Topology of boundaries of (hyperbolic) groups

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I. CAT(0)-spaces and polyhedra

Roughly, a space which is "nonpositively curved" and simply connected.

C = "Comparison" or "Cartan"

A = "Aleksandrov"

T = "Toponogov"

Some definitions. Let (X, d) be a metric space. A path $c : [a, b] \to X$ is a *geodesic* (or a *geodesic segment*) if $d(c(s), c(t)) = |s - t|$ for all $s, t \in [a, b]$.

A *geodesic ray* $c : [0, \infty) \to X$ is similarly defined. (X, d) is a *geodesic space* if any two points can be connected by a geodesic segment.

The CAT(κ)-**inequality.** For $\kappa \in \mathbb{R}$, $\mathbb{X}^2_\kappa$ is the simply connected, complete, Riemannian 2-manifold of constant curvature κ:

- $\mathbb{X}_0^2$ is the Euclidean plane $\mathbb{E}^2$.

- If $\kappa > 0$, then $\mathbb{X}_\kappa^2 = \mathbb{S}^2$ with its metric rescaled so that its curvature is κ (i.e., it is the sphere of radius $1/\sqrt{\kappa}$).

- If $\kappa < 0$, then $\mathbb{X}_\kappa^2 = \mathbb{H}^2$, the hyperbolic plane, with its metric rescaled.

A *triangle* T in X is a configuration of three geodesic segments (the "edges") connecting three points (the "vertices") in pairs. A *comparison triangle* for T is a triangle T^* in $\mathbb{X}^2_\kappa$ with the same edge lengths.

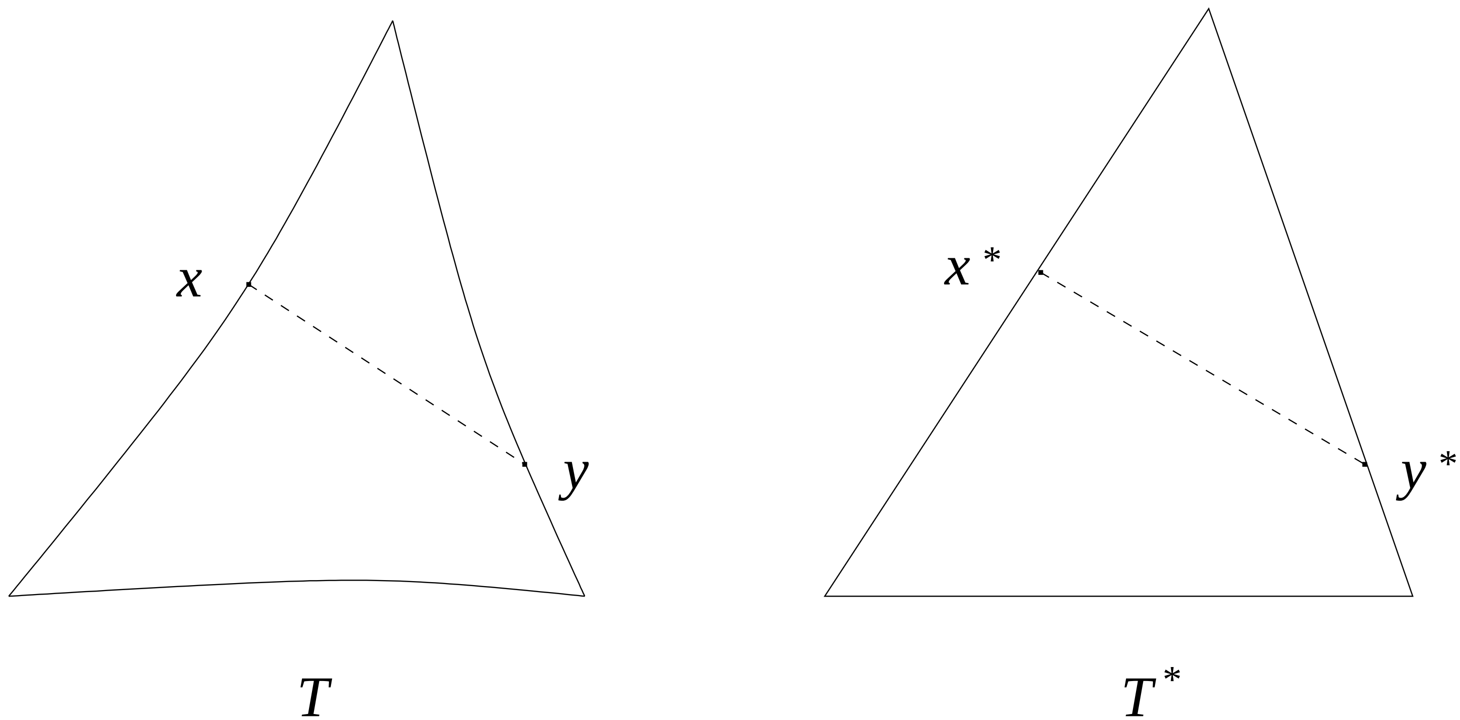
A metric space X *satisfies* CAT(κ) (or is a CAT(κ)-*space*) if the following two conditions hold:

- If $\kappa \leq 0$, then X is a geodesic space, while if $\kappa > 0$, it is required there be a geodesic segment between any two points $< \pi/\sqrt{\kappa}$ apart.

- (*The* CAT(κ) *inequality*). For any triangle T (with $l(T) < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$) and any two points $x, y \in T$, we have

$$d(x,y) \le d^*(x^*, y^*),$$

where x^*, y^* are the corresponding points in the comparison triangle T^* and d^* is distance in $\mathbb{X}^2_\kappa$.



Definition. A metric space X *has curvature* $\leq \kappa$ if the CAT(κ) inequality holds locally.

Observations.

- CAT(0) $\Longrightarrow$ contractible.

- curvature ≤ 0 $\Longrightarrow$ aspherical.

- Γ acts properly and cocompactly on a CAT(-1)-space $\Longrightarrow$ Γ word hyperbolic.

CAT(0)-polyhedra. Suppose X is a finite dimensional cell complex. Give it a "piecewise Euclidean metric" by declaring each cell to be a convex cell in Euclidean space and then measure the length of paths using Euclidean arc length. For example, X might be a cubical cell cx with each n-cell a regular Euclidean n-cube of edge length one. (To avoid more hypotheses assume Isom(X) acts cocompactly.) "Piecewise spherical" and "piecewise hyperbolic" metrics are defined similarly.

If v is a vertex of a convex cell, its *link* is the $\{$inward-pointing directions at $v\}$. The *link*, $L(v)$, of a vertex in X is the union of its links in the cells which contain it. $L(v)$ is naturally a piecewise spherical cell cx.

Theorem. (Gromov, 1987). *A piecewise constant curvature polyhedron has curvature $\leq \kappa$ iff the link of each vertex is* CAT(1).

Theorem. (Gromov). *A piecewise Euclidean cubical cell cx X has curvature ≤ 0 iff the link of each vertex is a flag cx.*.

Definition. A simplicial cx L is a *flag cx* iff every finite set of vertices which are pairwise connected by edges spans a simplex of L.

Remark. The barycentric subdivision of any cell cx is a flag cx. So, the condition that a polyhedron L be a flag cx places no restriction on its topology.

II. The visual boundary of a $\mathrm{CAT}(0)$-space

Idea: adjoin a space ∂X of "ideal points" to a complete CAT(0)-space X obtaining $\overline{X} = X \cup \partial X$. When X is locally compact, $\overline{X}$ will be a compactification of X.

Fix a base point $x_0 \in X$. Rough idea: $\overline{X}$ is formed by adding an "endpoint" $c(\infty)$ to each geodesic ray $c : [0, \infty) \to X$, which begins at x_0. ∂X is the set of such endpoints.

$\overline{X}$ has the "inverse limit topology." Consider the system of closed balls centered at x_0, $\{\overline{B}(x_0, r)\}_{r \in [0,\infty)}$. For each $s > r$, there is a retraction $p_{s,r} : \overline{B}(x_0, s) \to \overline{B}(x_0, r)$, defined as the "nearest point projection." In other words, if c is a geodesic segment starting from x_0, then $p_{s,r}$ takes $c(t)$ to itself when $0 \leq t \leq r$ and to $c(r)$ when $r < t \leq s$.

$$\overline{X} := \varprojlim \overline{B}(x_0, r).$$

$$X \subset \overline{X} \quad \text{and} \quad \partial X := \overline{X} - X.$$

Example. If $X = \mathbb{E}^n$ or $\mathbb{H}^n$, then $(\overline{X}, \partial X) = (D^n, S^{n-1})$. Same is true for X the universal cover of any nonpositively curved, complete Riemannian n-mfld.

