

Bloomington

April 8

Action dimensions of simple
complexes of groups

- ① Background
- ② Simple complexes of groups
 - $K(G, 1)$ - Question
 - Examples
- ③ Coarse van Kampen obstruction
(after Bestvina-Kapovich-Kleiner)
 - Configurations of subgroups & sheets
 - Polyhedral joins of spheres
- ④ Action dimensions of graph products,
Artin groups, hyperplane complements

① Definitions. $G =$ discrete, torsion-free gp

$BG =$ its classifying space

$EG =$ universal cover

Def (geometric dimension & action dimension)

$$\text{gdim}(G) = \min\{\dim BG\}$$

$\text{actdim}(G) =$ min dimension of thickening
of BG to mfd

$$\text{gdim}(G) \leq \text{actdim}(G) \leq 2\text{gdim}(G) + 1$$

If $BG \sim$ closed n -mfd, then

$$\text{gdim}(G) = \text{actdim } G = n$$

Papers:

- Bestvina, Kapovich, Kleiner, Inventiones 2002
- Avramidi, Davis, Okun, Schreue, BLMs 2015
- D, Huang, BLMs 2017

• D, Le, Schreie, arXiv 1803.07095

Thm There is God-given action of G on contractible mfd. (not cocompact)

↳ actdim (G) as predicted

Ex, $G =$ lattice in semisimple Lie gp. Then actdim $(G) =$ dim symmetric space

• $G =$ (torsion-free subgroup) of Mod_g
actdim $G =$ dim (Teichmüller space)

② Simple complex of groups (following Bridson-Haefliger)

Q a poset

A simple cx of groups is a

functor $G: Q \rightarrow$ from Q to category of groups & monomorphisms. i.e.

→ →

GQ is $\{G_\sigma\}_{\sigma \in Q}$ &

if $\tau < \sigma$, $i_{\sigma\tau}: G_\tau \rightarrow G_\sigma$,

functorial. The G_σ are local groups.

$$\lim GQ = \varinjlim G_\sigma = \text{direct limit}$$

Aspherical realizations: Given models

for BG_σ we can glue together

to get a space

$$BGQ = \bigcup_{\sigma \in Q} BG_\sigma$$

If $|Q|$ is 1-connected, then

$$\pi_1 BGQ = \lim GQ = G$$

$K(G, 1)$ -Question for GQ : Is

$$BGQ \sim BG \quad \text{i. e., is}$$

BG- \mathbb{Q} aspherical?

Idea: Given mflds with bdry

$M_\sigma \sim BG_\sigma$, glue them together

to get a mfld model for

BG. Of course this only

works when answer to $\pi(G, 1)$

Question is positive.

Examples. From now on

$$\mathbb{Q} = S(L)$$

= poset of simplices in
simplicial complex L ,
including \emptyset .

Graph Product Cx. Given a graph

L' & a collection $\{G_\nu\}_{\nu \in V(L')}$

The graph product $G = \prod_{L'} G_\nu$

is quotient of $\ast G_\nu$ by

normal subgroup generated by

$$[g_\nu, g_\omega], \quad \{\nu, \omega\} \in E(L').$$

$L =$ flag cx determined by L'
(= "clique cx")

if $\sigma \in \mathcal{S}(L)$, put

$$G_\sigma = \prod_{\nu \in \sigma} G_\nu$$

if $\tau \subset \sigma$, natural inclusion

$$i_{\sigma\tau} : G_\tau \hookrightarrow G_\sigma. \quad \text{So}$$

$$\{G_\sigma; \sigma \in \Sigma\} = \mathcal{G}\mathcal{S}(L)$$

is a complex of gps,

the graph product complex

• $K(L, 1)$ - Question? Yes

(provided $L = \text{flag } CX$)

Artin Complex: $(m_{s,t}) = \text{Coxeter matrix}$

$(W, S) = \text{Coxeter system}$

$L = \text{nerve}$

$$= \{ \sigma \subset S \mid W_\sigma \text{ is finite} \}$$

$$= \{ \text{spherical subsets of } S \}$$

Artin gp A_L .

