplicity 1, and we can choose a positive eigenvector (that is an eigenvector with all positive coordinates) corresponding to λ

We formulate three immediate consequences of the theorem

COROLLARY 9.1. *Let A be a positive definite, irreducible, normalized almost negative matrix. Then all entries of the inverse A^{-1} of A are positive.*

PROOF. For a nonnegative $n \times n$-matrix $M = (a_{ij})$, irreducibility means that for any pair (i, j) of indices there exists a sequence $i(1) = i, i(2), \ldots, i(n) = j$ of indices with $a_{i(k)i(k+1)} > 0 (k = 1, \ldots, n-1)$. This implies that all entries of M^n are positive. A is positive definite, so $\lambda > 0$, therefore $\mu = 1 - \lambda < 1$, and by statement 1 of the Frobenius-Perron Theorem, this implies that the euclidean norm of M is < 1. So the matrix $A^{-1} = (\mathrm{ID} - M)^{-1}$ is the sum of the convergent series $\text{ID} + M + M^2 + \cdots + M^n + \cdots$, where all terms are nonnegative and there are strictly positive terms. ☐

COROLLARY 9.2. *Let A be an irreducible, normalized almost negative matrix, and let w be a positive eigenvector corresponding to the minimal eigenvalue λ of A. Assume that A is not positive definite (that is, $\lambda \leq 0$), then $\langle w, x \rangle \leq 0$ for every $x \in \mathbf{N}(A)$.*

PROOF. It suffices to prove that $\langle w, v_i \rangle \leq 0 (i = 1, \ldots, n)$ where $\{v_1, \ldots, v_n\}$ is the vertex set of $\mathbf{N}(A)$. Write $w = \alpha_1 v_1 + \cdots + \alpha_n v_n$, where $\alpha_i > 0 (i = 1, \ldots, n)$, then

$$\langle w, v_i \rangle = \alpha_1 \langle v_1, v_i, + \rangle \cdots + \alpha_n \langle v_n, v_i, = \rangle \lambda \alpha_i \leq 0$$

☐

We say that an almost negative $n \times n$-matrix is *parabolic*, if all principal submatrices of order $\leq (n-1)$ of A are positive definite, and $\det A = 0$. Parabolic matrices are automatically irreducible. If $I \subset \{1, \ldots, n\}$ corresponds to the vertex set of a simplex in $\mathbf{N}(A)$, then parabolic submatrices of $\text{LK}(I, A)$ are in $1-1$ correspondence with parabolic principal submatrices of A.

COROLLARY 9.3. *Let A be an irreducible, normalized almost negative $n \times n$-matrix. If the smallest eigenvalue of A is 0, then A is parabolic.*

PROOF. Statement 4 of the Frobenius-Perron Theorem implies that all eigenvalues of principal submatrices of order $\leq (n-1)$ are positive. ☐

LEMMA 9.4. *Let $\langle , \rangle$ be a bilinear form in a real vector space V. Let u, v, and z be three nonzero vectors in V satisfying the following requirements:*

$$u = \alpha v + \beta z, \text{ where } \alpha, \beta > 0$$
$$\langle v, v \rangle \leq 0$$
$$\langle v, z \rangle \leq 0, \text{ and}$$
$$\langle u, u \rangle = \langle z, z \rangle = 1.$$

Then $\langle u, z \rangle \geq 1$. If $\langle v, v \rangle < 0$ and $\alpha \neq 0$, then $\langle u, z \rangle > 1$.

PROOF. The restriction of the form $\langle , \rangle$ to the two-dimensional subspace W spanned by v and z is not positive definite (since $\langle v, v \rangle \leq 0$). In such a space the scalar product of any two vectors of length 1 is either ≥ 1 or ≤ -1. Since

$$1 = \langle u, u \rangle = \alpha \langle u, v \rangle + \beta \langle u, z \rangle = \alpha^2 \langle v, v \rangle + \alpha\beta \langle z, v \rangle + \beta \langle u, z \rangle \leq \beta \langle u, z \rangle$$

implies that $\langle u, z \rangle > 0$, we have $\langle u, z \rangle \leq 1$. If $\langle v, v \rangle < 0$, then the form is indefinite on W, and the scalar product of any two independent vectors of length 1 is either > 1 or < -1. Thus, if $\langle v, v \rangle < 0$ and $\alpha \neq 0$, then $\langle u, z \rangle > 1$. ☐

In the proofs of the next two lemmas we use the notations introduced in Section 7.

LEMMA 9.5. *Let A be an almost negative $n \times n$-matrix and u be a vector in V with nonnegative coordinates and with $\langle u, u \rangle = 1$. Then there exists a vector $z \in \mathbf{N}(A)$ with $\langle u, z \rangle \geq 1$. If A has no parabolic principal submatrices and $u \notin \mathbf{N}(A)$, then z can be chosen with $\langle u, z \rangle > 1$.*

Proof. Let Δ denote the affine $(n-1)$-simplex spanned by (the endpoints of) the basis vectors $u_1, \ldots, u_n$ in V. By induction on the partially ordered set of faces of Δ we define an affine simplicial subdivision on Δ as follows:

The subdivision of $\emptyset$ is itself.

Suppose that Δ' is a face of Δ and we have defined a simplicial subdivision of the boundary $\partial\Delta'$ of Δ'. Let A' denote the principal submatrix of A corresponding to the index set $I \subset \{1, \ldots, n\}$, where the vertex set of Δ' is $\{u_i | i \in I\}$. We have the following four cases:

(1) A' is positive definite. Then the subdivision of Δ' is itself. This is compatible with the subdivision of $\partial\Delta'$, since all principal submatrices of A' are positive definite.

(2) There is an index $i \in I$ with $\langle u_i, u_i \rangle \leq 0$. Then choose any such i, and define the subdivision of Δ' as the affine cone of the subdivision of the face of Δ' opposite the vertex u_i, with cone point u_i. (Note that the cone of $\emptyset$ with cone point u_i is u_i.) This is compatible with the subdivision of $\partial\Delta'$ because, by induction, it is the affine join of the face with vertices $\{u_i | \langle u_i, u_i \rangle \leq 0\}$ and the opposite face.

(3) A' is irreducible, not positive definite, and $\langle u_i, u_i \rangle > 0$ for all $i \in I$. Then introduce the unique positive eigenvector $w_I \in \Delta'$ of the normalized matrix associated to A' as a new vertex in the interior of Δ', and subdivide Δ' as the affine cone of $\partial\Delta'$ with cone point w_I.

(4) A' is reducible, not positive definite, and $\langle u_i, u_i \rangle > 0$ for all $i \in I$. Then choose A_1 and A_2 with non-empty index sets I_1 and I_2 and with $A' = A_1 \oplus A_2$, and define the subdivision fo Δ' as the affine join of the subdivisions of the faces corresponding to the index sets I_1 and I_2. This is compatible with the subdivision of $\partial\Delta'$, since, by induction, whenever Δ'' is a face of Δ' with an index set J with non-empty intersections with I_1 and I_2, the principal submatrix corresponding to Δ'' is reducible and the subdivision of Δ'' is the join of subdivisions of the faces corrsponding to $J \cap I_1$ and $J \cap I_2$.

Let F denote the set of vertices in the subdivision of Δ. Then $F = F_1 \cup F_2 \cup F_3$, where

$$\begin{aligned}
F_1 &= \{u_i | \langle u_i, u_i \rangle > 0\} \\
F_2 &= \{u_j | \langle u_j, u_j \rangle \leq 0\} \\
F_3 &= \{w_I | \text{ the principal minor corresponding to } I \text{ is irreducible,} \\
&\qquad \text{not positive definite, and } \langle u_i, u_i \rangle > 0 \text{ for all } i \in I\}.
\end{aligned}$$

We show that

$$\text{if } w \in F_2 \cup F_3, \text{ then } \langle w, z \rangle \leq 0 \text{ for all } z \in \Delta \tag{4}$$

Indeed, for $w = u_j \in F_2$, the almost negativeness of A implies 4, and for $w = w_I \in F_3$, Corollary 9.2 implies that $\langle w_I, u_i, \leq \rangle 0$ for $i \in I$, and the almost negativeness of A implies that $\langle w_I, u_j \rangle \leq 0$ for $j \notin I$.

Let u be a vector in V with nonnegative coordinates and with $\langle u, u \rangle = 1$. If $u \in \mathbf{N}(A)$, then $z = u$ is as required, so assume that $u \notin \mathbf{N}(A)$. Then u can uniquely be written as a linear combination of some elements of F with positive coefficients:

$$u = \sum_{w \in F'} \alpha_w w,$$

where $F' \subset F$ is the vertex set of a simplex in the subdivision of Δ, and $\alpha_w > 0$. $(w \in F')$. Property 4 implies that $F' \cap F_1 \neq \emptyset$, and $u \notin \mathbf{N}(A)$ implies that $F' \cap (F_2 \cup F_3) \neq \emptyset$. Then $u = v + z'$, where

$$v = \sum_{w \in F' \cap (F_2 \cup F_3)} \alpha_w w \quad \text{and} \quad z' = \sum_{w \in F' \cap F_1} \alpha_w w$$

$F \cap F_1$ is the index set of a simplex in $\mathbf{N}(A)$, so $z' = \beta z$ for some $z \in \mathbf{N}(A)$ and $\beta > 0$. Then $u, v, z, \alpha = 1$ and β satisfy the requirements in Lemma 9.4, and so $\langle u, z \rangle \geq 1$.

