Generalized moment-angle complexes, graph products of groups and related constructions

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November 9, 2010

Data

- A simplicial complex L with vertex set I.
  (Apology: The letter K will be used for something else.)
- A family of pairs of spaces $(\mathbf{A}, \mathbf{B}) = \{(A_i, B_i)\}_{i \in I}$, indexed by I.

Notation

- $\mathcal{S}(L) := \{\text{vertex sets of simplices in } L\}$ (including $\emptyset$)
- For $\mathbf{x} := (x_i)_{i \in I}$, a point in $\prod_{i \in I} A_i$, put

$$\mathrm{Supp}(\mathbf{x}) := \{i \in I \mid x_i \in A_i - B_i\}.$$

Definition (cf. Denham - Suciu)

The *polyhedral product* $Z_L(\mathbf{A},\mathbf{B})$ is the subset of $\prod_{i\in I} A_i$ consisting of those $\mathbf{x}$ such that $\operatorname{Supp}(\mathbf{x}) \in \mathcal{S}(L)$.
$Z_L(\mathbf{A},\mathbf{B})$ is also called the *generalized moment angle complex*.

Alternate definition

For each $J \in \mathcal{S}(L)$, put

$$Z_J(\mathbf{A},\mathbf{B}) := \prod_{i\in J} A_i \times \prod_{i\in I-J} B_i \quad \text{and}$$

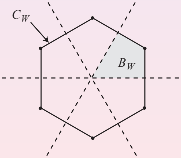
$$Z_L(\mathbf{A},\mathbf{B}) = \bigcup_{J\in\mathcal{S}(L)} Z_J(\mathbf{A},\mathbf{B}).$$

If all (A_i, B_i) are the same, say (A, B), then we omit the boldface and write $Z_L(A, B)$ instead of $Z_L(\mathbf{A}, \mathbf{B})$.

Example

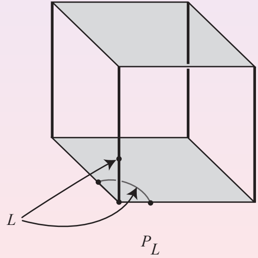
- Suppose $(A, B) = ([0, 1], 1)$.
- Put $K(L) := Z_L([0, 1], 1)$ (and call it a *chamber*).
- $K(L)$ is a subcomplex of the cube, $\prod_{i \in I} [0, 1]$.

$K(L)$ is homeo to the cone on L (the empty set provides the cone point). If L is the boundary complex of a simplicial polytope, then $K(L)$ can be identified with the dual polytope.



Example

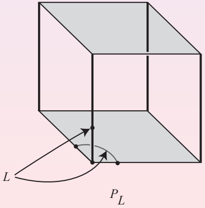
- $(A, B) = (D^1, S^0)$ $(= ([-1, 1], \{\pm 1\}))$.
- $\mathbf{C}_2$ $(= \{\pm 1\})$ is the cyclic group of order 2. It acts on $[-1, 1]$. Hence, $(\mathbf{C}_2)^I \curvearrowright [-1, 1]^I$.
- $Z_L(D^1, S^0)$ is a $(\mathbf{C}_2)^I$-stable subspace.

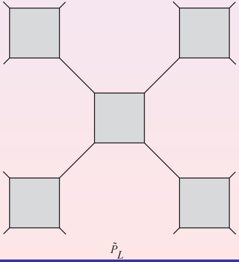


Remarks

- The space $Z_L(D^1, S^0)$ is generally not simply connected.
- In fact, the group of all lifts of the $(\mathbf{C}_2)^I$-action to the universal cover is the right-angled Coxeter gp (or RACG), W, corresponding to the 1-skeleton of L. Moreover,

$$1 \to \pi_1(Z_L(D^1, S^0)) \to W \to (\mathbf{C}_2)^I \to 1$$





Classical moment-angle complex

Example

- $(A, B) = (D^2, S^1)$.
- The gp S^1 $(= SO(2))$ acts on D^2.
- Put $m = \text{Card}(I)$, $T^m = (S^1)^I$. Then $T^m \curvearrowright Z_L(D^2, S^1)$.

Remarks

- $K(L)$ $(= Z_L([0,1], 1)$ is the orbit space of $(\mathbf{C}_2)^m$ on $Z_L(D^1, S^0)$, as well as, the orbit space of T^m on $Z_L(D^2, S^1)$.
- If L is a triangulation of S^{n-1}, then $K(L)$ is an n-disk, $Z_L(D^1, S^0)$ is a n-mfld and $Z_L(D^2, S^1)$ is an $(n+m)$-mfld.