Generators: $\{x_s\}_{s \in S}$

Relations:

$$\underbrace{x_\alpha x_\lambda \dots}_{m_{\alpha\lambda}} = \underbrace{x_\lambda x_\alpha \dots}_{m_{\alpha\lambda}}$$

If $\sigma \in \mathcal{S}(L)$, then

$A_\sigma = \langle \{x_n\}_{n \in \sigma} \rangle$ is
spherical

$$A\mathcal{S}(L) = \{A_\sigma\}_{\sigma \in \mathcal{S}(L)}$$

$K(G, 1)$ -Question for

$A\mathcal{S}(L)$ is still open

But

Thm (Charney-Deuis) If L is
a f.l.c.g. c.x., then $K(G, 1)$ -
Question has answer = Yes.

③ Coarse van Kampen obstruction

Def. X, Y metric spaces.

$f: X \rightarrow Y$ is coarse embedding

if \exists functions $\rho_-, \rho_+ : [0, \infty) \rightarrow [0, \infty)$

$\rho_- \leq \rho_+$, $\lim_{t \rightarrow \infty} \rho_-(t) = \infty$, s.t

$$\rho_-(d(x, x')) \leq d(f(x), f(x')) \leq \rho_+(d(x, x'))$$

• Given a simplicial cx K

its van Kampen obstruction $vk^n(K)$

is a certain cohomology class

in $H^n(\cdot; \mathbb{Z}_2)$ which is an

obstruction to embedding $K \subset \mathbb{R}^n$

Method of BKK For lower

bounds for actdim

• $\text{Cone}_\infty(K) = (K \times [0, \infty)) / (K \times 0)$

$\nu K^n(K)$ is an obstruction to finding a coarse embedding

of $\text{Cone}_\infty(K) \longrightarrow$ Contractible $(n+1)$ -m.f.d. W^{n+1}

1) Find K and coarse embedding

$\text{Cone}_\infty(K) \longrightarrow EG$ with

$$\nu K^n(K) \neq 0$$

2) If $G \curvearrowright W^{n+1}$ then $EG \rightarrow W^{n+1}$, so $\nu K^n(K)$.

obstructs such an action

3) $\text{obdim} G = \max \{n+2 \mid \exists \text{ obstruction } K \text{ as above}\}$

Thm (BKK): $\text{act dim } G \geq \text{obdim } G$.

Configurations of Sheets and

Subgps. (How do you find such H ?)

Inclusion of subgp $H \hookrightarrow G$
is a coarse embedding

If $H = \pi_1$ (closed spherical mfd)

then $E H$ is contractible mfd
looks like

$$E H = \text{Cone}_\infty S^{m-1} \quad (\text{a sheet})$$

If we have a collection of such subgroups we get a configuration of subgps

$$\bigcup H_x \hookrightarrow G \quad \#$$

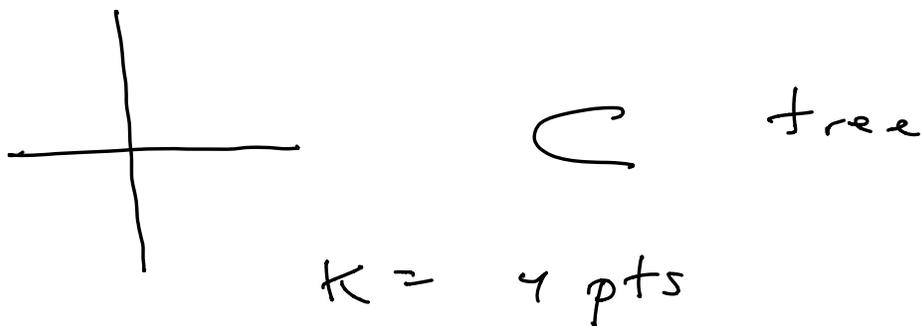
configuration of sheets

$$\bigcup_{\alpha} EH_{\alpha} \hookrightarrow EG$$

$\text{Cone}_{\infty}(K)$ where K is
some configuration of spheres

Motivating example $G = F_2$

Take 2 infinite cyclic subgps



$$G = \underbrace{F_2 \times \dots \times F_2}_{d+1}$$

$$\text{Cone}_{\infty}(K \ast \dots \ast K) = \text{Cone}_{\infty}(K_{4, \dots, 4})$$

