Orbifolds 1

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Definitions

An action of a topological gp G on a space X is a (continuous) map $G \times X \to X$, denoted $(g, x) \to gx$, st

- g(hx)=(gh)x,
- 1x=x.

(Write $G \curvearrowright X$.)

Given $g \in G$, define $\theta_g : X \to X$ by $x \to gx$. Since $\theta_g \circ \theta_{g^{-1}} = 1_X = \theta_{g^{-1}} \circ \theta_g$, the map θ_g is a homeomorphism and the map $\Theta : G \to \text{Homeo}(X)$ defined by $g \to \theta_g$ is a homomorphism of groups.

Given $x \in X$, $G_x := \{g \in G \mid gx = x\}$ is the *isotropy subgroup*. The action is *free* if $G_x = \{1\}$, $\forall x \in X$.

More Definitions

$G(x) := \{gx \in X \mid g \in G\}$ is the *orbit* of x. Note $G(x) \cong G/G_x$. The action is *transitive* if there is only one orbit. Given $x \in X$, the natural map $G/G_x \to G(x)$ defined by $gG_x \to gx$ is a continuous bijection.

The *orbit space* X/G is the set of orbits in X with the quotient topology (wrt the natural map $X \to X/G$).

A map $f : X \to Y$ of G-spaces is *equivariant* (or a *G-map*) if $f(gx) = gf(x)$

More definitions

Suppose $H \subset G$ is a subgp and Y is a H-space. Then H acts on $G \times Y$ via $h \cdot (g, x) = (gh^{-1}, hx)$. The orbit space is denoted $G \times_H Y$ and called the *twisted product*. The image of (g, x) in $G \times_H Y$ is denoted $[g, x]$. Note G acts on $G \times_H Y$.

A *slice* at a point $x \in X$ is a G_x-stable subset U_x st the map $G \times_{G_x} U_x \to X$ is an equivariant homeomorphism onto a neighborhood of $G(x)$. If U_x is homeomorphic to a disk, then $G \times_{G_x} U_x$ is an *equivariant tubular neighborhood* of $G(x)$.

Remark

A neighborhood of the orbit in X/G is homeomorphic to U_x/G_x.

The Differentiable Slice Theorem

Theorem

Suppose a compact Lie gp acts differentiably (= "smoothly") on a manifold M. Then every orbit has a G-invariant tubular neighborhood. More precisely, $\exists$ *a linear representation of* G_x *on a vector space S st that* $G \times_{G_x} S$ *is a tubular neighborhood of* $G(x)$.

Proof.

By integrating over the compact Lie group G we can find a G-invariant Riemannian metric. Then apply the usual proof using the exponential map. □

Proper actions of discrete groups

Γ a discrete gp, X a Hausdorff space and $\Gamma \curvearrowright X$.
The Γ-acton is *proper* if given any 2 points $x, y \in X$, $\exists$ open nbhds U of x and V of y st $\gamma U \cap V \neq \emptyset$ for only finitely many γ.

Exercise

Show that a Γ-action on X is proper iff

- X/Γ is Hausdorff,
- each isotropy subgp is finite
- each point $x \in X$ has a slice, ie, $\exists$ Γ_x-stable open nbhd U_x st $\gamma U_x \cap U_x = \emptyset, \quad \forall \gamma \in \Gamma - \Gamma_x$.

(This means that $\Gamma \times_{\Gamma_x} U_x$ is a nbhd of the orbit of x.)

Actions on manifolds

Suppose a discrete gp Γ acts properly on an n-dim mfld M^n.

A slice U_x at $x \in M^n$ is *linear* if $\exists$ a linear Γ_x-action on $\mathbf{R}^n$ st U_x is Γ_x-equivariantly homeomorphic to a Γ_x-stable nbhd of the origin in $\mathbf{R}^n$. The action is *locally linear* if every point has a linear slice.

Proposition

If $\Gamma \curvearrowright M^n$ properly and differentiably, then action is locally linear.

Proof.