If A has no parabolic principal submatrices, then $\langle w, w \rangle < 0$ for all $w \in F' \cap (F_2 \cup F_3)$, so $\langle v, v \rangle < 0$, and, by Lemma 9.4 *ipuz* > 1. □

COROLLARY 9.6. *Let A be an almost negative $n \times n$-matrix and let u be a vector in the orthogonal sum $V \oplus \mathbb{R}^k$, such that $\langle u, u \rangle = 1$ and the first n coordinates of u be nonnegative. Then there exists a vector $z \in \mathbf{N}(A) * \mathbb{S}^{k-1}$ with $\langle u, z \rangle \geq 1$. If A has no parabolic principal submatrices and $u \notin \mathbf{N}(A) * \mathbb{S}^{k-1}$, then z can be chosen with $\langle u, z \rangle > 1$.*

PROOF. If $u \in V^{\perp}$, then $u \in \mathbb{S}^{k-1}$ and $z = u$ as required, otherwise Lemma 9.5 applied to $\langle u', u' \rangle^{-1/2} u'$, where u' is the orthogonal projection of u in V, guarantees a vector $z' \in \mathbf{N}(A)$ with $\langle z', u' \rangle \geq 1$ (or > 1), and then $z = (u - u') + \langle u', u' \rangle^{1/2} z'$ is as required. □

LEMMA 9.7. *Let A be an almost negative $n \times n$-matrix and $x, y \in \mathbf{N}(A)$ with $\langle x, y \rangle > -1$. Then $d(x, y) \leq \cos^{-1}\langle x, y \rangle$, where d denotes the intrinsic metric on $\mathbf{N}(A)$. If A has no parabolic principal submatrices of order ≥ 3 and there is no simplex in $\mathbf{N}(A)$ containing both x and y, then $d(x, y) < \cos^{-1}\langle x, y \rangle$.*

PROOF. Given $y \in \mathbf{N}(A)$, define the open set $U \subset \mathbf{N}(A)$ and the function $f : U \to \mathbb{R}$ by

$$\begin{aligned} U &= \{x \in \mathbf{N}(A) | \langle x, y \rangle > -1\} \text{ and} \\ f(x) &= \cos^{-1}\langle x, y \rangle \quad (x \in U). \end{aligned}$$

By Corollary 8.5, f is a well-defined continuous nonnegative function with $f(x) = 0$ if and only if $x = y$. It automatically satisfies condition 3 in Lemma 4.5. We show that f satisfies condition 4.

Let $x \in U$, $x \neq y$. Let $\{v_i | i \in I\}$ be the vertex set of the support B of x in $\mathbf{N}(A)$, $k = \dim B$. By Corollary 8.5, $\langle x, y \rangle < 1$, and so

$$u = \frac{y - \langle x, y \rangle x}{\sqrt{1 - \langle x, y \rangle^2}}$$

is a unit vector in $x^{\perp} = B^{\perp} \oplus \mathbb{R}^k$ with nonnegative coordinates (with respect to the basis $\{\phi_U(u_i) | i \in I^A *\}$) in the $B^{\perp}$-component. Corollary 9.6 applied to the matrix $\mathrm{LK}(I, A)$, which is almost negative by Lemma 8.3, and to the vector u guarantees

$$z \in \mathbf{N}(\mathrm{LK}(I, A)) * \mathbb{S}^{k-1} = \mathrm{LK}(B, \mathbf{N}(A)) * \mathbb{S}^{k-1} = \mathrm{LK}(x, \mathbf{N}(A)) \text{ with } \langle u, z \rangle \geq 1.$$

Now $r(x) = \exp_x(\epsilon z) \in \mathrm{St}(x, \mathbf{N}(A)) \cap U$ is defined for some $\epsilon > 0$, and we have

$$\begin{aligned} d(x, r(x) + f(r(x)) &= \epsilon + \cos^{-1}\langle x \cos\epsilon + z \sin\epsilon, \langle x, y \rangle x + \sqrt{1 - \langle x, y \rangle^2} \rangle u \\ &= \epsilon + \cos^{-1}(\langle x, y \rangle \cos\epsilon + \sqrt{1 - \langle x, y \rangle^2} \langle z, u \rangle \sin\epsilon) \\ &\geq \epsilon + (\cos^{-1}\langle x, y \rangle - \epsilon) \\ &= f(x). \end{aligned}$$

Then Lemma 4.5 implies that $d(x,y) \le f(x) = \cos^{-1}\langle x,y\rangle$.

If A has no parabolic principal submatrices of order ≥ 3, then $\mathrm{LK}(I,A)$ has no parabolic principal submatrices at all, and if x and y are not contained in a common simplex, then $u \notin \mathrm{LK}(x,\mathbf{N}(A)) = \mathbf{N}(\mathrm{LK}(I,A)) * \mathbb{S}^{k-1}$, so Corollary 9.6 gives z with $\langle u,z\rangle \ge 1$, and we have strict inequality in the above calculation, which implies that in this case $d(x,y) < \cos^{-1}\langle x,y\rangle$. $\square$

The following statement is a variation of Lemma 5.4.

LEMMA 9.8. *Let $\mathbf{N} = \mathbf{N}(A)$ be the nerve of an almost negative matrix A and v be a vertex of $\mathbf{N}$. Let $p : [0,\lambda] \to \mathbf{N}$ be the path associated to a local geodesic segment in $\mathbf{N}$. Let P denote the image of p and suppose that $P \cap \mathrm{OST}(v,\mathbf{N}) \ne \emptyset$, $p(0) \notin \mathrm{OST}(v,\mathbf{N})$, $p(\lambda) \notin \mathrm{OST}(v,\mathbf{N})$. Then the cone $\mathbf{C}\mathrm{LK}(v,\mathbf{N})$ (which is contained in $\mathrm{ST}(v,\mathbf{N})$ contains a portion of length π of P.*

PROOF. We proceed by induction on $\dim N$. For $\dim N \le 1$ the lemma is obvious, since all edges of N have length $\ge \pi/2$. Suppose that $\dim N > 1$ and we have proved the lemma for nerves of smaller dimensions.

First, observe that if L is an $\mathbb{S}$-complex and $R \subset L$ is the image of a geodesic segment r of length $\mu < \pi$, then the suspension $\mathbf{S}R \subset \mathbf{S}L$ of R is isometric to an "orange peel" sector of width μ in $\mathbb{S}^2$, in particular, the only pair of points in $\mathbf{S}R$ at a distance $\ge \pi$ is the pair of vertices. Indeed, an allowable chain representing r in L gives $\mathbf{S}R$ an $\mathbb{S}$-complex structure consisting of a finite number of orange peels, successively pasted together.

By Corollary 8.2 the cone $\mathbf{C}\mathrm{LK}(v,\mathbf{N})$ is contained in $\mathrm{ST}(v,\mathbf{N})$, so we can assume that $v \notin P$. Let $pr : (\mathrm{ST}(v,\mathbf{N}) - \{v\}) \to \mathrm{LK}(v,\mathbf{N})$ denote the radial projection from v, then for every $x \in \mathrm{OST}(v,\mathbf{N}) - \{v\}$, $\mathrm{LK}(x,\mathbf{N})$ is isometrically isomorphic to the suspension $\mathbf{S}\mathrm{LK}(pr(x),\mathrm{LK}(v,\mathbf{N}))$, since the differential dpr_x of pr at x takes the orthogonal complement of $\mathrm{KER}(dpr_x)$ in $T_x\mathbf{N}$ isomorphically onto $T_{pr(x)}\mathrm{LK}(v,\mathbf{N})$.

Without loss of generality we can assume that $p(t) \in \mathrm{OST}(v,\mathbf{N})$ for $t \in (0,\lambda)$. If P is represented by some local geodesic segment $C = (x_0,\dots,x_m)$, then $pr(C) = (pr(x_0),\dots,pr(x_m))$ is an allowable chain in $\mathrm{LK}(v,\mathbf{N})$. We show that $pr(C)$ is a local geodesic segment in $\mathrm{LK}(v,\mathbf{N})$. Let $1 \le k \le m-1$, $x = x_k$, let d' and d'' denote the intrinsic metric in $\mathrm{LK}(x,\mathbf{N}) = \mathbf{S}\mathrm{LK}(pr(x),\mathrm{LK}(v,\mathbf{N}))$ and in $\mathrm{LK}(pr(x),\mathrm{LK}(v,\mathbf{N}))$ respectively, and let $v^{\mathrm{in}}, v^{\mathrm{out}} \in \mathrm{LK}(x,\mathbf{N})$ and $u^{\mathrm{in}}, u^{\mathrm{out}} \in \mathrm{LK}(pr(x),\mathrm{LK}(v,\mathbf{N}))$ denote the tangent vectors at x and at $pr(x)$ respectively. Since C is a local geodesic segment in $\mathbf{N}$, $d'(v^{\mathrm{in}},v^{\mathrm{out}}) \ge \pi$ by Lemma 4.1. If $d''(u^{\mathrm{in}},u^{\mathrm{out}}) < \pi$, then the initial observation applied to $L = \mathrm{LK}(pr(x),\mathrm{LK}(v,\mathbf{N}))$ and a geodesic segment from u^{in} to u^{out} shows that v^{in} and v^{out} are the two vertices of the orange peel, which is impossible, since $v \notin R$.

