

The Euler characteristic of a polyhedral product

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Abstract Given a finite simplicial complex L and a collection of pairs of spaces indexed by the vertices of L , one can define the “polyhedral product” of the collection with respect to L . We record a simple formula for its Euler characteristic. In special cases the formula simplifies further to one involving the h -polynomial of L .

Keywords Charney–Davis conjecture · Euler characteristic · Graph product · Polyhedral product

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The purpose of this note is to record a simple formula for the Euler characteristic of the polyhedral product of a collection of pairs of spaces. To define this notion, start with a finite simplicial complex L with vertex set $[m]$, where $[m] := \{1, \dots, m\}$. Let $\mathcal{S}(L)$ be the poset (of vertex sets) of simplices in L , including the empty simplex. Next suppose we are given a family of pairs of finite CW complexes, $(\mathbf{A}, \mathbf{B}) = \{(A_i, B_i)\}_{i \in [m]}$. Denote the product $\prod_{i \in [m]} A_i$ by $\prod \mathbf{A}$. For $\mathbf{x} := (x_i)_{i \in [m]}$, a point in $\prod \mathbf{A}$, put $\text{Supp}(\mathbf{x}) := \{i \in [m] \mid x_i \notin B_i\}$. Define the *polyhedral product*, $\mathcal{Z}_L(\mathbf{A}, \mathbf{B})$, by

$$\mathcal{Z}_L(\mathbf{A}, \mathbf{B}) = \left\{ \mathbf{x} \in \prod \mathbf{A} \mid \text{Supp}(\mathbf{x}) \in \mathcal{S}(L) \right\}.$$

(The terminology comes from [1].) Define $[m]$ -tuples $\mathbf{e}(\mathbf{A})$ and $\mathbf{e}(\mathbf{B})$ by

$$\mathbf{e}(\mathbf{A}) = (\chi(A_i))_{i \in [m]} \quad \text{and} \quad \mathbf{e}(\mathbf{B}) = (\chi(B_i))_{i \in [m]}$$

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Let $\mathcal{P}(I)$ denote the power set of a finite set I . Given an I -tuple $\mathbf{t} = (t_i)_{i \in I}$ and $J \in \mathcal{P}(I)$, define a monomial \mathbf{t}_J by

$$\mathbf{t}_J = \prod_{j \in J} t_j$$

The following is a version of the Binomial Theorem,

$$(\mathbf{s} + \mathbf{t})_I = \sum_{J \in \mathcal{P}(I)} \mathbf{s}_J \mathbf{t}_{I-J}. \tag{1}$$

Theorem

$$\chi(\mathcal{Z}_L(\mathbf{A}, \mathbf{B})) = \sum_{J \in \mathcal{S}(L)} (\mathbf{e}(\mathbf{A}) - \mathbf{e}(\mathbf{B}))_J \mathbf{e}(\mathbf{B})_{[m]-J},$$

Proof For $I \in \mathcal{S}(L)$, the Binomial Theorem (1) gives

$$\chi\left(\prod_{i \in I} A_i\right) = \mathbf{e}(\mathbf{A})_I = ((\mathbf{e}(\mathbf{A}) - \mathbf{e}(\mathbf{B})) + \mathbf{e}(\mathbf{B}))_I = \sum_{J \in \mathcal{P}(I)} (\mathbf{e}(\mathbf{A}) - \mathbf{e}(\mathbf{B}))_J \mathbf{e}(\mathbf{B})_{I-J}.$$

In other words, the contribution to the Euler characteristic from the part of $\mathcal{Z}_L(\mathbf{A}, \mathbf{B})$ corresponding to the open simplex on I is $(\mathbf{e}(\mathbf{A}) - \mathbf{e}(\mathbf{B}))_I \mathbf{e}(\mathbf{B})_{[m]-I}$. The formula follows. □

The f -polynomial of L is the polynomial in indeterminates $\mathbf{t} = (t_i)_{i \in [m]}$ defined by

$$f_L(\mathbf{t}) = \sum_{J \in \mathcal{S}(L)} \mathbf{t}_J.$$

The \hat{h} -polynomial of L is defined by

$$\hat{h}_L(\mathbf{t}) := (\mathbf{1} - \mathbf{t})_{[m]} f_L\left(\frac{\mathbf{t}}{\mathbf{1} - \mathbf{t}}\right).$$

If \mathbf{t} is the constant indeterminate given by $t_i = t$, then f_L is a polynomial in one variable. Denote it $f_L(t)$. For $d = \dim L + 1$, the h -polynomial is defined by $h_L(t) = \hat{h}_L(t)/(1 - t)^{m-d} = (1 - t)^d f_L(\frac{t}{1-t})$.

