Orbifolds 2

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May 14, 2014
https://people.math.osu.edu/davis.12/slides.html
We know what is meant by the Euler characteristic of a closed mfld or a finite CW complex (the alternating sum of the number of cells). A key property is that it is multiplicative under finite covers: if $M' \to M$ is an $m$-fold cover, then

$$
\chi(M') = m \chi(M).
$$

The Euler characteristic of an orbfld should be a rational number with same multiplicative property, i.e., if $M \to Q$ is an $m$-fold cover and $M$ is a mfld, then it should have

$$
\chi(M) = m \chi^{orb}(Q).
$$

That is,

$$
\chi^{orb}(Q) = \frac{1}{m} \chi(M).
$$

("$m$-fold cover" means $\text{Card}(p^{-1}(\text{generic pt})) = m$.)

The Euler characteristic of an orbifold

Suppose $Q$ is an orbifold which cellulated as a CW complex so that the local gp is constant on each open cell $c$. Let $G(c)$ be the local gp at $c$ and $|G(c)|$ its order.

$$
\chi^{orb}(Q) := \sum_{\text{cells } c} \frac{(-1)^{\dim c}}{|G(c)|}
$$

Exercise

Suppose $\Gamma \actson M$ properly, cocompactly, locally linearly and $\Gamma' \subset \Gamma$ is a subgp of index $m$. Show

$$
\chi^{orb}(M/\Gamma') = m \chi^{orb}(M/\Gamma).
$$
Alternate formula

Each stratum $S$ of a compact orbifold $Q$ is the interior of a compact mfld with bdry $\hat{S}$. Define $e(S) := \chi(\hat{S}) - \chi(\partial \hat{S})$.

$$\chi^{orb}(Q) = \sum_{\text{strata } S} \frac{e(S)}{|G(S)|}$$

Example

Suppose $|Q| = D^2$ and $Q$ has $k$ mirrors and $k$ corner reflectors labeled $m_1, \ldots, m_k$. Then

$$\chi^{orb}(Q) = 1 - \frac{k}{2} + \left( \frac{1}{2m_1} + \cdots + \frac{1}{2m_k} \right) = 1 - \frac{1}{2} \sum_i \left( 1 - \frac{1}{m_i} \right)$$
Example

Suppose $|Q| = S^2$ and $Q$ has $l$ cone points labeled $n_1, \ldots, n_l$. Then

$$\chi^{orb}(Q) = 2 - l + \left( \frac{1}{n_1} + \cdots + \frac{1}{n_l} \right) = 2 - \sum_i \left( 1 - \frac{1}{n_i} \right)$$

(This is twice the previous example, as it should be.)

Example (The general formula)

Suppose $|Q|$ is a surface with bdry, $Q$ has $k$ corner reflectors labeled $m_1, \ldots, m_k$ and $l$ cone points labeled $n_1, \ldots, n_l$. Then

$$\chi^{orb}(Q) = \chi(|Q|) - \frac{1}{2} \sum_{i=1}^k \left( 1 - \frac{1}{m_i} \right) - \sum_{i=1}^l \left( 1 - \frac{1}{n_i} \right).$$
Remark

\[ \chi^{\text{orb}}(Q) \leq \chi(|Q|) \] with equality iff there are no cone points or corner reflectors.

Notation

If a 2-dim orbifold has \(k\) corner reflectors labeled \(m_1, \ldots, m_k\) and \(l\) cone points labeled \(n_1, \ldots, n_l\), we will denote this by

\[ (n_1, \ldots, n_l; m_1, \ldots, m_k). \]

If \(\partial|Q| = \emptyset\), then there can be no mirrors or corner reflectors and we simply write \((n_1, \ldots, n_l)\).
Recall that closed surfaces are classified by orientability and Euler characteristic:

- $\chi(M^2) > 0 \implies M^2 = S^2$ or $\mathbb{R}P^2$ (positive curvature).
- $\chi(M^2) = 0 \implies M^2 = T^2$ or the Klein bottle (flat).
- $\chi(M^2) < 0 \implies$ arbitrary genus $> 1$ (negative curvature).

The idea is to classify orbifolds $Q^2$ by their Euler characteristics. Since $\chi^{orb}(\ )$ is multiplicative under finite covers, this will tell us which manifolds can finitely cover a given orbifold. For example, if $Q = S^2 \sqcup G$, with $G$ finite, then $\chi^{orb}(S^2 \sqcup G) > 0$. Conversely, if $Q$ is developable and $\chi^{orb}(Q) > 0$, then its universal cover is $S^2$. 
Exercise

List the 2-dim orbiflds $Q$ with $\chi^{orb}(Q) \geq 0$.