Since Γ_x is finite, we can find a Γ_x-invariant Riemannian metric on M. The exponential map, $exp : T_xM \to M$ is Γ_x-equivariant and takes a small disk about the origin homeomorphically onto a neighborhood U_x of x. If the disk is small enough, U_x is a slice. □

Definition

An *orbifold chart* on a space X is a 4-tuple $(\widetilde{U}, G, U, \pi)$ where

- U is open subset of X
- $\widetilde{U}$ is open in $\mathbf{R}^n$ and G is finite gp of homeomorphisms of $\widetilde{U}$
- $\pi : \widetilde{U} \to U$ is a map which can be factored as $\pi = \overline{\pi} \circ p$ where $p : \widetilde{U} \to \widetilde{U}/G$ is the orbit map and $\overline{\pi} : \widetilde{U}/G \to U$ is a homeo.

The chart is *linear* if $G \curvearrowright \mathbf{R}^n$ linearly.

For $i = 1, 2$, suppose $(\widetilde{U}_i, G_i, U_i, \pi_i)$ are orbifold charts on X. The charts are *compatible* if given points $\tilde{u}_i \in \widetilde{U}_i$ with $\pi_1(\tilde{u}_1) = \pi_2(\tilde{u}_2)$, $\exists$ homeo h from nbhd of $\tilde{u}_1$ in $\widetilde{U}_1$ onto nbhd of $\tilde{u}_2$ in $\widetilde{U}_2$ st $\pi_1 = \pi_2 \circ h$ on the nbhd.

Definition

An *orbifold atlas* on X is a compatible collection $\{(\widetilde{U}_i, G_i, U_i, \pi_i)\}_{i\in I}$ of orbifold charts which cover X.

Definition

An orbifold Q consists of an *underlying space* $|Q|$ together with a maximal atlas of charts.

A *smooth* orbifold means the groups act via diffeomorphisms and the charts are compatible via diffeomorphisms. A *locally linear* orbifold means all charts are equivalent to linear ones.

From now on, all orbifolds will be locally linear

Exercise

Suppose $\Gamma \curvearrowright M^n$ properly. By choosing slices we can cover M/Γ by orbifold charts. Show this gives the underlying space M/Γ the structure of orbifold which we denote by $M/\!\!/\Gamma$.

The local group

There is more info in an orbifold than just its underlying space. For example, if $q \in |Q|$ and $x = \pi^{-1}(q)$ is a point in the inverse image of q in some local chart, then the isotropy subgp G_x is independent of the chart up to isomorphism of gps. With this ambiguity, we call it the *local group at q* and denote it G_q.

A manifold is an orbifold in which each local gp is trivial.

Strata

In transformation gps, if $G \curvearrowright X$ and $H \subset G$, then

$$X_{(H)} := \{x \in X \mid G_x \text{ is conjugate to } H\}$$

is the set of points of *orbit type* G/H. The image of $X_{(H)}$ in X/G is a *stratum* of X/G. This image can be described as follows. First, take the fixed set X^H $(:= \{x \in X \mid hx = x, \forall h \in H\})$. Remove points x with $G_x \supsetneq H$ to get $X^H_{(H)}$. Then divide by the free action of $N(H)/H$ to get $X^*_{(H)}$, the *stratum of type* (H) in X/G. In an orbifold, Q, a *stratum of type* (H) is the subspace of $|Q|$ consisting of all points with local gp iso to H.

Proposition

If Q is a locally linear orbifold, then each stratum is a mfld.

Proof.

Suppose a finite gp $G \curvearrowright \mathbf{R}^n$ linearly and $H \subset G$. Then $(\mathbf{R}^n)^H$ is a linear subspace; hence, $(\mathbf{R}^n)^H_{(H)}$ is a mfld. Dividing by the free action of $N(H)/H$, we see that $(\mathbf{R}^n)^*_{(H)}$ is a mfld. □

A consequence of Differentiable Slice Thm

Suppose

- G is a cpt Lie gp; $G \curvearrowright M^n$.
- Each isotropy subgp G_x is finite. So, M is locally $G \times_{G_x} \widetilde{U}$.
- So, M/G is locally $\widetilde{U}/\!\!/G_x$, ie, M/G is an orbifold.

