Coxeter groups and buildings

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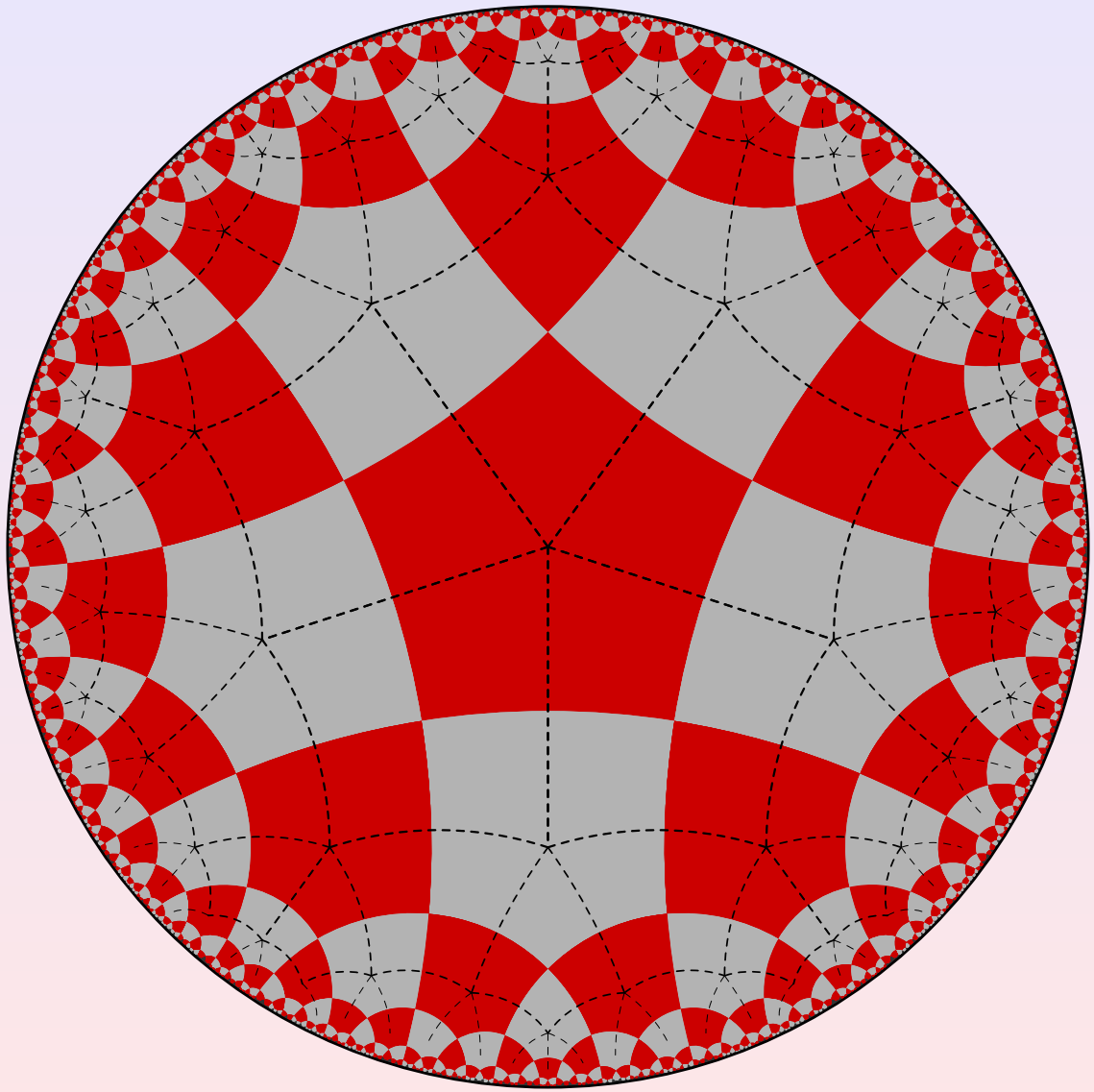

The theory of abstract reflection groups or Coxeter groups was developed by J. Tits around 1960. This is a much larger class of groups than the classical examples of geometric reflection gps on spaces of constant curvature. Nevertheless, associated to any Coxeter gp W there is a contractible space Σ on which W acts properly and cocompactly as a group generated by reflections. My main goal today is to describe Σ.