Let $pr' : (\mathrm{ST}(v,\mathbf{N}) - \{v\}) \to (\mathrm{ST}(v,\mathbf{N}) - \mathrm{OST}(v,\mathbf{N}))$ denote the radial projection from v, and let $y_k = pr'(x_k) (k = 0,\dots,m)$. For $1 \le k \le m-1$ let $u_k, v_k, w_k \in \mathrm{LK}(y_k,\mathbf{N})$ be the tangent vectors pointing in the direction of y_{k-1}, v and y_{k+1} respectively, then $C_k = (u_k,v_k,w_k)$ is an allowable chain in $\mathrm{LK}(y_k,\mathbf{N})$. Since $\mathrm{LK}(v_k,\mathrm{LK}(y_k,\mathbf{N}))$ is isometrically isomorphic to $\mathrm{LK}(pr(x_k),\mathrm{LK}(v,\mathbf{N}))$ under a map that takes the tangent vectors of C_k at v_k to the tangent vectors of $pr(C)$ at $pr(x_k)$, it follows that C_k is a local geodesic segment in $\mathrm{LK}(y_k,\mathbf{N})$. Then, from the induction hypothesis applied to a subdivision of $\mathrm{LK}(y_k,\mathbf{N}) * \mathbb{S}^{\dim B-1}$ with a vertex at $\langle \phi_B(v_k),\phi_B(v_k)\rangle^{-1/2}\phi_B(v_k)$, where B is the support of y_k in $\mathbf{N}$, it follows that $\lambda(C_k) \ge \pi$.

The wrapping map f along P described in the proof of Lemma 5.4 takes v to the north pole of $\mathbb{S}^2$, P onto a great circle arc in $\mathbb{S}^2$, and $pr'(P)$ onto a piecewise geodesic broken line in $\mathbb{S}^2$. $f(P)$ and $f(pr'(P))$ enclose a region D of $\mathbb{S}^2$. Interior angles of D at $f(y_k)$ are $\lambda(C_k) \geq \pi$ $(k = 1, \ldots, m-1)$, therefore $f(P)$ cannot be contained in the southern hemisphere. So, the intersection of P and the interior of the cone $\mathbf{C}\mathrm{LK}(v, \mathbf{N})$ is non-empty, and, as in the proof of Lemma 5.4, $f(P)$ must contain an antipodal pair in $\mathbb{S}^2$. $\square$

A subcomplex K of a simplicial M-complex $\mathbf{N}$ is called a *full subcomplex*, if $B \subset K$ whenever B is a simplex of N with all vertices in K. We say that a full subcomplex K is *spanned* by its vertices. If $\mathbf{N}$ is the nerve of an almost negative matrix, tehn every full subcomplex K of $\mathbf{N}$ is the nerve of an almost negative matrix; namely, the nerve of the principal submatrix corresponding to the set of vertices of K.

COROLLARY 9.9. *Let K be a full subcomplex of the nerve $\mathbf{N}(A)$ of an almost negtaive matrix A. If $x, y \in K$ and $d(x, y) < \pi$, then $d_K(x, y) = d(x, y)$, where d and d_K denote the intrinsic metric on $\mathbf{N}(A)$ and K respectively.*

PROOF. By Lemma 9.8, a geodesic segment from x to y in $\mathbf{N}(A)$ must be contained in the complement of $\mathrm{OST}(v, \mathbf{N}(A))$ for every vertex $v \in \mathbf{N}(A) - K$, that is, in K. $\square$

COROLLARY 9.10. *If A is an almost negative matrix, then all simplices in $\mathbf{N}(A)$ are geodesically convex in $\mathbf{N}(A)$. In other words, the restriction of the intrinsic pseudometric to every simplex is the original spherical metric of the simplex.*

LEMMA 9.11. *Let A be an almost negative matrix, and suppose that v is a vertex of $\mathbf{N} = \mathbf{N}(A)$ that is connected to all other vertices of $\mathbf{N}$ by edges of $\mathbf{N}$. Let $x, y \in \mathrm{ST}(v, \mathbf{N}) - \mathrm{OST}(v, \mathbf{N})$ and assume that $d(x, y) < \pi$, where d denotes the intrinsic metric on $\mathbf{N}$. then $d'(x, y) \leq d(x, y)$, where d' denotes the intrinsic metric on the suspension $\mathbf{S}\mathrm{LK}(v, \mathbf{N})$. If A has no parabolic principal submatrices of order ≥ 3 and no simplex of $\mathrm{ST}(v, \mathbf{N})$ contains both x and y, then $d'(x, y) < d(x, y)$.*

PROOF. We proceed by induction on the order n of A. For $n \leq 2$ the statement is obvious. Suppose that $n > 2$ and the lemma is true for matrices of order $< n$.

Case 1. There is a simplex B in $\mathbf{N}$ containing both x and y. Then by Corollary 9.10, $\langle x, y \rangle = \cos d(x, y) > -1$, and Lemma 9.7 applied to the complex $\mathbf{S}\mathrm{LK}(v, N)$ proves the statement.

Case 2. No simplex of $\mathbf{N}$ contains both x and y. Let $p : [0, \lambda] \to \mathbf{N}$ be a geodesic segment in $\mathbf{N}$ from x to y. Without loss of generality we can assume that $p(t) \notin \mathrm{ST}(v, \mathbf{N})$ for $t \in (0, \lambda)$. Let K denote the full subcomplex of $\mathbf{N}$ spanned by the vertex set $\{v\} \cup G \cup H$, where G and H are the sets of vertices of the support in $\mathbf{N}$ of x and y respectively. Now $K \neq \mathbf{N}$, since otherwise $G \cup H$ would span a simplex containing x and y. By Corollary 9.9, $d_K(x, y) = d(x, y)$, and the induction hypothesis applied to K proves the lemma. $\square$

10. Closed geodesics in $\mathbf{N(A)}$

The following proposition is the main result of Chapter 2.

PROPOSITION 10.1. *If A is an almost negative matrix, then the girth of $\mathbf{N}(A)$ is at least 2π.*

Proof. We proceed by induction on the order n of A. If $n \leq 2$, then $\mathbf{g}(\mathbf{N}(A)) = \infty$, so assume that $n > 2$ and $\mathbf{g}(\mathbf{N}(A')) \geq 2\pi$ for all almost negative matrices A' of order $< n$.

Let $Q \subset \mathbf{N}(A)$ be the image of a closed geodesic in $\mathbf{N}(A)$. Let K be the full subcomplex of $\mathbf{N}(A)$ spanned by the vertices v of $\mathbf{N}(A)$ with $Q \cap \mathrm{Ost}(v, \mathbf{N}(A)) \neq \emptyset$. Then $Q \subset K$ and Q is the image of a closed geodesic in K. If $K \neq \mathbf{N}(A)$, then by the induction hypothesis $\mathbf{g}(K) \geq 2\pi$, and so the length of Q is $\geq 2\pi$. Thus, we can assume that $K = \mathbf{N}(A)$.

Case 1. There is a pair of distinct vertices v, w of $\mathbf{N}(A)$ without an edge connecting them. Then $\mathrm{Ost}(v, \mathbf{N}(A)) \cap \mathrm{Ost}(w, \mathbf{N}(A)) = \emptyset$, and the length of Q is $\geq 2\pi$ by Lemma 9.8.

Case 2. There is a vertex v of $\mathbf{N}(A)$ such that $Q \subset \mathrm{St}(v, \mathbf{N}(A))$. By the proof of Lemma 9.8, either $Q \subset \mathrm{lk}(v, \mathbf{N}(A))$, in which case we can apply the induction hypothesis to $\mathrm{lk}(v, \mathbf{N}(A))$, or the segment $P = Q \cap \mathbf{C}\mathrm{lk}(v, \mathbf{N}(A))$ of Q has length π. If the complementary segment $P' =$ closure of $(Q - P)$ were shorter than π, then Lemma 5.4 applied to the cone in $\mathbf{S}\mathrm{lk}(v, \mathbf{N}(A))$ complementary to $\mathbf{C}\mathrm{lk}(v, \mathbf{N}(A))$ would imply that P' is contained in $\mathrm{lk}(v, \mathbf{N}(A))$, a contradiction.