Thurston's big improvement over Satake's earlier version was to show that the theory of covering spaces and fundamental groups worked for orbifolds. (When I was a grad student a few years before, this was "well-known" *not* to work.)

The local model for a covering projection between n-dimensional mflds is the identity map $id : U \to U$ on an open subset $U \subset \mathbf{R}^n$. Similarly, the local model for an *orbifold covering projection* is the natural map $\mathbf{R}^n/H \to \mathbf{R}^n/G$ where a finite gp G acts on $\mathbf{R}^n$ and $H \subset G$ is a subgp.

Proposition

If $\Gamma \curvearrowright M$ properly and $\Gamma' \subset \Gamma$ is a subgp, then $M/\!/\Gamma' \to M/\!/\Gamma$ is an orbifold covering projection.

Definition

An orbfld Q is *developable* if it is covered by a mfld. As we will see, this is equivalent to the condition that Q is the quotient of a discrete gp acting properly on a mfld. (In Thurston's terminology, Q is a "good" orbfld.)

Remark

Not every orbfld is developable (eg, later we will describe the "tear drop," the standard counterexample).

Definition

Q is *simply connected* if it is connected and does not admit a nontrivial orbfld covering, ie, if $p : Q' \to Q$ is a covering with $|Q'|$ connected, then p is a homeomorphism.

Fact

Any connected orbifold Q admits a simply connected orbfld covering $\pi : \widetilde{Q} \to Q$. This has the usual universal property, ie, if we pick a "generic" base point $q \in Q$ and $p : Q' \to Q$ is another covering with base points $q' \in Q'$ and $\tilde{q} \in \widetilde{Q}$ lying over q, then π factors through Q' via a covering proj $\widetilde{Q} \to Q'$ taking $\tilde{q}$ to q'. In particular, $\widetilde{Q} \to Q$ is a regular covering in the sense that its group of deck transformations is simply transitively on $\pi^{-1}(q)$.

Definition of the orbifold fundamental group

Definition

$\pi_1^{orb}(Q)$ is the group of deck transformations of the universal orbfld cover, $p : \widetilde{Q} \to Q$

Later I will give a definition in terms of generators and relations. A third defn: "homotopy classes" of loops $[0, 1] \to Q$. (To do this we must first define what is meant by a map from a space to Q - it should be a continuous map to $|Q|$ together with a choice of a "local lift" (up to equivalence) for each orbfld chart for Q. A fourth defn: if $\mathcal{G}_Q$ is the groupoid associated to Q and $B\mathcal{G}_Q$ is its classifying space, then $\pi_1^{orb}(Q) := \pi_1(B\mathcal{G}_Q)$.

Developability and the local group

For each $x \in |Q|$, let G_x denote the local gp at x. (It a subgp of $GL(n, \mathbf{R})$, well-defined up to conjugation. We can identify G_x with the fundamental group of a nbhd of the form $\widetilde{U}_x/G_x$ where $\widetilde{U}_x$ is a ball in some linear representation. So, G_x is the "local fundamental group" at x. The inclusion of the nbhd induces a homomorphism $G_x \to \pi_1^{orb}(Q)$.

Proposition

Q is developable $\iff$ each local gp injects (ie, for each $x \in |Q|$, the map $G_x \to \pi_1^{orb}(Q)$ is injective).

Dimension 1

The only finite gp which acts linearly (and effectively) on $\mathbf{R}^1$ is the cyclic gp of order 2, C_2. It acts via the reflection $x \mapsto -x$. The orbit space $\mathbf{R}^1/C_2$ is identified with $[0, \infty)$.

It follows that every 1-dimensional orbfld Q is either a 1-mfld or a 1-mfld with boundary. If Q is compact and connected, then it is either a circle or an interval (say, $[0, 1]$).