Corollary 1 *Suppose each B_i is a point $*_i$. Then*

$$\chi(\mathcal{Z}_L(\mathbf{A}, *)) = \sum_{J \in \mathcal{S}(L)} \tilde{\mathbf{e}}(\mathbf{A})_J = f_L(\tilde{\mathbf{e}}(\mathbf{A})),$$

where $\tilde{\mathbf{e}}(\mathbf{A}) := \mathbf{e}(\mathbf{A}) - \mathbf{1} = (\chi(A_i) - 1)_{i \in [m]}$ is the m -tuple of reduced Euler characteristics.

Corollary 2 *For each $i \in [m]$, suppose $B_i = E_i$ a finite set of cardinality $q_i + 1$ and $A_i = \text{Cone } E_i$. Put $(\mathbf{Cone } \mathbf{E}, \mathbf{E}) = \{(\text{Cone } E_i, E_i)\}_{i \in [m]}$. Then $\chi(\mathcal{Z}_L(\mathbf{Cone } \mathbf{E}, \mathbf{E})) = \hat{h}_L(-\mathbf{q})$.*

Proof Any cone is contractible and hence, has Euler characteristic 1. So, by the theorem,

$$\begin{aligned} \chi(\mathcal{Z}_L(\mathbf{Cone } \mathbf{E}, \mathbf{E})) &= \sum_{J \in \mathcal{S}(L)} (-1)^{|J|} \mathbf{q}_J (\mathbf{1} + \mathbf{q})_{[m]-J} \\ &= (\mathbf{1} + \mathbf{q})_{[m]} f_L\left(\frac{-\mathbf{q}}{\mathbf{1} + \mathbf{q}}\right) = \hat{h}_L(-\mathbf{q}). \end{aligned}$$

□

Corollary 3 For each $i \in [m]$, suppose A_i is an odd-dimensional manifold with nonempty boundary B_i . Then, for $d = \dim L + 1$,

$$\chi(\mathcal{Z}_L(\mathbf{A}, \mathbf{B})) = \mathbf{e}(\mathbf{B})_{[m]} \frac{h_L(-1)}{(-2)^d}.$$

Proof By Poincaré duality, $\chi(B_i) = 2\chi(A_i)$. So, $\mathbf{e}(\mathbf{A}) - \mathbf{e}(\mathbf{B}) = -\frac{1}{2}\mathbf{e}(\mathbf{B})$. By the theorem,

$$\begin{aligned} \chi(\mathcal{Z}_L(\mathbf{A}, \mathbf{B})) &= \sum_{J \in \mathcal{S}(L)} (-1/2)^{|J|} \mathbf{e}(\mathbf{B})_{[m]} \\ &= \mathbf{e}(\mathbf{B})_{[m]} f_L(-1/2) = \mathbf{e}(\mathbf{B})_{[m]} \frac{h_L(-1)}{(-2)^d}. \end{aligned}$$

□

Remark Here are some applications of these corollaries to aspherical spaces. The basic fact is proved in [3]: if the following three conditions hold, then $\mathcal{Z}_L(\mathbf{A}, \mathbf{B})$ is aspherical.

- For each $i \in [m]$, A_i is aspherical.
- For each $i \in [m]$, each path component of B_i is aspherical and any such component maps π_1 -injectively into A_i .
- L is a flag complex.

When these conditions hold, $\mathcal{Z}_L(\mathbf{A}, \mathbf{B})$ is the classifying space of a group \mathcal{G} , called the “generalized graph product” of the $\pi_1(A_i)$ (cf. [3]). In the next two paragraphs $\mathbf{G} = (G_i)_{i \in [m]}$ is an m -tuple of discrete groups and L is a flag complex.