\widehat{EG}

$K_{4, \dots, 4}$ does not embed in \mathbb{R}^{2d+1}

So " $\text{act dim}(F_2)^{d+1} = 2d+2$
 " ob dim

Examples 1) $G = A_L = \text{RAAG}$

Use configuration of standard free abelian subgps

$$\cup H_\sigma = \cup_{\sigma \in \beta(L)} \mathbb{Z}^\sigma \subset A_L$$

2) If G is graph product ^{closed} of G_α , each $G_\alpha = \pi_1$ (aspherical) _{mfld}

Use $M_\sigma = BG_\sigma = \text{PTM}_\alpha$

$$\cup H_\sigma = \cup_{\sigma \in \beta(L)} G_\sigma \subset EG$$

3) G a general Artin gp.

Use a configuration of "standard abelian subgps"

indexed by simplices α in some subdivision L_n of

L.

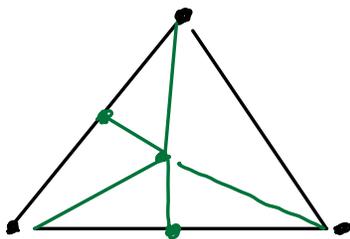
L has a vertex for each irreducible subdiagram of Coxeter diagram

Example: $\sigma = 2$ -simplex

$A_2 =$ irreducible spherical Artin gp of rank 3

 Coxeter diagram

σ



triangle gives a \mathbb{Z}^3

Polyhedral Joins: L a simplicial

ex. $V = \text{Vert}(L)$

Given $\{K_\nu\}_{\nu \in V}$ finite
 simplicial complexes

$$K_\sigma = \bigstar_{\nu \in \sigma} K_\nu$$

$$\bigstar_L K_\nu = \bigcup_{\sigma \in L} K_\sigma$$

If each $K_\nu = S^{m-1}$ we
 write $\mathcal{O}_m(L)$ for polyhedral
 join of $(m-1)$ -spheres. In
 our examples obstructor

has form $\text{Cone}_\infty \mathcal{O}_m L$

In the end we reduce
 to following

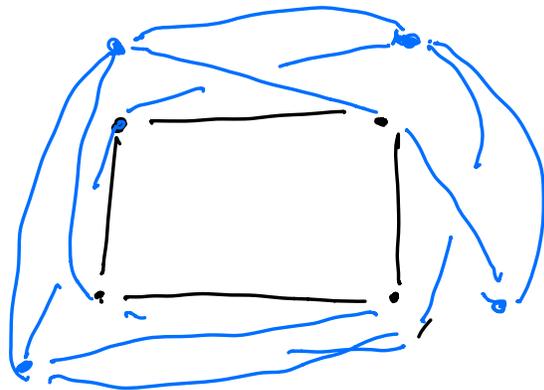
Problem: Compute largest n such that $v_k^n(O_m L) \neq 0$

Examples

1) A_L is a RAAG

obstructor = $O_1 L$

a join of 0-spheres



$$i = L$$

2) $A_L =$ Artin gp

obstructor = $O_1 L$ (with a circled question mark)

...

3) $G = \Gamma \backslash L \times G_N$ graph product

$G_N = \pi_1(M_N)$, M_N closed aspherical manifold

Obstructor = $O_m L$

Thm 0 L^d a flag cx B

$$\delta = \dim O_m L = m(d+1) - 1$$

If $H_d(L; \mathbb{Z}_2) \neq 0$, then

$$\forall k^{\delta+d}(O_m L) \neq 0.$$

Thm 1 (ADOS) A_L is a RAAG

$H_d(L^d; \mathbb{Z}_2) \neq 0$. Then

$$\text{act dim } A_L = \text{ob dim } A_L = 2d+2$$

Thm 2 (D-Huang). A_L Artin
gp for which $K(h,1)$ -Question
is Yes. If $H_d(L; \mathbb{Z}/2) \neq 0$
then $\text{act dim } A_L = 2d + 2$

Thm 3 (D-Le-Schneve)

$G =$ graph product of G_n
 $G_n = \Pi_1(M_n^m)$.

If $H_d(L; \mathbb{Z}_2) \neq 0$, then
 $\text{act dim } G = (m+1)(d+1)$