The infinite dihedral gp, D_∞ is the gp generated by 2 distinct affine reflections on $\mathbf{R}^1$. $\mathbf{R}^1/D_\infty \cong [0, 1]$.

2-dimensional linear groups

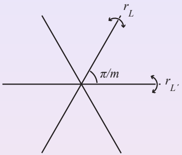
A finite gp $G \curvearrowright \mathbf{R}^n$ linearly. Then G is conjugate to subgp of $O(n)$. (Pf: By averaging we get an invariant inner product). Hence, G acts on the unit sphere $S^{n-1} \subset \mathbf{R}^n$.

Suppose $G \subset O(2)$. Then $S^1 /\!\!/ G = S^1$ or $S^1 /\!\!/ G = [0, 1]$.

- In the first case, $S^1 \to S^1 /\!\!/ G = S^1$ is an n-fold cover, where $n = |G|$, and G is the cyclic gp C_n acting by rotations.
- In the second case, the composition, $\mathbf{R}^1 \to S^1 \to S^1 /\!\!/ G = [0, 1]$, is the univ cover with gp of deck transformations D_∞. It follows that $G = D_m$ (the dihedral gp of order $2m$) or $G = C_2$ $(= D_1)$ acting by reflection across a line.

Theorem of Leonardo da Vinci

Any finite subgp of $O(2)$ is conjugate to either C_n or D_m.



Question

What does $\mathbf{R}^2/\!/G$ look like?

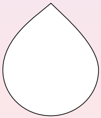
- $\mathbf{R}^2$ $(G = \{1\})$
- cone $(G = C_n)$
- half-space $(G = D_1)$
- sector $(G = D_m)$

In half-space case, codim 1 stratum is a *mirror*. In the sector case, codim 2 stratum is a *corner reflector*

2-dimensional orbifolds

Here is the picture: the underlying space of a 2-dim orbfld Q is a 2-mfld, possibly with bdry. Certain points in the interior of the $|Q|$ are "cone points" labeled by an integer n_i specifying that the local gp is C_{n_i}. The codim 1 strata are the mirrors; their closures cover $\partial|Q|$. The closures of two mirrors intersect in a corner reflectors (where local gp is D_{m_i}).

The picture to the right is possible.
It is not developable.



General orbifolds

- $G \subset O(n)$, $D^n \subset \mathbf{R}^n$ the unit disk, $G \curvearrowright D^n$
- $D^n = \text{Cone}(S^{n-1})$, so $D^n /\!\!/ G = \text{Cone}(S^{n-1} /\!\!/ G)$
  $\therefore$ a point in a general orbfld has conical nbhd of this form.

Example

Suppose $G = C_2$ acting via antipodal map, $x \mapsto -x$. Then $D^n /\!\!/ C_2 = \text{Cone}(\mathbf{R}P^{n-1})$

Suppose Q is an n-dim orbfld, $Q_{(2)}$ is the complement of the strata of codim > 2. The description of $Q_{(2)}$ is similar to a 2-dim orbfld. $|Q_{(2)}|$ is an n-mfld with bdry; the bdry is a union of (closures of) mirrors; the codim 2 strata in the interior are codim 2 submflds.

Examples of orbifold coverings

Suppose $X \to |Q|$ is an ordinary covering of topological spaces. Pullback strata of Q to strata in X obtaining an orbfld Q'. (Specific example: Q is $\mathbf{R}P^2$ with one cone point labeled n. $S^2 \to \mathbf{R}P^2$ is the double cover. The single cone point pulls back to two cone points in S^2 labeled n.

Double $|Q|$ along its bdry to get a 2-fold orbfld covering $Q' \to Q$ w/o codim 1 strata. (Eg, if Q is a triangle, Q' is a 2-sphere with 3 cone points.)

The n-fold branched cover of Q along a codim 2 stratum labeled by the cyclic gp of order n.