- (1) Suppose \mathcal{G} is the graph product of the G_i with respect to the 1-skeleton of L . Then $B\mathcal{G}$ is the polyhedral product of the classifying spaces BG_i of the G_i (cf. [3–5]). Suppose each G_i is type FL (so that its Euler characteristic is defined). Applying Corollary 1 to the case $(\mathbf{A}, *) = \{(BG_i, *)\}_{i \in [m]}$, we get

$$\chi(B\mathcal{G}) = f_L(\tilde{\mathbf{e}}(\mathbf{G})).$$

- (2) Suppose each G_i is finite of order $q_i + 1$. Let \mathcal{G}_0 be the kernel of the natural epimorphism $\mathcal{G} \rightarrow \prod_{i \in [m]} G_i$. Then $B\mathcal{G}_0 = \mathcal{Z}_L(\mathbf{Cone} \mathbf{E}, \mathbf{E})$, where $E_i = G_i$. Applying Corollary 2, we get the following formula for the rational Euler characteristic of \mathcal{G} (cf. [2, p. 308]),

$$\chi(\mathcal{G}) = \frac{\chi(B\mathcal{G}_0)}{(\mathbf{1} + \mathbf{q})_{[m]}} = \frac{\hat{h}_L(-\mathbf{q})}{(\mathbf{1} + \mathbf{q})_{[m]}}.$$

- (3) The Euler Characteristic Conjecture asserts that if N^{2k} is a closed aspherical $2k$ -manifold, then $(-1)^k \chi(N^{2k}) \geq 0$ (cf. [2, p. 310]). The Charney–Davis Conjecture (cf. [2, p. 313]) asserts that if L is a flag triangulation of a $(2c - 1)$ -sphere or even a “generalized homology sphere” (as defined in [2, p. 192]), then $(-1)^c h_L(-1) \geq 0$. The point here is that if the Charney–Davis Conjecture is true, then one cannot find a counterexample to the Euler Characteristic Conjecture by using the construction in Corollary 3. Indeed, suppose $A_i = M_i$, a $(2k_i + 1)$ -manifold with (nonempty) boundary and $B_i = \partial M_i$. Write $(\mathbf{M}, \partial \mathbf{M})$ for $\{(M_i, \partial M_i)\}_{i \in [m]}$. By Corollary 3,

$$\chi(\mathcal{Z}_L(\mathbf{M}, \partial \mathbf{M})) = \mathbf{e}(\partial \mathbf{M})_{[m]} \frac{h_L(-1)}{(-2)^d}. \tag{2}$$

If L is a triangulation of a $(d - 1)$ -sphere (or generalized homology sphere and at least one k_i is positive), then $\mathcal{Z}_L(\mathbf{M}, \partial\mathbf{M})$ is a closed manifold of dimension

$$d + \sum_{i=1}^m 2k_i.$$

This is an even integer if and only if d is even; so, let us assume $d = 2c$. Suppose, in addition, $(\mathbf{M}, \partial\mathbf{M})$ and L satisfy the three conditions at the beginning of this remark, so that $\mathcal{Z}_L(\mathbf{M}, \partial\mathbf{M})$ is a closed aspherical manifold. Suppose that each ∂M_i satisfies the Euler Characteristic Conjecture, i.e., $(-1)^{k_i} \chi(\partial M_i) \geq 0$. So, the sign of $\mathbf{e}(\partial\mathbf{M})_{[m]}$ is $(-1)^{\sum k_i}$. The Euler Characteristic Conjecture for $\mathcal{Z}_L(\mathbf{M}, \partial\mathbf{M})$ is

$$(-1)^{c+\sum k_i} \chi(\mathcal{Z}_L(\mathbf{M}, \partial\mathbf{M})) \geq 0.$$

By (2), the sign of the left hand side is the sign of $(-1)^c h_L(-1)$. So, the Euler Characteristic Conjecture for $\mathcal{Z}_L(\mathbf{M}, \partial\mathbf{M})$ is equivalent to $(-1)^c h_L(-1) \geq 0$, which is the Charney–Davis Conjecture.

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