Toric manifolds (or “quasi-toric mflds”)

- Let L be a triangulation of S^{n-1} (eg L could be ∂(simplicial polytope) in which case $K(L)$ is dual polytope.
- Suppose $\exists$ epimorphism $\lambda : T^m \to T^n$ s.t. the kernel N acts freely on $Z_L = Z_L(D^2, S^1)$. Then

$$M^{2n} := Z_L/N$$

  is called the *toric mfld* associated to L and λ.
- $T^n \curvearrowright M^{2n}$ and $M^{2n}/T^n = K(L)$.
- This is generalization of Delzant’s construction of Hamiltonian T^n-action on symplectic mfld where $K(L) \subset \mathbb{R}^n$ is simple convex polytope with edges parallel to rational vectors and where $\lambda = (\lambda_1, \ldots, \lambda_m)$ is defined by using normals to the facets.

Small covers

- Similarly, if $\exists$ epimorphism $\lambda : (\mathbf{C}_2)^m \to (\mathbf{C}_2)^n$ s.t. the kernel N acts freely on $Z_L(D^1, S^0)$, then $M^n := Z_L(D^1, S^0)/N$ is called the associated *small cover* of $K(L)$ (thought of as an orbifold).
- $(\mathbf{C}_2)^n \curvearrowright M^n$ and $M^n/(\mathbf{C}_2)^n = K(L)$.
- Later we will see that $\pi_1(M^n)$ is the kernel of $W_L \to (\mathbf{C}_2)^m$ where W_L is the RACG associated to L^1. Moreover, if L is a flag cx, then M^n is a $K(\pi, 1)$.

DJ - space

$ES^1 \to BS^1 = \mathbb{C}P^\infty$ and $E\mathbf{C}_2 \to B\mathbf{C}_2 = \mathbb{R}P^\infty$ are the universal S^1- and $\mathbf{C}_2$-bundles, respectively. Similarly, $ET^m \to BT^m = (\mathbb{C}P^\infty)^m$ and $E(\mathbf{C}_2)^m \to B(\mathbf{C}_2)^m = (\mathbb{R}P^\infty)^m$.

Definition

The Borel construction on $Z_L(D^2, S^1)$ is *Davis-Januszkiewicz space*, $DJ(L)$, i.e.,

$$DJ(L) := Z_L(D^2, S^1) \times_{T^m} ET^m.$$

Similiarly, *real DJ space* is

$$\mathbb{R}DJ(L) := Z_L(D^1, S^0) \times_{(\mathbf{C}_2)^m} E(\mathbf{C}_2)^m$$

Let $D(\xi)$ be canonical D^2-bundle over BS^1 and $S(\xi)$ $(= ES^1)$ the S^1-bundle. So, $D(\xi) \sim BS^1$ and $S(\xi) \sim *$.

Alternate definition of DJ - space

$$DJ(L) = Z_L(D(\xi), S(\xi)) \sim Z_L(BS^1, *).$$

Similarly,

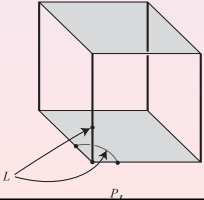
$$\mathbb{R}DJ(L) \sim Z_L(B\mathbf{C}_2, *)$$

RAAG s

Recall $Z_L(A, B) \subset \prod_I A$.

Example

- $(A, B) = (S^1, *)$
- Then $Z_L(S^1, *)$ is a subcx of the torus T^m, where $m = \text{Card}(I)$. It is a union of subtori determined by L.
- The fundamental group of $Z_L(S^1, *)$ is the right-angled Artin group ($=$ RAAG) determined by the graph L^1



Identify opposite faces to get $T^2 \vee T^1$.

The face ring (or Stanley-Reisner ring)

- L a simplicial cx with vertex set I.
- R a commutative ring, $\mathbf{x} = (x_i)_{i \in I}$, and $R[\mathbf{x}]$ is the polynomial algebra.
- The *face ring* $R[L]$ is the quotient of $R[\mathbf{x}]$ by the ideal $\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by all square free monomials of the form $x_{i_1} \cdots x_{i_k}$, with $\{i_1, \ldots i_m\} \notin \mathcal{S}(L)$.

Recall $DJ(L) = Z_L(BS^1, *)$ and $H^*(BT^m) \cong \mathbb{Z}[\mathbf{x}]$.

Theorem

$$H^*(DJ(L)) \cong \mathbb{Z}[L] := \mathbb{Z}[\mathbf{x}]/\mathcal{I}$$

Recall $H^*(T^m) = \bigwedge[\mathbf{x}]$ (the exterior algebra).