Case 3. Every pair of distinct vertices of $\mathbf{N}(A)$ is connected by an edge in $\mathbf{N}(A)$, and we can choose a vertex v of $\mathbf{N}(A)$ such that Q is not contained in $\mathrm{St}(v, \mathbf{N}(A))$. Now by Lemma 9.8, $R = Q \cap \mathrm{St}(v, \mathbf{N}(A))$ is a segment of Q of length $\geq \pi$, let x and y denote the endpoints of R. Let a and b denote the endpoints of the subsegment $P = R \cap \mathbf{C}\mathrm{lk}(v, \mathbf{N}(A))$ of R. By the proof of Lemma 9.8, the length of P is π. We show that the length of the complementary segment P' is $\geq \pi$. Suppose that the length of P' is $< \pi$, then P' is the image of a geodesic segment in $\mathbf{N}(A)$.

We claim first that $\{x, y\} \neq \{a, b\}$; that is, either $d(v, x) > \pi/2$ or $d(v, y) > \pi/2$. Let K be the smallest full subcomplex of $\mathbf{N}(A)$ containing P', then by Lemma 9.8, K is spanned by the union of the vertex sets of supports of x and y in $\mathbf{N}(A)$. If $d(v, x) = d(v, y) = \pi/2$, then $\langle v, w \rangle = 0$ for all vertices of K, therefore $P' \subset \mathrm{lk}(v, \mathbf{N}(A))$, a contradiction.

Lemma 9.11 applied to $\mathbf{N}(A)$, v, x and y implies that there exists an allowable chain $C = (a, x, \ldots, y, b)$ in $\mathbf{S}\mathrm{lk}(v, \mathbf{N}(A))$ with $\lambda(C) \leq$ length of $P' < \pi$. But then Lemma 5.4 applied to the cone in $\mathbf{S}\mathrm{lk}(v, \mathbf{N}(A))$ from a to b must lie entirely in $\mathrm{lk}(v, \mathbf{N}(A))$, and has length $< \lambda(C) \leq$ length of P', contradicting that P' represents the distance between a and b in $\mathbf{N}(A)$. $\square$

Corollary 10.2. *If A is an almost negative matrix and B is a simplex in $\mathbf{N}(A)$, then $\mathbf{g}(\mathrm{lk}(B, \mathbf{N}(A))) \geq 2\pi$.*

Proof. By Lemma 8.3, $\mathrm{lk}(B, \mathbf{N}(A))$ is the nerve of an almost negative matrix, and Proposition 10.1 applies. $\square$

Lemma 10.3. *Let A be an almost negative matrix. If $\mathbf{g}(\mathbf{N}(A)) = 2\pi$, then A either has a parabolic principal submatrix of order ≥ 3, or a reducible principal submatrix $A_1 \oplus A_2$, where A_i is not positive definite $(i = 1, 2)$.*

Proof. We prove Lemma 10.3 by induction on the order of A.

In Case 1. of the proof of Proposition 10.1 the length of Q is greater than 2π unless there exists a pair of points $x, y \in Q$ with $\{x, y\} = \mathrm{St}(v, \mathbf{N}(A)) \cap \mathrm{St}(w, \mathbf{N}(A)) \cap Q$. Let $H_1 = \{v, w\}$ and let H_2 denote the union of the vertex sets of supports of x and y, and let A_i be the principal submatrix corresponding to $H_i (i = 1, 2)$. Then A_1 and A_2 cannot be positive definite, since no simplex of $\mathbf{N}(A)$ contains both v

and w, or both x and y. Furthermore, $d(v,x) = d(v,y) = d(w,x) = d(w,y) = \pi/2$ implies that the principal submatrix corresponding to $H_1 \cup H_2$ is the direct sum of A_1 and A_2.

In Case 2. of the proof of Proposition 10.1 the length of Q is greater than 2π unless Q is a closed geodesic in the suspension $\mathbf{S}\mathrm{LK}(v, \mathbf{N}(A))$, which, by the induction hypothesis, contradicts Corollary 5.6.

If we assume that A has no parabolic principal submatrices of order ≥ 3, then in Case 3. of the proof of Proposition 10.1 it follows that the length of P' is strictly greater than π. Indeed, if we assume that the length of P' is π, then Lemma 9.11 implies that the distance of a and b in $\mathrm{LK}(v, \mathbf{N}(A))$, therefore in $\mathbf{N}(A)$ is strictly less tha π, a contradiction, since both halves of the closed geodesic Q between a and b have length π. $\square$

COROLLARY 10.4. *Let A be an almost negative matrix and B a simplex in $\mathbf{N}(A)$. If $\mathbf{g}(\mathrm{LK}(B, \mathbf{N}(A))) = 2\pi$, then A either has a parabolic principal submatrix of order ≥ 3, or a reducible principal submatrix $A_1 \oplus A_2$, where A_i is not positive definite* $(i = 1, 2)$.

PROOF. Let $\{v_i | i \in I\}$ be the vertex set of B.

If the principal submatrix of $\mathrm{LK}(I, A)$ corresponding to the index set $I' \subset I^*$ is parabolic, then the index set $I' \cup I$ determines a parabolic principal submatrix of order ≥ 3 of A.

Now observe that for any index set $I' \subset I$ that determines a simplex in $\mathbf{N}(\mathrm{LK}(I, A))$, the inverse of the matrix $\mathrm{LK}(I, A_{I \cup I'})$ is the principal submatrix of the inverse matrix $A_{I \cup I'})^{-1}$ corresponding to I'. Therefore, if $\mathrm{LK}(I, A_{I \cup I'})$ is reducible, say $I' = I'_1 \cup I'_2$, where the principal submatrices of $\mathrm{LK}(I, A_{I \cup I'})$ corresponding to I'_1 and I'_2 are orthogonal, then by Corollary 9.3, $A_{I \cup I'}$ is reducible with some partition $I = I''_1 \cup I''_2$ such that the principal submatrices $A_{I'_1 \cup I''_2}$ and $A_{I''_1 \cup I'_2}$ of A are orthogonal.

If $I'_1, I'_2 \in I^*$ are minimal index sets that determine not positive definite and orthogonal principal submatrices of $\mathrm{LK}(I, A)$, then choose minimal index sets $I''_i \subset I$ such that $I_i = I'_i \cup I''_i$ correspond to not positive definite principal submatrices of A $(i = 1, 2)$. Then the above observation implies that the principal submatrices of A corresponding to I_1 and I_2 are orthogonal. $\square$

CHAPTER 3

The Main Construction

We introduce an $\mathbb{E}$-complex structure on a well-known construction related to Coxeter groups to prove the central result (Theorem 14.1).

11. Coxeter groups

We review some of the well-known facts about finitely generated Coxeter groups. Further details and proofs can be found in [**2**] or [].

Suppose that a group W has a presentation of the form

$$\langle S | (ss')^{m(s,s')} = 1 \rangle,$$

where S is a finite set, and the function $m : S \times S \to \mathbb{N} \cup \{\infty\}$ satisfies the conditions

$$\begin{aligned} m(s,s') &= m(s',s), \\ m(s,s) &= 1, \\ m(s,s') &\geq 2 \text{ if } s \neq s'. \end{aligned}$$

(If $m(s,s') = \infty$, then the corresponding relation is omitted.) Then the natural map $S \to W$ maps S bijectively onto a set of elements of order 2 in W, and, after identifying elements of S with their images, the order of ss' in W is $m(s,s')$. The pair (W,S) is a *Coxeter system*, and W is a *Coxeter group*. The *rank* of (W,S) is the cardinality of S. We say that the Coxeter system (W,S) is *finite* or *infinite* if the group W is finite or infinite respectively. For any subset $T \subset S$, let W_T denote the subgroup generated by T in W, then (W_T, T) is a Coxeter system. A Coxeter system is *irreducible*, if it cannot be written as a product of nontrivial Coxeter systems, where $(W,S) \times (W',S') = (W \times W', S \times \{1\} \cup \{1\} \times S')$. Every Coxeter group[1] can uniquely be written as the product of its nontrivial irreducible subsystems. The presentation of a Coxeter system (W,S) is customarily given by the *Coxeter graph* of (W,S): a graph $G_{(W,S)}$ with one node for each $s \in S$ and one edge between s and s' if $m(s,s') > 2$, labeled by $m(s,s')$ if $m(s,s') > 3$. Then nontrivial irreducible subsystems of (W,S) are in 1-1 correspondence with connected components of $G_{(W,S)}$.

The *nerve* $\mathbf{N}(W,S)$ of a Coxeter system (W,S) is the abstract simplicial complex with vertex set S, and a nonempty subset $T \subset S$ is a simplex in $\mathbf{N}(W,S)$ if and only if the group W_T is finite.