Sample calculation

Suppose $|Q| = D^2$ with $(; m_1, \ldots, m_k)$. Recall

$$\chi^{orb}(Q) = 1 - \frac{1}{2} \sum_{i=1}^{k} (1 - (m_i)^{-1})$$

Since $1 - (m_i)^{-1} \geq 1/2$, we see that if $k \geq 4$, then $\chi^{orb}(Q) \leq 0$ with equality iff $k = 4$ and all $m_i = 2$. Hence, if $\chi^{orb}(Q) > 0$ then $k \leq 3$. 
More calculations

Suppose $|Q| = D^2$ and $k = 3$ (st $Q$ is a triangle). Then

$$
\chi^{orb}(Q) = \frac{1}{2}(-1 + (m_1)^{-1} + (m_2)^{-1} + (m_3)^{-1})
$$

So, as $(\pi/m_1 + \pi/m_2 + \pi/m_3)$ is $>,$ $=,$ or $< \pi$, $\chi^{orb}(Q)$ is, respectively, $>,$ $=,$ or $< 0$. For $\chi^{orb} > 0$ we see the only possibilities are: $(; 2, 2, m), (; 2, 3, 3), (; 2, 3, 4), (; 2, 3, 5)$. The last 3 correspond to the symmetry gps of the Platonic solids. For $\chi^{orb} = 0$, the only possibilities are: $(; 2, 3, 6), (; 2, 4, 4)$ $(; 3, 3, 3)$.
Euler characteristics
Classification of 2-orbifolds
Spaces of constant curvature
Geometric reflection groups

$\chi^{orb}(Q) > 0$

Nondevelopable orbifolds:
- $|Q| = D^2$: ( ; m), ( ; m_1, m_2) with $m_1 \neq m_2$.
- $|Q| = S^2$: (n), (n_1, n_2) with $n_1 \neq n_2$.

Spherical orbifolds:
- $|Q| = D^2$: ( ; ), ( ; m, m), ( ; 2, 2, m), ( ; 2, 3, 3), ( ; 2, 3, 4), ( ; 2, 3, 5), (2; m), (3; 2).
- $|Q| = S^2$: ( ), (n, n), (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5).
- $|Q| = \mathbb{R}P^2$: ( ), (n)
The list of 2-dim spherical orbifolds is the list of finite subgroups of $O(3)$.

Every 3-dim orbifold is locally isomorphic to the cone on one of the spherical 2-orbifolds.

For example, if $|Q| = S^2$ with 3 cone points, $(n_1, n_2, n_3)$, then $\text{Cone}(Q)$ has underlying space an open 3-disk. The 3 cone points yield 3 codim 2 strata labeled $m_1, m_2, m_3$ and the origin is labeled by the corresponding finite subgroup of $O(3)$. 
The 17 wallpaper groups

Flat orbifolds: $\chi^{orb}(Q) = 0$

- $|Q| = D^2: (2, 3, 6), (2, 4, 4), (3, 3, 3), (2, 2, 2, 2), (2, 2, 2), (3, 3), (4, 2), (2, 2; )$.
- $|Q| = S^2: (2, 3, 6), (2, 4, 4), (3, 3, 3), (2, 2, 2, 2)$.
- $|Q| = \mathbb{R}P^2: (2, 2)$.
- $|Q| = T^2: ( )$.
- $|Q| = $ Klein bottle: ( )
- $|Q| = $ annulus: ( ; )
- $|Q| = $ Möbius band: ( ; ).
\( \chi^{orb}(Q) < 0 \)

It turns out that all remaining 2-dim orbifolds are developable and can be given a hyperbolic structure. The triangular orbifolds (ie, \( |Q| = D^2 \); \( m_1, m_2, m_3 \)) have a unique hyperbolic structure. The others have a positive dimensional moduli space.
In each dimension $n$, there are 3 simply connected spaces of constant curvature: $S^n$ (the sphere), $\mathbb{E}^n$ (Euclidean space) and $\mathbb{H}^n$ (hyperbolic space).

**Minkowski space**

Let $\mathbb{R}^{n,1}$ denote $\mathbb{R}^{n+1}$ equipped with the indefinite symmetric bilinear form:

$$\langle x, y \rangle := x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}.$$ 

The hypersurface defined by $\langle x, x \rangle = -1$ is a hyperboloid of two sheets. The component with $x_{n+1} > 0$ is $\mathbb{H}^n$. 

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Orbifolds 2
Riemannian metric on $\mathbb{H}^n$

As in the case of a sphere, given $x \in \mathbb{H}^n$, $T_x\mathbb{H}^n = x^\perp$. Since $\langle x, x \rangle < 0$, the restriction of $\langle \cdot, \cdot \rangle$ to $T_x$ is positive definite. So this defines a Riem metric on $\mathbb{H}^n$. It turns out this metric has constant curvature $-1$. 
Suppose $G$ is a gp of isometries acting real analytically on a mfld $X$. (The only examples we will be concerned with are $X^n = S^n$, $E^n$ or $H^n$ and $G$ the full isometry group.)