Theorem

$H^*(Z_L(S^1, *)) \cong \bigwedge[L] := \bigwedge[\mathbf{x}]/\mathcal{I}$, the "exterior face ring".

Theorem (Bahri, Bendersky, Cohen, Gitler)

Suppose that for each (A_i, B_i), the space B_i is contractible. Let k be a field. Then

$$H^*(Z_L(\mathbf{A}, \mathbf{B}); k) \cong \bigotimes H^*(A_i; k) \,/\, \mathcal{I}.$$

Data

- A simplicial graph L^1 with vertex set I.
- A family of discrete gps $\mathbf{G} = \{G_i\}_{i \in I}$

Definition

The *graph product* of the G_i is the group Γ formed quotienting the free product of the G_i by the normal subgroup generated by all commutators of the form $[g_i, g_j]$ where $\{i, j\} \in \mathrm{Edge}(L^1)$, $g_i \in G_i$ and $g_j \in G_j$.

Example

- If all $G_i = \mathbf{C}_2$, then Γ is the RACG determined by L^1.
- If all $G_i = \mathbb{Z}$, then Γ is the RAAG determined by L^1.

Relative graph products

More data

For each $i \in I$, besides G_i, suppose given a G_i-set E_i.
Put $(\mathrm{Cone}\,\mathbf{E}, \mathbf{E}) := \{(\mathrm{Cone}\,E_i, E_i)\}_{i \in I}$.

- Form the polyhedral product $Z_L(\mathrm{Cone}\,\mathbf{E}, \mathbf{E})$. It is not simply connected. Let

$$\widetilde{Z}_L(\mathrm{Cone}\,\mathbf{E}, \mathbf{E}) := \text{ the univ cover of } Z_L(\mathrm{Cone}\,\mathbf{E}, \mathbf{E}).$$

- $G = \prod_{i \in I} G_i \curvearrowright Z_L(\mathrm{Cone}\,\mathbf{E}, \mathbf{E})$. Let Γ be the gp of all lifts of G-action to $\widetilde{Z}_L(\mathrm{Cone}\,\mathbf{E}, \mathbf{E})$. Γ is the *graph product of the G_i relative to the E_i*. (Only the 1-skeleton, L^1, matters in this defn.) (This defn needs to be tweaked if G does not act effectively on $\prod E_i$.)

Example

If each $G_i = \mathbf{C}_2$, then $Z_L(\mathrm{Cone}\,\mathbf{C}_2, \mathbf{C}_2)$ is the space $Z_L(D^1, S^0)$ considered previously.

Remarks

- If each $E_i = G_i$, then the group of lifts, Γ, agrees with the first definition of graph product.
- The inverse image of $\prod_{i \in I} E_i$ in $\widetilde{Z}_L(\mathrm{Cone}\,\mathbf{E}, \mathbf{E})$ is the set of (centers of) chambers in a "right-angled building" (a RAB).
- If L is a flag complex (to be defined later), then $\widetilde{Z}_L(D^1, S^0)$ is the standard contractible complex for the RACG W and $\widetilde{Z}_L(\mathrm{Cone}\,\mathbf{E}, \mathbf{E})$ is the standard realization of the RAB.

- Let $(\mathbf{A}, \mathbf{B}) = (A_i, B_i)_{i \in I}$. Suppose each A_i is path connected. Let $p_i : \widetilde{A}_i \to A_i$ be the univ cover.
- Put $G_i = \pi_1(A_i)$ and let E_i be the set of path components of $p_i^{-1}(B_i)$ in $\widetilde{A}_i$. So, E_i is a G_i-set.

Proposition

$\pi_1(Z_L(\mathbf{A}, \mathbf{B})) = \Gamma$, where Γ is the relative graph product of the (G_i, E_i) and G is their direct product.

Remember: $G = \prod G_i$ acts on $Z_L(\mathrm{Cone}\,\mathbf{E}, \mathbf{E})$ and Γ is gp of lifts of G-action to $\widetilde{Z}_L(\mathrm{Cone}\,\mathbf{E}, \mathbf{E})$.

Proof of Proposition.

$(\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}) := (\widetilde{A}_i, p_i^{-1}(B_i))$. $Z_L(\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}) \to Z_L(\mathbf{A}, \mathbf{B})$ is an intermediate covering space and G is the gp of deck transformations. The univ cover $\widetilde{Z}_L(\mathbf{A}, \mathbf{B}) \to Z_L(\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}})$ is induced from $\widetilde{Z}_L(\text{Cone}\,\mathbf{E}, \mathbf{E}) \to Z_L(\text{Cone}\,\mathbf{E}, \mathbf{E})$. □

When is a polyhedral product aspherical?