Example. If X is a regular tree (valence > 2), then ∂X is a Cantor set.

Example. If X is the universal cover of a compact hyperbolic

3-mfld with totally geodesic bdry, then ∂X is a Sierpinski curve.

The definition of ∂X can be made to be independent of the choice of basepoint x_0. Define two geodesic rays (with different initial points) to be *parallel* (or “asymptotic”) if they remain a bounded distance apart. We could have defined

$$\partial X = \{\text{parallel classes of geodesic rays}\}.$$

Problem. ∂X is not a quasi-isometry invariant. So, even if $\Gamma \subset \mathrm{Isom}(X)$ acts cocompactly, ∂X is not an invariant of Γ.

However, if X is CAT(-1) so that (Γ is word hyperbolic), then it is: $\partial X = \partial\Gamma$, where $\partial\Gamma$ is defined below.

Boundary of a word hyperbolic group Γ. Here is one possible definition. Let Ω be its Cayley graph. Then

$$\partial\Gamma := \{\text{parallel classes of geodesic rays in } \Omega\}.$$

***Z*-sets**. A closed subset Y of a compact ANR $\overline{X}$ is a *Z-set* if for every open subset $U \subset \overline{X}$, the inclusion $U - Y \hookrightarrow U$ is a homotopy equivalence. Standard example: $\overline{X}$ is a compact manifold with boundary and Y is a closed subset of its boundary.

Theorem. *Suppose X is a complete, locally compact,* CAT(0)-*space. Then ∂X is a Z-set in $\overline{X}$.*

Theorem. (Bestvina–Mess, 1991). *If Γ is word hyperbolic, then there is a Z-set compactification of its Rips complex X with*

$\overline{X} - X = \partial\Gamma.$

So, if Γ is word hyperbolic or if it acts cocompactly on a CAT(0)-space X, then

$$H^*_c(X) \cong H^*(\overline{X}, \partial X) \cong \check{H}^{*-1}(\partial X) \quad (= \check{H}^{*-1}(\partial\Gamma) \ ),$$

where $\check{H}^*(\ )$ means reduced Cěch-cohomology.

CAT(0)-polyhedra with isolated PL singularities. Let X be a CAT(0) or CAT(-1)-polyhedron.

Theorem. (D. Stone, 1976). *If X is a PL n-manifold, then $(\overline{X}, \partial X) \cong (D^n, S^{n-1})$.*

Suppose that the link of each vertex is a PL $(n-1)$-manifold. Choose a base point $x_0 \notin \mathrm{Vert}(X)$. In the next theorem we consider the inverse system of metric spheres $\{\partial B(x_0, r)\}_{r \in [0,\infty)}$ centered at x_0.

Theorem. (D. -Januszkiewicz, 1991). *Suppose* $v_1, \ldots, v_m$ *are the vertices of* X *which lie in* $B(x_0, r)$. *Then*

(i) $\partial B(x_0, r) = L(v_1) \sharp \cdots \sharp L(v_m)$, *where* $\sharp$ *means connected sum.*

(ii) *The inverse system of metric spheres is equivalent to the inverse system* $\{L(v_1) \sharp \cdots \sharp L(v_m)\}$, *where the bonding maps are the obvious ones. This gives* $\partial X = \varprojlim (L(v_1) \sharp \cdots \sharp L(v_m))$.

Key point in the proof. The link L of any vertex is piecewise spherical and CAT(1). A metric ball of radius $< \pi$ in such an L is contractible. If L is a PL mfld, then such a metric ball is a PL disk. $\square$

Later I will explain how to construct a nonpositively curved, cubical cell complex Y such that the link of any vertex is any given finite simplicial complex. The construction can be modified to make the curvature ≤ -1.