Tits coined the term “Coxeter gp” in connection with the theory of buildings. Actually, the bldgs of interest to algebraists all are associated either to finite reflection gps on spheres (leading to spherical bldgs) or to Euclidean reflection gps (leading to affine bldgs).

In fact, $\exists$ bldgs associated to general Coxeter gps and each has an associated contractible cx Λ, similar to Σ. These give interesting examples for geometric group theory.

1. Introduction
2. Abstract reflection groups and abstract buildings
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   - Chamber systems
   - Buildings
3. Their geometric realizations
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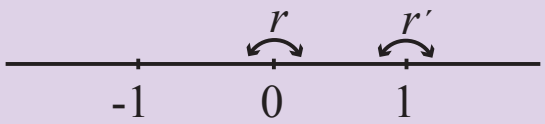
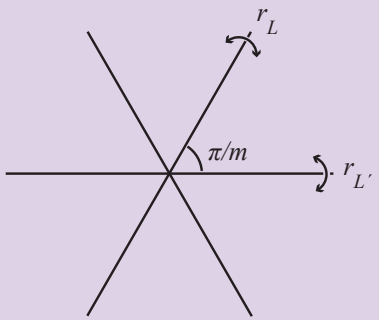
   - via chambers
   - Examples

Dihedral groups



For $m \neq \infty$, $\mathbf{D}_m$ can be represented as the subgp of $O(2)$ which is generated by reflections across lines L, L', making an angle of π/m.



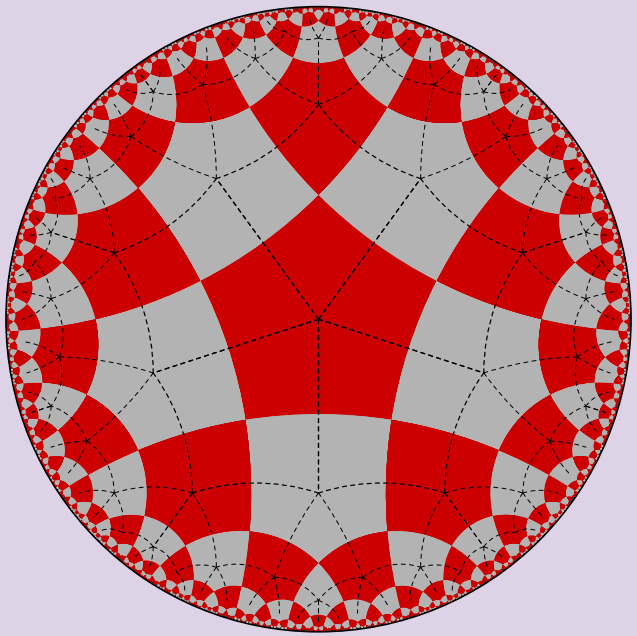
Some history

- In 1852 Möbius determined the finite subgroups of $O(3)$ generated by isometric reflections on the 2-sphere.
- The fundamental domain for such a group on the 2-sphere was a spherical triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, $\frac{\pi}{r}$, with p, q, r integers ≥ 2.
- Since the sum of the angles is $> \pi$, we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.
- For $p \geq q \geq r$, the only possibilities are: $(p, 2, 2)$ for any $p \geq 2$ and $(p, 3, 2)$ with $p = 3$, 4 or 5.
- Later work by Riemann and Schwarz showed there were discrete gps of isometries of $\mathbb{E}^2$ or $\mathbb{H}^2$ generated by reflections across the edges of triangles with angles integral submultiples of π. Poincaré and Klein: similarly for polygons in $\mathbb{H}^2$.

- Around 1850, Schläfli classified regular polytopes in $\mathbf{R}^{n+1}$, $n > 2$. The symmetry group of such a polytope was a finite gp generated by reflections and as in Möbius' case, the projection of a fundamental domain to S^n was a spherical simplex with dihedral angles integral submultiples of π.
- Around 1890, Killing and E. Cartan classified complex semisimple Lie algebras in terms of their root systems. In 1925, Weyl showed the symmetry gp of such a root system was a finite reflection gp.
- These two lines were united by Coxeter in the 1930's. He classified discrete groups reflection gps on $\mathbb{S}^n$ or $\mathbb{E}^n$.