The *cosine matrix* of (W,S) is an $S \times S$-matrix $A = (a_{ss'})$, where

$$a_{ss'} = \left\{ \begin{array}{ll} -\cos \frac{\pi}{m(s,s')} & \text{if } m(s,s') < \infty \\ -1 & \text{otherwise} \end{array} \right\}$$

[1]False as stated: he means Coxeter system

Then A is a normalized almost negative matrix. As in Chapter 2, let V be the real vector space equipped with the bilinear form $\langle , \rangle$ with matrix A in a basis $\{v_s | s \in S\}$. The form $\langle , \rangle$ is positive definite if and only if the group W is finite. Therefore, the nerve $\mathbf{N}(A)$ of the matrix A is simplicially isomorphic to $N(W, S)$. The Coxeter system (W, S) is irreducible if and only if its cosine matrix A is irreducible. (W, S) is called *affine*, if A is parabolic. There is a well-known classification of all finite and affine Coxeter systems, cf. [**2**].

The *canonical representation* $\rho : W \to O(V, \langle , \rangle)$ of W is defined by

$$\rho(s)(x) = x - 2\langle x, v_s \rangle v_s \quad (s \in S, x \in V).$$

Here, for $s \in S$, $\rho(s)$ is the orthogonal reflection across the hyperplane$v_s^\perp$ in V. ρ is a faithful and descrete representation of W. The images of elements of S under the *dual representation* $\rho^* : W \to GL(V^*)$, given by

$$\rho^*(w)(\alpha)(x) = \alpha(\rho(w^{-1})(x)) \quad (w \in W, \alpha \in V^*, x \in V),$$

are linear reflections across the faces of a simplicial cone $C = \cap\{H_s | s \in S\}$, where H_s is the half-space $\{\alpha \in V^* | \alpha(v_s) \geq 0\}$. A theorem of J. Tits says that translates of C under different elements of W have no common interior point.

A theorem of E. Vinberg (valid for any representation of W as a linear reflection gorup, cf. []) says that $D = \cup\{\rho^*(w)(C) | w \in W\}$ is a convex cone in V^* (called the *Vinberg cone* of (W, S)), the interior $\int D$ of D consists precisely of the points of D with finite isotropy subgroups, and W acts properly on $\int D$ with a convex fundamental domain $C^f = C \cap \int D$.

12. Mirror structures and the universal space construction

Following the terminology of [**7**], we say that a *mirror structure* on a Haussdorff topological space X is a family $M = \{X_s | s \in S\}$ of closed subspaces, called *mirrors*, where S is an arbitrary finite index set. For $T \subset S$, put $X_T = \cap\{X_s | s \in T\} \cap X$, and for $x \in X$, put $S(x) = \{s \in S | x \in X_s\}$. Suppose that (W, S) is a Coxeter system. The following construction of the *universal space* $U = U(W, X, M)$ and the *universal action* of W on U was described by E. Vinberg (cf. []), and used extensively by M. Davis (cf. [**6**],[**7**]):

Let $\sim$ denote the equivalence relation defined on $W \times X$ by

$$(w, x) \sim (w', x') \text{ if and only if } x = x' \text{ and } w^{-1}w' \in W_{S(x)}$$

U is defined as the quotient space $(W \times X)/\sim$. For $(w, x) \in W \times X$, let $[w, x]$ denote the image of (w, x) under the canonical projection of $W \times X$ onto U. We identify the closed subspace $\{[1, x] | x \in X\}$ of U with X. The universal action of W on U is defined by $w'[w, x] = [w'w, x]$, then X is a fundamental domain for the universal action. W acts as a reflection group on U in the sense that the fixed point set of each conjugate of each element of S separates U. If X is a CW-complex and all mirrors are subcomplexes, then U has a natural CW-complex structure with X as a subcomplex. The W-space U has the following universality property:

> If Y is any W-space, then any continuous function $f : X \to Y$ satisfying the condition $sf(x) = f(x)(x \in X_s, s \in S)$ uniquely extends to a continuous equivariant map $U \to Y$.

For example, Vinberg showed that, with the notations of Section 11, $\int D = U(W, C^f, M)$ where $M = \{C_x | s \in S\}$ is the mirror structure on C^f defined by $C_s = \{\alpha \in C^f | \alpha(v_s) = 0\}$ (cf. []).

The universal action of W on $U(W,X,M)$ is proper if and only if for any $x \in X$, $S(x)$ is a simplex in $\mathbf{N}(W,S)$ (that is, all isotropy subgroups are finite). Then U/W is homeomorphic to X. A fundamental result of M. Davis (cf. [**6**]) says that the universal space $U(W,X,M)$ is contractible if and only if X is contractible and the subspace X_T fo X is non-empty and acyclic for each simplex T in the nerve $\mathbf{N}(W,S)$.

Now we define the topological W-space $K(W) = K(W,S)$ for any Coxeter system (W,S). In Sections 13 and 14 we shall give $K(W)$ an $\mathbb{E}$-complex structure.

For each $s \in S$, let X_s denote the closed star of the vertex s in the barycentric subdivision $\mathbf{N}' = \mathbf{N}(W,S)'$ of the simplicial complex $\mathbf{N}(W,S)$. Let M be the mirror structure $\{X_s | s \in S\}$ on the cone $\mathbf{CN}'$ on $\mathbf{N}'$. The W-space $K(W)$ is defined as the universal space $U(W, \mathbf{CN}', M)$ with the universal W-action. $K(W)$ is a locally finite simplicial complex and W acts simplicially. If $T \subset S$ and W_T is finite, then X_T is the dual cell in $\mathbf{N}'$ of the simplex T of $\mathbf{N}(W,S)$, therefore by Davis' theorem, $K(W)$ is contractible. W acts properly on $K(W)$, since a collection of closed starts in $\mathbf{N}'$ of vertices of $\mathbf{N}(W,S)$ has a non-empty intersection if and only if these vertices form a simplex in $\mathbf{N}(W,S)$. W acts with compact quotient, since $K(W)/W$ is homeomorphic to $\mathbf{CN}'$.

Proposition 12.1. *$K(W)$ is equivariantly homotopy equivalent to the interior of the Vinberg cone.*

Proof. It suffices to construct a homotopy equivalence between the spaces C^f and $\mathbf{CN}'$ that respect mirrors, since then an application of the universality property gives an equivariant homotopy equivalence.

Vertices $\{T\}$ of $\mathbf{CN}'$ are in $1-1$-correspondence with subsets T of S with W_T finite. Namely, $\{\emptyset\}$ is the cone point in $\mathbf{CN}'$, and for $T \neq \emptyset$, $\{T\}$ is the barycenter of the simplex T on $\mathbf{CN}$. For each vertex $\{T\}$ of $\mathbf{CN}'$, define $\alpha_T \in C^f \subset V^*$ by

$$\alpha_T(v_s) = \left\{ \begin{array}{ll} 0 & \text{if } s \in T \\ \frac{1}{|S-T|} & \text{if } s \in S - T \end{array} \right\}$$

where $|S-T|$ denotes the cardinality of $S-T$. This map $\{\{T\}|T \subset S\} \to C^f$ extends linearly to an embedding $g : \mathbf{CN}' \to C^f$ that respects mirrors.

Let Δ denote the affine simplex

$$\{\alpha \in C | \sum_{s \in S} \alpha(v_s) = 1\}$$

in V^*. The cone $\mathbf{C}\Delta$ of Δ with the origin of V^* as cone point has the natural mirrors $\{\alpha \in \mathbf{C}\Delta | \alpha(v_s) = 0\} (s \in S\}$, and the map $r : C \to \mathbf{C}\Delta$ defined by

$$r(\alpha) = \left\{ \begin{array}{ll} \alpha & \text{if } \sum_{s \in S} \alpha(v_s) \leq 1 \\ \frac{alpha}{\sum_{s \in S} \alpha(v_s)} & \text{otherwise} \end{array} \right\}$$

obviously is a mirror preserving deformation retraction of C onto $\mathbf{C}\Delta$; moreover, r restricts to a mirror preserving deformation retraction of C^f onto $\mathbf{C}\Delta^f = C^f \cap \mathbf{C}\Delta$. Triangulate $\mathbf{C}\Delta$ as the cone $\mathbf{C}\Delta'$ on the barycentric subdivision Δ' of Δ. The simplices $\Delta_{T_0,\ldots,T_k}$ of $\mathbf{C}\Delta'$ are indexed by increasing sequences $T_0, \ldots, T_k (k \geq 0)$ of distinct subsets of S. An element $\alpha \in V^*$ is in the interior of the simplex $\Delta_{T_0,\ldots,T_k}$ if and only if it satisfies the conditions

(1) $\alpha(v_s) = \alpha(v_{s'})$ if $s, s' \in T_{j+1} - T_j$,
(2) $\alpha(v_s) > \alpha(v_{s'}$ if $s' \in T_{j+1} - T_j$ and $s \in T_j - T_{j-1}$, and

(3) $\alpha(v_s) = 0$ if $s \in S - T_k$.