By a $(G, X)$-structure we mean that each of the charts $(\tilde{U}, H, U, \pi)$ has $\tilde{U} \subset X$, that $H$ is a finite subgp of $G$ and the overlap maps (= compatibility maps) are required to be restrictions of isometries in $G$. 
Convex polytopes in $\mathbb{X}^n$

- A hyperplane or half-space in $\mathbb{S}^n$ or $\mathbb{H}^n$ is the intersection of a linear hyperplane or half-space with the hypersurface. The unit normal vector $u$ to a hyperplane means that the hyperplane is the orthogonal complement, $u^\perp$, of $u$ (orthogonal wrt the standard bilinear form, in the case of $\mathbb{S}^n$, or the form $\langle , \rangle$, in the case of $\mathbb{H}^n$).

- A half-space in $\mathbb{H}^n$ bounded by the hyperplane $u^\perp$ is a set of the form $\{x \in \mathbb{H}^n \mid \langle u, x \rangle \geq 0\}$ and similarly, for $\mathbb{S}^n$.

- A convex polytope in $\mathbb{S}^n$ or $\mathbb{H}^n$ is a compact intersection of a finite number of half-spaces.
Suppose $u$ is unit vector in $\mathbb{R}^{n+1}$. Reflection across the hyperplane $u \perp$ (either in $\mathbb{R}^{n+1}$ or $S^n$) is given by

$$x \mapsto x - 2(x \cdot u)u.$$ 

Similarly, suppose $u \in \mathbb{R}^{n,1}$ satisfies $\langle u, u \rangle = 1$. Reflection across the hyperplane $u \perp$ in $H^n$ is given by

$$x \mapsto x - 2\langle x, u \rangle u.$$
Suppose $K$ is a convex polytope in $\mathbb{X}^n (= \mathbb{S}^n, \mathbb{E}^n$ or $\mathbb{H}^n)$ such that if two codim 1 faces have nonempty intersection, then the dihedral angle between them has form $\pi/m$ for some integer $m \geq 2$. (This condition is familiar: it means that each codim 2 face has the structure of a codim 2 corner reflector.) Let $W$ be the subgp of $\text{Isom}(\mathbb{X}^n)$ generated by reflections across the codim 1 faces of $K$.

Some basic facts

- $W$ is discrete and acts properly on $\mathbb{X}^n$
- $K$ is a strict fundamental domain in the sense that the restriction to $K$ of the orbit map, $p : \mathbb{X}^n \to \mathbb{X}^n/W$, is a homeomorphism. It follows that $\mathbb{X}^n/W \cong K$ and hence, $K$ is an orbifold with an $\mathbb{X}^n$-structure.

(Neither fact is obvious.)
- In 1852 Möbius determined the finite subgroups of $O(3)$ generated by isometric reflections on the 2-sphere.
- The fundamental domain for such a group on the 2-sphere was a spherical triangle with angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$, with $p, q, r$ integers $\geq 2$.
- Since the sum of the angles is $> \pi$, we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.
- For $p \geq q \geq r$, the only possibilities are: $(p, 2, 2)$ for any $p \geq 2$ and $(p, 3, 2)$ with $p = 3, 4$ or $5$. The last three cases are the symmetry groups of the Platonic solids.
- Later work by Riemann and Schwarz showed there were discrete gps of isometries of $E^2$ or $H^2$ generated by reflections across the edges of triangles with angles integral submultiples of $\pi$. Poincaré and Klein: a similar result for polygons in $H^2$. 
In 2nd half of the 19th century work began on finite reflection gps on $S^n$, $n > 2$, generalizing Möbius’ results for $n = 2$. It developed along two lines.

- Around 1850, Schlafli classified regular polytopes in $\mathbb{R}^{n+1}$, $n > 2$. The symmetry group of such a polytope was a finite gp generated by reflections and as in Möbius’ case, the projection of a fundamental domain to $S^n$ was a spherical simplex with dihedral angles integral submultiples of $\pi$.

- Around 1890, Killing and E. Cartan classified complex semisimple Lie algebras in terms of their root systems. In 1925, Weyl showed the symmetry gp of such a root system was a finite reflection gp.

- These two lines were united by Coxeter in the 1930’s. He classified discrete groups reflection gps on $S^n$ or $E^n$. 
Let $K$ be a fundamental polytope for a geometric reflection gp. For $S^n$, $K$ is a simplex. For $E^n$, $K$ is a product of simplices. For $H^n$ there are other possibilities, eg, a right-angled pentagon in $H^2$ or a right-angled dodecahedron in $H^3$. 
Conversely, given a convex polytope $K$ in $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ st all dihedral angles have form $\pi$/integer, there is a discrete gp $W$ generated by isometric reflections across the codim 1 faces of $K$.