Properties

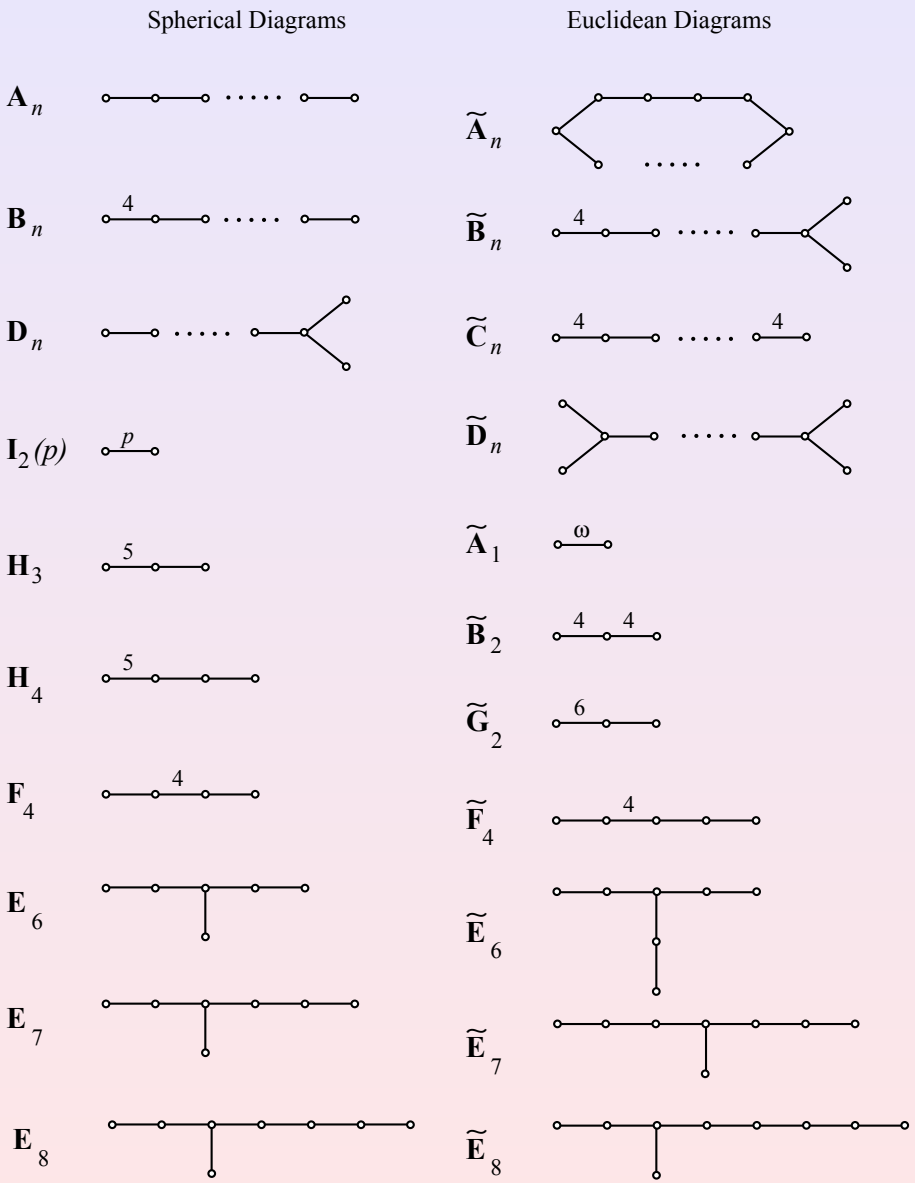
Let K be a fundamental polytope for a geometric reflection gp. For $\mathbb{S}^n$, K is a simplex. For $\mathbb{E}^n$, K is a product of simplices. For $\mathbb{H}^n$ there are other possibilities, eg, a right-angled pentagon in $\mathbb{H}^2$ or a right-angled dodecahedron in $\mathbb{H}^3$.



- Conversely, given a convex polytope K in $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ st all dihedral angles have form $\pi/$integer, there is a discrete gp W generated by isometric reflections across the codim 1 faces of K.
- Let S be the set of reflections across the codim 1 faces of K. For $s, t \in S$, let $m(s, t)$ be the order of st. Then S generates W, the faces corresponding to s and t intersect in a codim 2 face iff $m(s, t) \neq \infty$, and for $s \neq t$, the dihedral angle along that face is $\pi/m(s, t)$.