The image $g(\mathbf{CN}')$ of $\mathbf{CN}')$ is the subcomplex of $\mathbf{C}\Delta'$ consisting of the simplices $\Delta_{T_0,\ldots,T_k}$ with W_{S-T_0} finite. We show that $g(\mathbf{CN}')$ is a mirror preserving deformation retract of $\mathbf{C}\Delta^f$. Let α be an arbitrary point in $\mathbf{C}\Delta^f$. Let $\Delta_{T_0,\ldots,T_k}$ be the support of α in the complex $\mathbf{C}\Delta'$. Condition 3 and $\alpha \in C^f$ together imply that W_{S-T_k} is finite. Let $j(\alpha) \le k$ be the smallest index j with W_{S-T_j} finite. Define for $0 \le j \le |S|$ the subset $L_j = \{\alpha \in \mathbf{C}\Delta^f | j(\alpha) \le j\}$ of $\mathbf{C}\Delta^f$, then $g(\mathbf{CN}') = L_0 \subset \cdots \subset L_{|S|} = \mathbf{C}\Delta^f$. For $j > 0$ and $\alpha \in L_j$ with support $\Delta_{T_0,\ldots,T_k}$, define $r_j(\alpha) \in L_{j-1}$ by the formula

$$r_j(\alpha)(v_s) = \left\{ \begin{array}{ll} \alpha(v_s) & \text{if } s \notin T_{j-1} \\ \alpha(v_{s'}) & \text{if } s \in T_{j-1} \end{array} \right\}$$

where $s' \in T_j - T_{j-1}$. (That is, r lowers the coordinates in $T_j - T_{j-1}$ until they coincide with the coordinates in $T_{j+1} - T_j$ (or $S - T_j$ if $j = k$), keeping the other coordinates unchanged.) Obviously r_j is a mirror preserving deformation retraction of L_j onto L_{j-1}. Finally, the composition $r_1 \circ \cdots \circ r_{|S|}$ is a mirror preserving deformation retraction of $\mathbf{C}\Delta^f$ onto $g(\mathbf{CN}')$. $\square$

Remark 12.2. Proposition 12.1 gives a second proof of the contractibility of $K(W)$.

13. Blocks

Throughout this section, let (W, S) be a fixed finite Coxeter system of rank n. Then $\mathbf{N}(W, S)$ is a simplex of dimension $n-1$. We give the cone $\mathbf{CN}' = \mathbf{CN}(W, S)'$ a metric by defining a euclidean cell $B = B(W)$, called the block corresponding to W, and a homeomorphism $f : B \to \mathbf{CN}'$.

Since the bilinear form $\langle , \rangle$ on the space V given by the cosine matrix A of W is positive definite, we can identify V with its dual space V^* by the correspondence $v \leftrightarrow \langle v, \cdot \rangle (v \in V)$. Let $\{u_s | s \in S\}$ be the basis in V dual to the basis $\{v_s | s \in S\}$, that is $\langle u_s, v_s \rangle = \delta_{ss'}$. The matrix of the form $\langle , \rangle$ in this basis is the inverse of A. For any $T \subset S$, the vectors $\phi_T(u_s)(s \in S - T)$, where $\phi_T : V \to \{u_s | s \in T\}^\perp$ is the orthogonal projection, form the basis dual to $\{v_s | s \in S - T\}$, and the link $\mathrm{LK}(S - T, A^{-1})$ is the inverse of the principal submatrix of A corresponding to the index set T.

We give two descriptions of the block B, the first of which is suitable for a modified construction, discussed in Chapter 4, using hyperbolic space instead of euclidean space.

In the first description of B, let M be one of the symbols $\mathbb{E}$ or $\mathbb{H}$, and let ϵ be a fixed positive number, where $\epsilon = 1$ if $M = \mathbb{E}$. Let p be any point in the space M^n, and identify the tangent space $T_p M^n$ with the euclidean vector space $(V, \langle , \rangle)$. The set of vectors $\{u_s | s \in S\}$ is called the *frame* for B at p. For each subset T of S, let C_T denote the image of the convex cone spanned by $\{u_s | s \in T\}$ in V under the exponential map $\exp_p : V \to M^n$. There is a unique point q in the interior of C_S such that the distance of q from $C_{S-\{s\}}$ equals ϵ for all $s \in S$. For each $T \subset S$, drop a perpendicular to C_T from q, and let q_T denote the foot of this perpendicular in C_T. In particular, $q_\emptyset = p$ and $q_S = q$. $B = B_\epsilon$ is defined as the convex hull of the set $\{q_T | T \subset S\}$. Then B is the ocmbinatorial n-cube with vertices q_T, $T \subset S$. We define the homeomorphism $f = f_W : B \to \mathbf{CN}'$ by recursion on n. For $n = 0$,

f is the unique map between one-point spaces. Suppose that $n > 0$ and we have defined f for all finite Coxeter systems of rank $< n$. Let T be a proper subset of S. Then the face B_T of B with vertices $q_{T'}$, $T \subset T' \subset S$, is canonically identified with the block $B(W_{S-T})$, since the plane B_T is the orthogonal complement of the plane of C_T at q_T, and parallel translation along the straight segment from p to q_T takes the set of vectors $\{\phi_T(u_s)|s \in S - T\}$ into the frame for B_T at q_T. Therefore the map $f_{W_{S-T}} : B_T \to \mathbf{CN}(W_{S-T}, S - T')'$ is defined, and, by induction, the maps $f_{W_{S-T}} : B_T \to \mathbf{CN}(W_{S-T}, S - T')' \subset \mathbf{CN}'$ for various non-empty subsets T of S agree on intersections, and define a homeomorphism between the union of faces of B containing q, and the cone of the boundary of the simplex S in $\mathbf{N}$. Let $f : B \to \mathbf{CN}'$ be the conical extension of this map, using p as a cone point in B and the barycenter $\{S\}$ of the simplex S as a cone point in $\mathbf{CN}'$.

The other description of B, valid only in the euclidean case, puts q in the origin. Keeping the usual notations, define

$$B = \{x \in V | \langle x, u_s\rangle \geq 0 \text{ and } \langle x, v_s, \leq\rangle 1 \text{ for all } s \in S\}$$

that is, B is the intersection of the convex cone spanned by the basis $\{u_s|s \in S\}$ with the half-spaces $\langle x, v_s\rangle \leq 1$, $s \in S$. The faces of B are the subsets

$$F_{T,U} = \{x \in B | \langle x, v_s\rangle = 1 \text{ for all } s \in T \text{ and } \langle x, u'_s\rangle = 0 \text{ for all } s' \in S - U\}$$

for $T \subset U \subset S$. The dimension of the face $F_{T,U}$ is the cardinality of $U - T$. In particular, $B = F_{\emptyset,S}$, and the vertices of B are $F_{T,T}$ ($= q_{s-T}$ in the first description of B) for $T \subset S$.

Faces of the form $F_{\emptyset,U}$ and $F_{T,S}$ are called *inside faces* and *outside faces* of B respectively. The link of the origin (the vertex $F_{\emptyset,\emptyset}$) in B is the nerve $\mathbf{N}(A)$ of the cosine matrix A of W, therefore for each $U \subset S$, the link $\textsc{lk}(F_{\emptyset,U}, B)$ of the inside face $F_{\emptyset,U}$ in B is the nerve $\mathbf{N}(\textsc{lk}(U, A)) = \textsc{lk}(\mathbf{N}(A_U, \mathbf{N}(A)))$, where A_U is the principal submatrix of A corresponding to the indices in U. The link of the vertex $F_{S,S}$ of B is the nerve $\mathbf{N}(A^{-1})$ of the inverse of A (that is, the spherical simplex associated to the simplicial cone spanned by the vectors u_s $s \in S$, in V), therefore for each $T \subset S$, the link $\textsc{lk}(F_{T,S}, B)$ fo the outside face $F_{T,S}$ in B is the nerve $\mathbf{N}(\textsc{lk}(S - T, A^{-1}) = \mathbf{N}(A_T^{-1})$.

For any face $F_{T,U}$ of B we have $F_{T,U} = F_{\emptyset,U} \cap F_{T,S}$, and here $F_{\emptyset,U}$ and $F_{T,S}$ are perpendicular along $F_{T,U}$ (that is, for all vectors u tangent to $F_{\emptyset,U}$ and normal to $F_{T,U}$, and all vectors v tangent to $F_{T,S}$ and normal to $F_{T,U}$ at any point in $F_{T,U}$, we have $\langle u, v\rangle = 0$.) Therefore $\textsc{lk}(F_{T,U}, B) = \textsc{lk}(F_{\emptyset,U}, B) * \textsc{lk}(F_{T,S}, B)$, and so $\textsc{lk}(F_{T,U}, B) = \textsc{lk}(\mathbf{N}(A_u), \mathbf{N}(A)) * \mathbf{N}(A_T^{-1})$.

14. The $\mathbb{E}$-complex structure on $\mathbf{K(W)}$

Let (W, S) be a Coxeter system. We define a W-invariant $\mathbb{E}$-complex structure on the W-space $K(W)$ defined in Section 12.