For example, we can find such a Y with the link of each vertex $\cong \mathbb{R}P^2$. Taking $\Gamma = \pi_1(Y)$ and $X = \widetilde{Y}$, we get:

Theorem. (Bestvina–Mess). *There are torsion-free, word hyperbolic groups* Γ *with* $\mathrm{cd}_{\mathbb{Z}}(\Gamma) = 3$ *and* $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = 2$*.*

Proof. $\partial\Gamma = \partial X$ is the inverse limit of nonorientable surfaces of increasing genus (a "Pontrjagin surface"). So, $\check{H}^2(\partial\Gamma;\mathbb{Z}) = \mathbb{Z}/2$ and $\check{H}^2(\partial\Gamma;\mathbb{Q}) = 0$. Then use the facts that $H^*(\Gamma;\mathbb{Z}\Gamma) = H^*_c(X) = \check{H}^{*-1}(\partial\Gamma)$ and $\operatorname{cd}(\Gamma) = \max\{n \mid H^n(\Gamma;\mathbb{Z}\Gamma) \neq 0\}$. □

Question. *Do there exist torsion free groups* Γ *with*

$\operatorname{cd}_{\mathbb{Q}}(\Gamma)/\operatorname{cd}_{\mathbb{Z}}(\Gamma) < 2/3$?

Next take Y so that the link of each vertex is a PL homology sphere M with nontrivial fundamental group π. Put $\Gamma = \pi_1(Y)$.

Theorem. *There are word hyperbolic groups* Γ *such that 1)* Γ *is the fundamental group of an aspherical* n*-manifold,* $n \geq 4$*, and 2)* Γ *is not simply connected at* ∞*. Moreover,* $\partial\Gamma$ *is a homology mfld with the same homology as* S^{n-1} *and it is not locally simply connected.*

Proof. Y is a homology mfld, homotopy equivalent to a mfld.

Since $\pi_1(\partial B(x_0, s)) \to \pi_1(\partial B(x_0, r))$ is onto, $\lim^1\{\pi_1(\partial B(x_0, r)\} = 0$. Since $\partial B(x_o, r)$ is a connected sum of copies of M, its π_1 is a free product of copies of π. It follows that

$$\pi_1^\infty := \varprojlim\{\pi * \cdots * \pi\} \neq 0.$$

So, $\overline{X}$ is not simply connected at ∞. $\square$

Corollary. *For $n \geq 4$, $\exists$ aspherical n-mflds not covered by $\mathbb{R}^n$.*

Remark. Fischer has studied boundaries of some of these examples with links PL manifolds. ∂X is often "homogeneous" in the sense that its homeomorphism group acts transitively.

References.

Bridson–Haefliger, *Metric Spaces of Non-positive Curvature*, 1999.

Bestvina–Mess, *The boundary of a negatively curved group*, JAMS, 1991.

Davis–Januszkiewicz, *Hyperbolization of polyhedra*, JDG, 1991.

Fischer, *Boundaries of right-angled Coxeter groups with manifold nerves*, Topology, 2003.

III. Constructing examples

A. Cubical complexes with prescribed links. Suppose L is a flag cx with vertex set S. We will construct a (nonpositively curved) cubical cx, Y_L, s.t. the link of each vertex is L. It is a subcomplex of the cube $[-1,1]^S$. For each simplex σ in L (including the empty simplex), let $\square^\sigma := [-1,1]^{\mathrm{Vert}(\sigma)}$. Put

$$Y_L := \bigcup_{\sigma \subset L} \text{faces parallel to } \square^\sigma.$$

B. Strict hyperbolization. Given a cubical cx Y, R. Charney and I showed how to convert it into a piecewise hyperbolic cx, $h(Y)$, s.t. the link of each vertex is either one of the original links in Y or a round sphere. The key idea was the following.

Lemma. (Charney–D., 1995). *For each n, $\exists$ a compact hyperbolic mfld with corners M^n s.t.*

(i) each stratum of M^n is totally geodesic and

(ii) The poset of nonempty intersections of strata is isomorphic

to the poset of faces of an n-cube.

To construct $h(Y)$ replace each n-cube of Y by a copy of M^n.

Corollary. *Given a flag cx L (with its natural piecewise spherical structure), $\exists$ a piecewise hyperbolic, finite cx Z s.t. the link of each vertex is L (or a round sphere). Hence, Z has curvature ≤ -1 and $\pi_1(Z)$ is word hyperbolic.*

More references

Charney–Davis, *Strict hyperbolization*, Topology, 1995.

Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Annals, 1983.