Coxeter diagrams

Associated to (W, S), there is a labeled graph Γ called its "Coxeter diagram." $\mathrm{Vert}(\Gamma) := S$.
Connect distinct elements s, t by an edge iff $m(s,t) \neq 2$. Label the edge by $m(s,t)$ if this is > 3 or $= \infty$ and leave it unlabeled if it is $= 3$.
(W, S) is *irreducible* if Γ is connected. The next slide shows Coxeter's classification of irreducible spherical and cocompact Euclidean reflection gps.


Spherical Diagrams
Euclidean Diagrams
A_n
B_n
4
D_n
$I_2(p)$
p
H_3
5
H_4
5
F_4
4
E_6
E_7
E_8
$\tilde{A}_n$
$\tilde{B}_n$
4
$\tilde{C}_n$
4
4
$\tilde{D}_n$
$\tilde{A}_1$
ω
$\tilde{B}_2$
4
4
$\tilde{G}_2$
6
$\tilde{F}_4$
4
$\tilde{E}_6$
$\tilde{E}_7$
$\tilde{E}_8$

Abstract reflection groups

Question

Given a gp W and a set S of involutions which generates it, when should (W, S) be called an "abstract reflection gp"?

Two answers

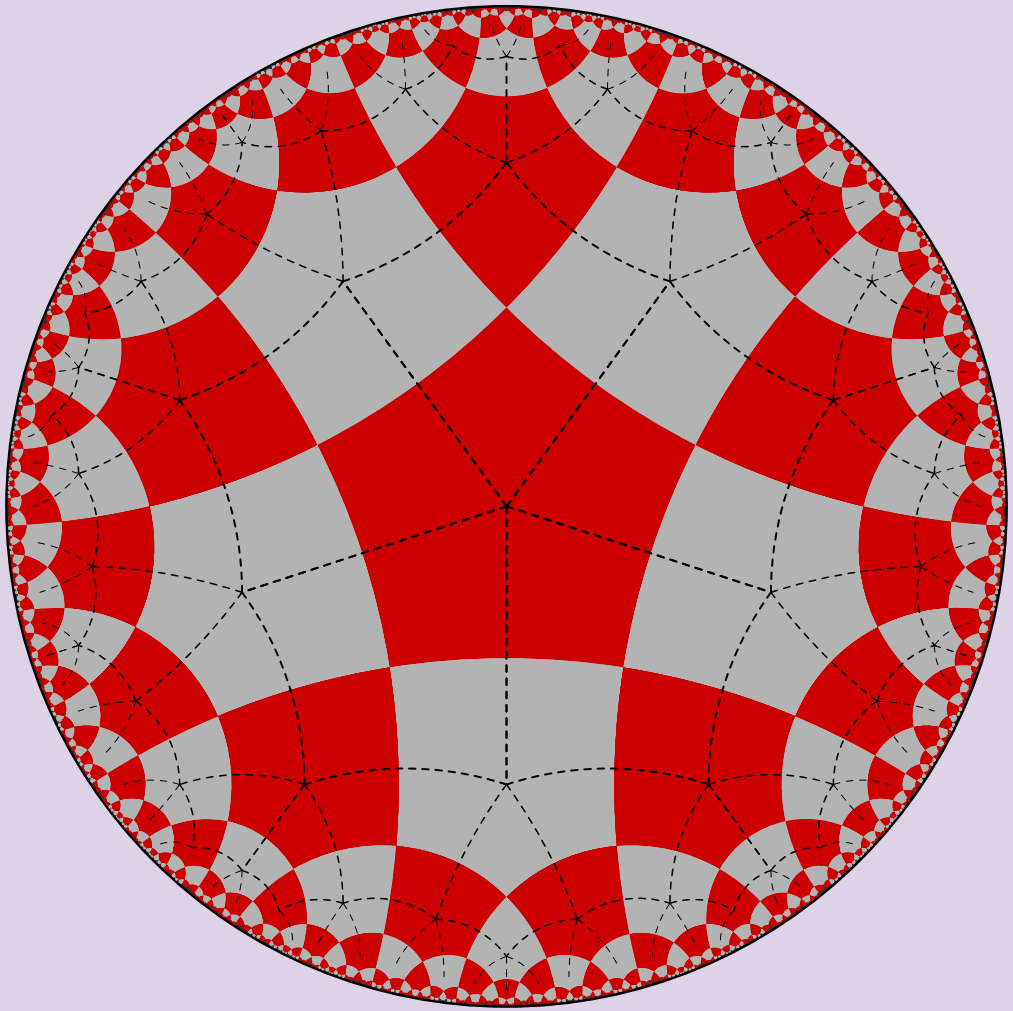
- Let Cay(W, S) be the Cayley graph (ie, its vertex set is W and $\{w, v\}$ spans an edge iff $v = ws$ for some $s \in S$). First answer: for each $s \in S$, the fixed set of s separates Cay(W, S).
- Second answer: W has a presentation of the form:

$$\langle S \mid (st)^{m(s,t)}, \text{ where } (s,t) \in S \times S \rangle.$$

Here $m(s, t)$ is any $S \times S$ symmetric matrix with 1s on the diagonal and off-diagonal elements integers ≥ 2 or the symbol ∞.

These two answers are equivalent!
In either case, (W, S) is called a *Coxeter system*

The dotted black edges give the Cayley graph



Definition

A *chamber system* is a set Φ (of "chambers") together with a family of equivalence relations indexed by some set S

Two chambers c and d are *s-adjacent* if they are s-equivalent and not equal.

A *gallery* in Φ is a finite sequence $(c_0, \ldots, c_n)$ of adjacent chambers. It is *type* $(s_1, \ldots, s_n)$ if c_i and c_{i-1} are s_i-adjacent. (So, its type is a word in S.)

Definitions

- Φ is *connected* if any two chambers can be connected by a gallery.
- Given a subset $T \subset S$, a gallery of type $(s_1, \ldots, s_n)$ is a *T-gallery* if each $s_i \in T$.
- A *residue* of *type* T is a T-gallery connected component of Φ.

Suppose (W, S) is a Coxeter system. For each $T \subset S$, $W_T := \langle T \rangle$.

Example

W has the structure of a chamber system over S.

- $v, w \in W$ are s-equivalent $\iff$ they belong to the same coset of W_s (the gp of order 2 generated by s).
- A gallery in W is an edge path in $\mathrm{Cay}(W, S)$. (Here a "chamber" is a vertex of $\mathrm{Cay}(W, S)$.)
- Its type is a word in S.
- A T-residue is a coset of W_T in W.

T is a *spherical* subset of S if $|W_T| < \infty$.
$\mathcal{S} :=$ the poset of spherical subsets.

Reduced expressions

Given a word $\mathbf{s} = (s_1, \ldots, s_n)$ in S, its *value* is $w(\mathbf{s})$ is the element $w \in W$ defined by $w := s_1 \cdots s_n$.
$\mathbf{s}$ is a *reduced expression* for w if $l(w) = n$.

Buildings

Suppose (W, S) is a Coxeter system.

Definition

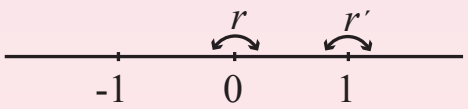
A chamber system Φ over S is a *building* of type (W, S) if each s-equivalence class has at least two elements and if it admits a *W-valued distance function* $\delta : \Phi \times \Phi \to W$.
This means that if $\mathbf{s} = (s_1, \ldots, s_n)$ is a reduced expression for w and if $c_0, c_n \in \Phi$, then there is a gallery of type $\mathbf{s}$ from c_0 to c_n $\iff$ $\delta(c_0, c_n) = w$

Example (The edge set of a tree)

Suppose W is the infinite dihedral group with generators s, t. Let $\mathcal{T}$ be a tree w/o terminal vertices. Any tree is bipartite. Call the colors s, t.

Let $\Phi = \text{Edge}(\mathcal{T})$. An edge path in $\mathcal{T}$ gives a gallery. The sequence of colors of the vertices it crosses gives a word in $\{s, t\}$. The word is a reduced expression if the path has no backtracking.

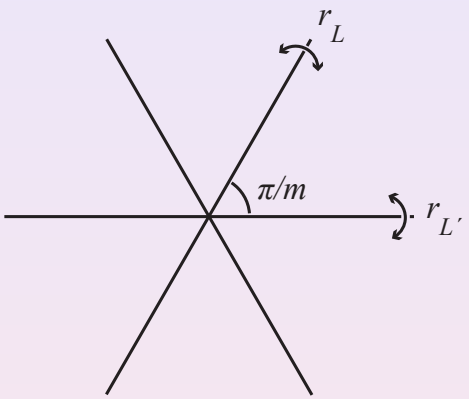
The W-distance between b,$c \in \Phi$ is defined as follows. Take the edge path w/o backtracking from b to c. Its type gives a word in $\{s, t\}$ and hence, an element $w \in W$ and $\delta(b, c) = w$.