First define an $\mathbb{E}$-complex structure on the fundamental domain $\mathbf{CN}'$ by pasting together the blocks $B(W_T)$ for simplices $T \subset S$ of $\mathbf{N}(W, S)$ along the maps $f_{W_T} : B(W_T) \to \mathbf{CN}(W_T, T)'$ defined in Section 13. Another way to descibe this $\mathbb{E}$-complex structured is as folllows:

Let L be the intersection of the subset (cone complex) $\mathbb{R}\mathbf{N}(A)$ of V with the half-spaces $\langle x, v_s\rangle \leq 1$, $s \in S$, in V, where $\mathbf{N}(A)$ is the nerve of the cosine matrix A of (W, S). L is the union of all blocks $B(W_T)$ for $T \subset S$, and is naturally an $\mathbb{E}$-complex, with the inclusions of blocks $B(W_T)$ into euclidean subspaces of

V as characteristic maps. Cells of L are the faces $F_{T,U}$ described in Section 13, where now $T \subset U$ and W_U finite. Therefore, for any face $F_{T,U}$ of L we have $\text{LK}(F_{T,U}, L) = \text{LK}(\mathbf{N}(A_U), \mathbf{N}(A)) * \mathbf{N}(A_T^{-1})$.

The mirrors on L are the subset $L_s = \{x \in L | \langle x, v_s \rangle = 1\}$ for $s \in S$, each L_s is a subcomplex of L, namely, L_s is the union of outside faces in all blocks containing the vertex v_s $(= F_{\{s\},\{s\}})$. Then the space $K(W)$ has a natural CW-complex structure, and the maps $w^{-1} : wB \to B$ for each cell wB of $K(W)$, where B is a cell in L, are characteristic maps for the natural $\mathbb{E}$-complex structure on $K(W)$. The group W acts by isometric isomorphisms. By Corollary 4.7, $K(W)$ is a geodesic metric space with its intrinsic metric. The stabilizer of a face $F_{T,U}$ is the subgroup generated by $\{s \in S | F_{T,U} \subset L_s\} = T$; therefore, the link of a face $F_{T,U} \subset L$ in $K(W)$ is

$$\begin{aligned}\text{LK}(F_{T,U}, K(W)) &= \text{LK}(F_{T,U}, W_T L) \\ &= W_T(\text{LK}(\mathbf{N}(A_U), \mathbf{N}(A)) * \mathbf{N}(A_T^{-1})) \\ &= \text{LK}(\mathbf{N}(A_U), \mathbf{N}(A)) * W_T(\mathbf{N}(A_T^{-1})) \\ &= \text{LK}(\mathbf{N}(A_U), \mathbf{N}(A)) * \mathbb{S}^{|T|-1}, \qquad (5)\end{aligned}$$

where in the last step we used that in the dual of the canonical representation of W_T, the group W_T acts on the unit sphere of the euclidean space with scalar product given by the matrix A_T^{-1} with the simplex $\mathbf{N}(A_T^{-1})$ as a fundamental domain. If F is any cell of $K(W)$, then $\text{LK}(F, K(W))$ is isometrically isomorphic to a link of the form 5, since a suitable group element $w \in W$ takes F into some $F_{T,U} \subset L$.

Then Corollaries 5.6 and 10.2 imply that the complex $K(W)$ satisfies the link axiom of Section 6. Thus, we have proved the following result:

THEOREM 14.1. *For every Coxeter group W there exists a contractible $\mathbb{E}$-complex $K(W)$ of curvature ≤ 0, on which W acts by isometric iisomorphisms effectively, properly, and with compact quotient.*

REMARK 14.2. In some special cases $K(W)$ is a well-known geometric object. If W is a finite Coxeter group of rank n, then $K(W)$ is isometric to an n-dimensional convex polyhedron. For example, if W is the dihedral group of order $2m$, then $K(W)$ is a regular $2m$-gon with the usual W-action. If W is an affine Coxeter gorup of rank n, then the complex $K(W)$ is a tessellation of the $(n-1)$-dimensional euclidean space and W acts as a symmetry group of this tessellation. If W is the free product of $n \geq 2$ copies of $\mathbb{Z}/2$ (that is, $m(s, s') = \infty$ for $s \neq s'$), then $K(W)$ is an infinite n-regular tree with uniform edge length 2.

15. An application

A theorem of Gromoll and Wolf states that the fundamental group π of a closed riemannian manifold M of non-positive sectional curvature can be solvable only if M is euclidean (cf. [**8**]). Then M is covered by a flat torus, and so π has an abelian subgroup of finite index.

In [**9**], M. Gromov generalized this theorem as follows. Let (X, d) be a convex space in the sense discussed in Section 6, and let Γ be a solvable group of isometries of X that satisfies the following condition:

(6) There is an $\epsilon > 0$, such that $d(x, \gamma(x)) \geq \epsilon$ for all $x \in X$ and $\gamma \in \Gamma - \{1\}$

Then Γ has an abelian subgroup of finite index.

An immediate application of Gromov's theorem to the space $K(W)$ yields the following corollaries:

Corollary 15.1. *If a subgroup G of a Coxeter group W is solvable, then G has an abelian subgroup of finite index.*

Proof. Let Γ be a torsion free subgroup of finite index in W. The existence of such a subgroup is guaranteed by Selberg's Lemma (cf. [**11**]), which states that every finitely generated subgroup of a matrix group is virtually torsion free, and by the canonical representation of W. The $\mathbb{E}$-complex $K(W)$ with its intrinsic metric d is a convex space. It follows from the construction of the universal space that two translates wL and $w'L$ of the fundamental domain L of $K(W)$ have a point x in common if and only if $w^{-1}w'$ is an element of the finite subgroup $W_{S(w^{-1}x)}$. Therefore, all translates of L by elements of Γ are pairwise disjoint. They form a discrete family of compact sets, and $\epsilon = \min\{d(L, wL) | w \in \Gamma - \{1\}\}$ shows that the action of $\Gamma \cap G$ on $K(W)$ satisfies condition 6 in Gromov's theorem. An abelian subgroup of finite index in $\Gamma \cap G$ has finite index in G. $\square$

Corollary 15.2. *With a finite number of exceptions, the irreducible Coxeter groups are not solvable.*

Proof. If W is solvable Coxeter group, then by Corollary 15.1, W is either finite or affine. Finite irreducible or affine Coxeter groups of sufficiently high rank necessarily contain a copy of the alternating group A_5 as a subgroup, so the statement follows from the classification of finite and affine Coxeter groups. $\square$

CHAPTER 4

Hyperbolicity

For a certain type of Coxeter groups, a modified version of the main construction yields a complex of curvature ≤ -1, and leads to a characterization of hyperbolic Coxeter groups (Theorem 17.1).

16. Hyperbolic metric spaces and groups

In [**10**] M. Gromov introduced a very general concept of hyperbolicity for metric spaces and finitely generated groups. We recall the definitions and some of the basic properties.

Let (X, ρ) be a metric space. We say that X is *hyperbolic*, if there exists a constant C, such that for every four points x, y, z, w of X the difference of the two largest of the real numbers $\rho(x,y)+\rho(z,w)$, $\rho(x,z)+\rho(y,w)$ and $\rho(x,w)+\rho(y,z)$ is less than C.

For example, the spaces $\mathbb{H}^n$ are hyperbolic, while $\mathbb{E}^n$ is not hyperbolic for $n \geq 2$. More generally, simply connected complete riemannian manifolds of section curvature $\leq \kappa$, where $\kappa < 0$ are hyperbolic. Even more generally, spaces satisfying the CAT(κ) axiom (see Section 6) with $\kappa < 0$ are hyperbolic.

Two metric spaces (X, ρ) and (Y, σ) are *quasi-isometric*, if there is a relation $\mathcal{R}$ in $X \times Y$ and there are positive constants A, B and C, such that

(1) for every $x \in X$ there are $x' \in X$ and $y' \in Y$ with $\rho(x,x') < A$ and $x'\mathcal{R}y'$,
(2) for every $y \in Y$ there are $x' \in X$ and $y' \in Y$ with $\sigma(y,y') < B$ and $x'\mathcal{R}y'$, and
(3) if $x\mathcal{R}y$ and $x'\mathcal{R}y'$, then
$$\frac{1}{C}\rho(x,x') - B \leq \sigma(y,y') \leq C\rho(x,x') + B$$

An important feature of hyperbolicity is that it is a quasi-isometry invariant property among geodesic metric spaces; that is, if X and Y are quasi-isometric geodesic metric spaces and X is hyperbolic, then Y is hyperbolic.

Let Γ be a group generated by a finite set F. The distance of the elements γ_1 and γ_2 in Γ in the *word metric* on Γ with respect to F is defined as the minimum length of the words in elements of F and their inverses which represent $\gamma_1^{-1}\gamma_2$. The Cayley graph of Γ with respect to F is a graph with elements of Γ as vertices, and with an edge between two distinct vertices γ_1 and γ_2 whenever $\gamma_1^{-1}\gamma_2 \in F$ or $\gamma_2^{-1}\gamma_1 \in F$. By declaring the edges to have length 1, we give the Cayley graph a 1-dimensional $\mathbb{E}$-complex structure, then the word metric on Γ is the restriction of the intrinsic metric on the Cayley graph to the vertex set. Γ acts properly as a group of isometries on the Cayley graph by a natural extension of the action of Γ on itself by left multiplications.