Suppose L is a flag cx (to be defined). Earlier, we said that $Z_L(A, B)$ is aspherical in the following cases:

- $Z_L(B\mathbf{C}_2, *) = BW_L$, where W_L is the associated RACG.
- $Z_L(S^1, *) = BA_L$, where A_L is the associated RAAG.
- $Z_L(D^1, S^0) = B\pi$, where $\pi = \text{Ker}(W_L \to (\mathbf{C}_2)^m)$.

What is the common generalization?

Definition

A simplicial cx L is a *flag complex* if any finite, nonempty set of vertices, which are pairwise connected by edges, bounds a simplex. Equivalently, L is a flag cx if every minimal nonface is a nonedge.

Definition

A pair of CW complexes (A, B) is *aspherical*, if A is aspherical, each path component of B is aspherical and the fundamental gp of any such component injects into $\pi_1(A)$.

Definition

A vertex i of a simplicial cx L is *conelike* if it is connected by an edge to every other vertex.

Theorem

$Z_L(\mathbf{A}, \mathbf{B})$ is aspherical $\iff$

(i) *Each A_i is aspherical.*

(ii) *For each non-conelike vertex $i \in I$, (A_i, B_i) is aspherical.*

(iii) *L is a flag cx.*

Corollary

If $(A_i, B_i) = (BG_i, *)$ *and* L *is a flag cx, then* $Z_L(\mathbf{A}, \mathbf{B}) = B\Gamma$*, the classifying space for the graph product* Γ*.*

Corollary

Suppose each $(A_i, B_i) = (M_i, \partial M_i)$ *is a mfld with bdry and an aspherical pair. Also suppose* L *is a flag triangulation of a sphere. Then* $Z_L(\mathbf{A}, \mathbf{B}) \subset \prod M_i$ *is a closed aspherical mfld.*

Ingredients for the proof

Retraction Lemma

Suppose $L' \subset L$ is a full subcx on vertex set I'. Then the map $r : Z_L(\mathbf{A}, \mathbf{B}) \to Z_{L'}(\mathbf{A}, \mathbf{B})$ induced by $\prod_{i \in I} A_i \to \prod_{i \in I'} A_i$ is a retraction.

RAB Lemma

Suppose $\mathbf{E} = (E_i)_{i \in I}$ is a collection of sets (each with the discrete topology). Then $\widetilde{Z}_L(\text{Cone}\,\mathbf{E}, \mathbf{E})$ is contractible $\iff$ L is a flag complex. Moreover, if this is the case, then $\widetilde{Z}_L(\text{Cone}\,\mathbf{E}, \mathbf{E})$ is the “standard realization” of a RAB of type W_L.

Theorem

$Z_L(\mathbf{A},\mathbf{B})$ is aspherical $\iff$

(i) *Each A_i is aspherical.*

(ii) *For each non-conelike vertex $i \in I$, (A_i, B_i) is aspherical.*

(iii) *L is a flag cx.*

Comment

What is the point of Condition (ii)? If L is flag, then the set of conelike vertices spans a simplex Δ and L decomposes as a join, $L = L' * \Delta$, and Z_L as a product:

$$Z_L(\mathbf{A},\mathbf{B}) = Z_{L'}(\mathbf{A},\mathbf{B}) \times \prod_{i \in \operatorname{Vert}\Delta} A_i,$$

so the B_i for conelike vertices do not enter the picture.

Theorem

$Z_L(\mathbf{A}, \mathbf{B})$ *is aspherical* $\iff$

(i) *Each* A_i *is aspherical.*

(ii) *For each non-conelike vertex* $i \in I$, (A_i, B_i) *is aspherical.*

(iii) *L is a flag cx.*

Sketch of proof in $\implies$ direction

- Retraction Lemma $\implies$ (i) and (ii).
- RAB Lemma $\implies$ (iii)

Γ is the graph product of G_i w.r.t. L^1. L is its flag cx.

Theorem (with Boris Okun)

Assume each G_i is infinite. Then

$$H^n(\Gamma; k\Gamma) = \bigoplus_{\substack{J \in \mathcal{S}(L) \\ p+q=n}} H^p(\mathrm{Cone}(\mathrm{Lk}\, J), \mathrm{Lk}\, J) \otimes [H^q(G_J; kG_J) \otimes_{G_J} k\Gamma]$$

where $G_J := \prod_{i \in J} G_i$ and

$$H^*(G_J; kG_J) = \bigotimes_{i \in J} H^*(G_i; kG_i).$$

M.W. Davis and B. Okun, *Cohomology computations for Artin groups, Bestvina-Brady groups and graph products*, arXiv:1002.2564.