Definitions

If W is

- finite, Φ is a *spherical bldg*.
- a cocompact Euclidean reflection gp, Φ is a *Euclidean* (or *affine*) *bldg*.
- a finite dihedral gp, Φ is a *generalized polygon*.



Example

W itself is a bldg. W-distance $\delta : W \times W \to W$ is defined by $\delta(v, w) = v^{-1}w$.

An *apartment* is the image of a W-isometric embedding $W \hookrightarrow \Phi$.

Geometric realizations

Suppose $\mathcal{P}$ is a poset. Its *geometric realization* $|\mathcal{P}|$ is the simplicial cx with one simplex for each chain $p_0 < \cdots < p_k$ in $\mathcal{P}$.

Recall T is a *spherical* subset of S if $|W_T| < \infty$ and
$\mathcal{S} :=$ poset of spherical subsets.
$W\mathcal{S} :=$ poset of spherical cosets $= \coprod W/W_T$. For a bldg Φ,
$\mathcal{C}$:= poset of spherical residues in Φ.

$$K := |\mathcal{S}| \qquad \Sigma := |W\mathcal{S}| \qquad \Lambda := |\mathcal{C}|.$$

Basic Fact

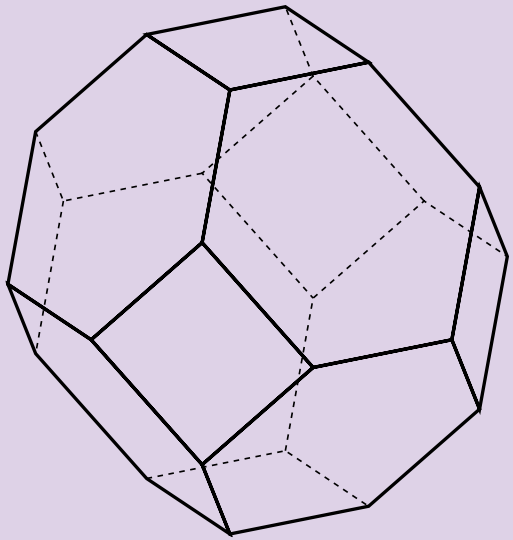
Λ (and in particular, Σ) is contractible (in fact, $\mathrm{CAT}(0)$).

Next I want to describe a polyhedral metric on these spaces which is "nonpositively curved" (or $\mathrm{CAT}(0)$) in sense of comparison triangles.

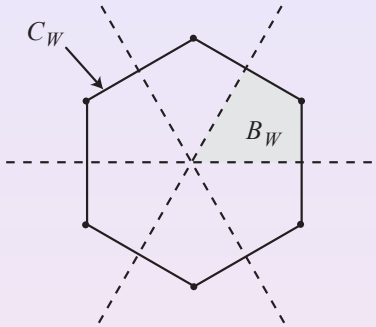
Coxeter zonotopes

If W is finite, then $W\mathcal{S}$ can be identified with the poset of faces of a convex polytope.

Permutohedron



For W finite, the intersection of its Coxeter zonotope with a fundamental simplicial cone is called a "Coxeter block"



For a general W, the chamber K $(= |\mathcal{S}|)$ is subdivided into Coxeter blocks. This defines a piecewise Euclidean metric on Λ and Σ.

Theorem (Moussong)

This metric is $\mathrm{CAT}(0)$ *(meaning that the space is simply connected and "nonpositively curved").*

Corollary

Λ (and Σ) are contractible.

Proof.