A finitely generated group Γ is called *word hyperbolic*, or simply *hyperbolic*, if Γ equipped with the word metric with respect to some finite set of generators (or equivalently, the Cayley graph of Γ with respect to some finite set of generators) is a hyperbolic metric space. Since different finite generating sets result in quasi-isometric word metrics on Γ and the Cayley graphs are geodesic metric spaces, a hyperbolic group Γ is a hyperbolic metric space with respect to any finite set of generators of Γ.

If a finitely generated group Γ acts properly on a locally simply connected geodesic metric space with compact quotient, then Γ and X are quasic-isometric, and hyperbolicity of one of Γ and X implies hyperbolicity of the other. Many examples of hyperbolic groups are obtained this way: fundamental groups of closed riemannian manifolds of negative sectional curvature, or discrete groups acting properly and with compact quotient on hyperbolic spaces $\mathbb{H}^n$ or on locallly finite $\mathbb{H}$-complexes satisfying the link axiom.

17. Hyperbolic Coexter groups

An application of the main construction described in Chapter 3 is the following characterization of hyperbolicity among Coxeter groups.

THEOREM 17.1. *For every Coxeter system (W, S) the following statements are equivalent:*

(1) *W is hyperbolic.*
(2) *W has no subgroups isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.*
(3) *There is no subset T of S such that (W_T, T) is an affine Coxeter system of rank ≥ 3, and there is no pair of disjoint subsets T_1, T_2 of S such that the subgroups W_{T_1} and W_{T_2} commute and are infinite.*

PROOF. $1 \Rightarrow 2$: A hyperbolic group cannot contain $\mathbb{Z} \oplus \mathbb{Z}$, cf.[**10**].
$2 \Rightarrow 3$: Obvious.
$3 \Rightarrow 1$: For each $T \subset S$ with W_T finite, build the blocks $B(W_T)_\epsilon$ in the hyperbolic space $\mathbb{H}^{|T|}$, and past them together to give $K(W)$ an $\mathbb{H}$-complex structure as described in Sections 13 and 14. As in the euclidean case, $K(W)$ is a geodesic metric space with its intrinsic metric. We show that for sufficiently small ϵ, the $\mathbb{H}$-complex $K(W)$ satisifies the link axiom. Then, using the theorems mentioned in the preceding section, it follows that W is hyperbolic.

Consider the $\mathbb{S}$-complex $\mathbf{N}(A)$, where A is the cosine matrix of (W, S). 3 implies that the matrix $\text{LK}(T, A)$ has no parabolic submatrices with not positive definite factors for any $T \subset S$ with W_T finite. Then by Corollaries 10.2 and 10.4, all girths $\mathbf{g}(\text{LK}(\mathbf{N}(A_T), \mathbf{N}(A)))$ are strictly greater than 2π. Therefore, by Lemma 5.11, there is a $\delta > 1$ such that all the corresponding links have girth $\geq 2\pi$ for any δ-change of the complex $\mathbf{N}(A)$.

Let $B = B(W_T)_\epsilon$ be a block in the fundamental domain L of $K(W)$. Since B is canonically combinatorially isomorphic to its euclidean counterpart, we can keep the notations for faces introduced in Section 13.

If $F_{T',T}$ is an outside face of B, then its link in B is the same as it was in the euclidean block, since the differential of the exponential map at the corner $p \in B$ identifies them. So $\text{LK}(F_{T',T}, B) = \mathbf{N}(A_{T'}^{-1})$.

Figure 1. Two hyperbolic Coxeter groups

If $F_{\emptyset,\emptyset} = q$ is the inside vertex of B, then let D_T be the spherical simplex $\text{LK}(q, B)$. For sufficiently small choice of $\epsilon = \epsilon_T > 0$, D_T is a δ-change of the simplex $\mathbf{N}(A_T)$, since $\exp_p : T_p\mathbb{H}^{|T|} \to \mathbb{H}^{|T|}$ is a near isometry near the origin.

Let ϵ be the smallest of the ϵ_T for all $T \subset S$ with W_T finite, then for each inside face $F_{\emptyset,U}$ of L, the link $\text{LK}(F_{\emptyset,U}, L)$ is a δ-change of $\text{LK}(\mathbf{N}(A_U), \mathbf{N}(A))$, and we have $\mathbf{g}(\text{LK}(F_{\emptyset,U}, L)) \geq 2\pi$.

Finally, if $F_{T,U}$ is any cell in L, then as in Section 14,

$$\begin{aligned} \text{LK}(F_{T,U}, K(W)) &= \text{LK}(F_{\emptyset,U}, L) * W_T\mathbf{N}(A_T^{-1}) \\ &= \text{LK}(F_{\emptyset,U}, L) * \mathbb{S}^{|T|-1}, \end{aligned}$$

and by Corollary 5.6, we have $\mathbf{g}(\text{LK}(F_{T,U}, K(W))) \geq 2\pi$. □

18. Some remarks and questions

It is of special interest to study the Coxeter groups W for which the complex $K(W)$ is a topological manifold. This happens if the nerve $\mathbf{N}(W)$ of W is a triangulation of a sphere. Then the $\mathbb{E}$-complex ($\mathbb{H}$-complex) $K(W)$ is a contractible, piecewise euclidean (piecewise hyperbolic), complete singular manifold of curvature ≤ 0 (≤ -1). Choosing torsion free subgroups Γ of finite index in W (Selberg's Lemma), we obtain closed, aspherical, piecewise euclidean or piecewise hyperbolic singular manifolds $K(W)/\Gamma$ with curvature ≤ 0 or ≤ -1.

For example, the nerve of reflection groups in $\mathbb{E}^n$ or $\mathbb{H}^n$ with bounded fundamental chamber (that is, of so-called crystallographic reflection groups) is automatically homoemorphic to the $(n-1)$-sphere. In case of affine Coxeter groups W, this is obviously the only way for $\mathbf{N}(W)$ to be a sphere. But a hyperbolic Coxeter group W with $\mathbf{N}(W)$ homeomorphic to a sphere need not be a crystallographic group, as the examples given by their Coxeter graphs in Figure 1 show.
In both case $\mathbf{N}(W)$ is a triangulation of the 3-sphere (simplicially isomorphic to the boundary of the 4-dimensional cyclic polytope on 6 and 7 vertices, cf. [**3**]), hyperbolicity of W is easily seen using condition 3 of Theorem 17.1, and the determinant of the cosine matrix of W in both cases is positive, unlike that of a hyperbolic crystallographic reflection group. It may be interesting to find intrinsic conditions on a Coxeter group W which, together with hyperbolicity, ensure that W is a hyperbolic crystallographic reflection group.

The range of hyperbolic crystallographic reflection groups, however, is limited, at least in dimension, if not in complexity, as E. Vinberg showed that they only exist in dimensions less than 30 (cf. []). Vinberg's proof only uses some combinatorial properties of triangulations of spheres (the Dehn-Sommerville relations and some estimates, cf. [**3**]), and the hyperbolicity condition 3 in Theorem 17.1, therefore it implies that piecewise hyperbolic singular manifolds constructed as $K(W)$ with a hyperbolic Coxeter group W can only exist in dimensions less than 30. Moreover, those combinatorial properties are valid for the so-called Cohen-Macaulay complexes (homology manifolds with the homology of a sphere, cf. [], [**5**]), so the proof of Vinberg's theorem implies the non-existence of piecewise hyperbolic homology manifolds constructed as $K(W)$ in dimensions above 29. In the special case of right-angled Coxeter groups W (that is, when $m(s, s') = 2$ or ∞), then the hyperbolicity condition 3 reduces to Siebenmann's no $\square$-condition (cf. [**10**], p. 123, or see below) in the nerve $\mathbf{N}(W)$, Vinberg's proof limits the dimension to 4.

Vinberg's result and some unsuccessful efforts to construct counterexamples suggest the following problem:

> Is there a limit on the virtual cohomological dimension of hyperbolic Coxeter groups?

In the special case of right-angled Coxeter groups, this question is related to the question of finding a limit on the homology of a certain type of simplicial complexes:

> Let K be a simplicial complex subject to the following two conditions:
>
> (1) K is determined by its 1-skeleton. That is, whenever all pairs of elements of a set T of vertices of K form edges in K, the set T forms a simplex in K;
>
> (2) the no $\square$-condition. That is, every 4-circuit in the edge graph of K has at least one of its diagonals as an edge in K.
>
> Is it true that K has trivial homology in dimensions abouve a limit independent of K?

An example with nontrivial homology in the highest dimension ($= 3$) known to us is the boundary complex of the 600-cell, a regular polyhedron in $\mathbb{E}^4$.

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