Generators and relations for $\pi_1^{orb}(Q)$

Remark

$\pi_1^{orb}(Q) = \pi_1^{orb}(Q_{(2)})$. (Pf: general position)

Let $\hat{Q}$ denote the complement in $|Q|$ of the strata of codim ≥ 2 (retain the mirrors on $\partial|Q|$). Choose a base point x_0 in interior $\hat{Q}$. We are going to construct $\pi_1^{orb}(Q)$ from $\pi_1(\hat{Q}, x_0)$ by adding generators and relations.

- For each component T of a codim 2 stratum in interior of $|Q|$, choose a loop α_T starting at x_0 which makes a small loop around T. Let $n(T)$ be the order of the cyclic gp labeling T.

Generators and relations, continued

- Suppose P is a codim 2 stratum contained in $M \cap N$ (st P is a corner reflector). Let $m(P)$ be the label on P (st the dihedral gp at P has order $2m(P)$).

- For each mirror M and homotopy class of paths γ_M from x_0 to M introduce a new generator $\beta_{(M,\gamma_M)}$.

Relations

$$[\alpha_T]^{n(T)} = 1, \quad [\beta_{(M,\gamma_M)}]^2 = 1, \text{ and } \quad ([\beta_{(M,\gamma_M)}][\beta_{(N,\alpha_N)}])^{m(P)} = 1,$$

where P is a component of $\overline{M} \cap \overline{N}$ and γ_M and α_N are homotopic as paths from x_0 to P.

Haefliger explained that the best way to define an orbifold was as a certain groupoid.

- There is the notion of a pseudo-group $\mathcal{P}$ of local homeomorphisms on a space X. Associated to $\mathcal{P}$ there is a (topological) groupoid $\Gamma_{\mathcal{P}}$ of germs of local homeomorphisms in $\mathcal{P}$.
- Suppose $\{(\widetilde{U}_i, G_i, U_i, \pi_i)\}_{\imath \in I}$ is an orbifold atlas on $|Q|$. Put $\widetilde{Q} = \coprod \widetilde{U}_i$.
- For $X = \widetilde{Q}$ we have the pseudo-gp of all local homeomorphisms $\widetilde{U}_i \to \widetilde{U}_j$ which commute with the projections π_i and π_j. The associated groupoid is Γ_Q.
- Actually, the correct definition of an orbfld should probably be that it is equal to the groupoid Γ_Q.

The classifying space of an orbifold

- There is the notion of a classifying space of a groupoid. When the groupoid is Γ_Q we get BQ the classifying space of the orbifold.
- When $Q = M/\!\!/\Gamma$ where $\Gamma \curvearrowright M$ properly, then BQ is homotopy equivalent to the Borel construction, $E\Gamma \times_\Gamma M$. As in this case, in the general case, we have a projection map $p : BQ \to |Q|$ so that $p^{-1}(x) \sim BG_x$.

Remark

We could have defined $\pi_1^{orb}(Q)$ to be $\pi_1(BQ)$.

Bundles over orbifolds

There is a notion of a "fiber bundle over an orbifold Q"; it is simply a bundle over $\widetilde{Q}$ together with a Γ_Q-action by bundle maps. For example, a smooth orbifold has a tangent bundle TQ. There is an associated bundle of orthonormal frames $PQ \to Q$; its local models have the form $O(n) \times \widetilde{U}_i$. The orbfold quotient of the local model has the form $O(n) \times_{G_i} \widetilde{U}_i$. It follows that $O(n) \curvearrowright PQ$ with finite isotropy subgps and with quotient orbfld Q.

Remark

Although not every orbifold is developable, the above shows that every orbifold can be writtten in the form $Q = M/G$, where G is a compact Lie group.

Another model for BQ

Let $V_{N,n}$ be Stiefel manifold of n-frames in $\mathbf{R}^N$. The space $V_{\infty,n}$ of n-frames in $\mathbf{R}^\infty$ is contractible. Consider the associated bundle

$$E = V_{\infty,n} \times_{O(n)} PQ.$$

There is a projection $E \to Q$ and the fiber over x is $V_{\infty,n}/G_x \sim BG_x$. It follows that E is a model for BQ.

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