Λ is $\mathrm{CAT}(0) \implies \exists$ unique geodesic between any 2 points $\implies$ Λ is contractible. $\square$

Corollary (Bruhat-Tits Fixed Point Theorem)

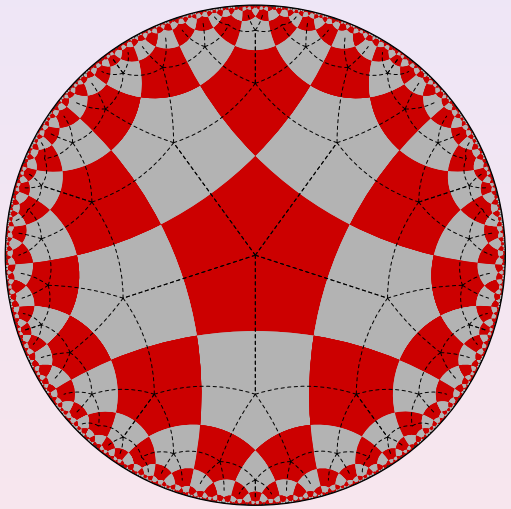
If a group G acts on a building $\mathcal{C}$ with a bounded orbit of chambers, then it has a fixed point on Λ. Hence, G is contained in the stabilizer of a spherical residue.

$$K := |\mathcal{S}| \qquad \Sigma := |W\mathcal{S}| \qquad \Lambda := |\mathcal{C}|.$$

W is gp generated by reflections in the faces of right-angled pentagon.

K is a pentagon.

Σ is the tessellation of $\mathbb{H}^2$ by pentagons



The $\mathrm{CAT}(0)$ metric is defined by declaring each quadrilateral to be a Euclidean square.

Another approach to defining the geometric realization of a chamber cx or bldg is to declare each chamber to be homeomorphic to a certain space X and then declare how adjacent chambers are glued together.

Usually we take $X = K$ (the geometric realization of $\mathcal{S}$). Another possibility is $X = \Delta$ a simplex with its $\{$codim 1 faces$\}$ indexed by S. In that case we get the *Coxeter complex*.

Here are the details:

Alternate description

- X a space, $\{X_s\}_{s\in S}$ a family of closed subspaces. $S(x) := \{s \in S \mid x \in X_s\}$.
- $\mathcal{U}(\Phi, X) := (\Phi \times X)/\sim$, where $(c, x) \sim (c', x') \iff$ $x = x'$ and c,c' belong to the same $S(x)$-residue.
- $\mathcal{U}(\Phi, X)$ is formed by pasting together copies of X, one for each chamber in Φ.
- Put $K_s := |\mathcal{S}_{\geq\{s\}}|$. Then $\Lambda = \mathcal{U}(\Phi, K)$.

When $\Phi = W$, two chambers ($=$ copies of X) meet along each face of type X_s. For a general bldg there are two or more along each X_s. (Think: line versus tree.)

Right-angled bldgs (= RABs)

A Coxeter gp is *right angled* if each m_{st} is $= 2$ or ∞.
A bldg is *right-angled* if its Coxeter gp is (ie all $m(s,t)$ are 2 or ∞).

Example (Product of trees)

If $W = (D_\infty)^n$, then K is an n-cube. A bldg of type (W,S) is a product of n trees.

Will show

$\forall$ right-angled (W,S), $\exists$ RABs of type (W,S) with chamber transitive automorphism gp and arbitrary thickness along faces of type s.

Graph products of type (W, S)

- (W, S) a right-angled Coxeter system.
- Let L^1 be a graph with $\mathrm{Vert}(L^1) = S$ and $\mathrm{Edge}(L^1) = \{\{s, t\} \mid m_{st} \neq \infty\}$.
- Let $\{G_s\}_{s \in S}$ be a family of finite gps. For each $T \in \mathcal{S}$ (i.e., for each complete subgraph of L^1), put

$$G_T := \prod_{s \in T} G_s.$$

- $G = \varinjlim G_T$ is the *graph product* of the G_s.

Put $\Phi = G$. If $g = g_{s_1} \cdots g_{s_n} \in G$, put $\delta(1, g) = s_1 \cdots s_n \in W$ and $\delta(h, g) = \delta(1, h^{-1}g)$. Then Φ is a RAB. The number of chambers along a face of type s is $\mathrm{Card}(G_s)$.

Conclusion

RABs of arbitrary thickness exist.

Kac-Moody gps and bldgs

Kac-Moody gps and bldgs

- Suppose each $m(s,t) \in \{1,2,3,4,6,\infty\}$ and $\mathbb{F}_q$ is a finite field.
- $\exists$ a "Kac-Moody gp" of type (W,S) over $\mathbb{F}_q$. It has structure of a "(B,N) pair" and hence, gives a bldg Φ with chamber transitive action of G.
- $q+1$ chambers meet along each face of type s.
- G acts properly on $\Lambda \times \Lambda$ (where Λ is the geometric realization of Φ.)