Let $S$ be the set of reflections across the codim 1 faces of $K$. For $s, t \in S$, let $m(s, t)$ be the order of $st$. Then $S$ generates $W$, the faces corresponding to $s$ and $t$ intersect in a codim 2 face iff $m(s, t) \neq \infty$, and for $s \neq t$, the dihedral angle along that face is $\pi/m(s, t)$. Moreover,

$$\langle S \mid (st)^{m(s,t)} \rangle, \text{ where } (s, t) \in S \times S$$

is a presentation for $W$. 
Lemma (Coxeter)

Suppose $K \subset \mathbb{S}^n$ is an $n$-dim convex polytope which is “proper” (meaning that it does not contain any pair of antipodal points). Further suppose that whenever two codim 1 faces intersect along a codim 2 face, the dihedral angle is $\leq \pi/2$. Then $K$ is a simplex.

A similar result holds for a polytope $K \subset \mathbb{E}^n$ which is not a product.
Corollary

The fundamental polytope for a spherical reflection gp is a simplex.

Proof.

For $m$ an integer $\geq 2$, we have $\pi/m \leq \pi/2$. 

Corollary

The fundamental polytope for a finite linear reflection gp on $\mathbb{R}^n$ is a simplicial cone.
Corollary

Suppose that a convex polytope $K \subset X^n$ is fund domain for reflection gp in $\text{Isom}(X^n)$ (where $X^n = S^n$, $E^n$ or $H^n$). Then $K$ is a simple polytope. (This means that exactly $n$ facets meet at each vertex.)

Corollary

Suppose $Q$ is an $n$-orbifold with all the local groups $\cong$ finite reflection gps on $R^n$. Then the underlying space of $Q$ is naturally a mfld with corners (meaning that it is locally modeled on the simplicial cone $[0, \infty)^n$).
Suppose $\sigma^n$ is a simplex in $X^n$. Let $u_0, \ldots, u_n$ be its inward pointing unit normal vectors. (The $u_i$ lie in $\mathbb{R}^{n+1}$, $\mathbb{R}^n$ or $\mathbb{R}^{n,1}$ as $X^n = S^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$.) The Gram matrix, $G$, of $\sigma$ is the symmetric $(n + 1) \times (n + 1)$ matrix $(g_{ij})$ defined by $g_{ij} = u_i \cdot u_j$. 

Gram matrix of a simplex in $X^n$
A symmetric matrix $G$ with 1’s on the diagonal is type

(1) if $G > 0$, 

(0) if $G$ is positive semidefinite with 1-dim kernel, each principal submatrix is $> 0$, and $\exists$ a vector $v \in \text{Ker } G$ with all its coordinates $> 0$. 

(-1) if $G$ has signature $(n, 1)$ and each principal submatrix is $> 0$. 

**Linear algebra fact**

The extra condition in type 0 (that Ker $G$ is spanned by a vector with positive coordinates) is automatic when $G$ is indecomposable and $g_{ij} \leq 0 \ \forall i \neq j$ (ie, when all dihedral angles are nonobtuse).
Theorem

Suppose $G$ is a symmetric $(n + 1) \times (n + 1)$ matrix with 1’s on the diagonal. Let $\varepsilon \in \{+1, 0, -1\}$. Then $G$ is the Gram matrix of a simplex $\sigma^n \subset X^n_{\varepsilon} \iff G$ is type $\varepsilon$.

Recall $X^n_{\varepsilon}$ is $\mathbb{S}^n$, $\mathbb{E}^n$, $\mathbb{H}^n$ as $\varepsilon = +1, 0, -1$.

Proof.

For $\mathbb{S}^n$: we can find basis vectors $u_0, \ldots, u_n$ in $\mathbb{R}^{n+1}$, well-defined up to isometry, st $(u_i \cdot u_j) = G$. (This is because $G > 0$.) Since the $u_i$ form a basis, the half-spaces, $u_i \cdot x \geq 0$, intersect in a simplicial cone and the intersection of this with $\mathbb{S}^n$ is $\sigma^n$. 
Proof, continued.

The proof for $\mathbb{H}^n$ is similar. For $\mathbb{E}^n$, the argument has some additional complications.
Suppose $\sigma^n \subset X^n$ is fund simplex for a geometric reflecton gp. Let \( \{u_0, \ldots, u_n\} \) be the inward-pointing unit normal vectors. Then

$$u_i \cdot u_j = -\cos\left(\frac{\pi}{m_{ij}}\right)$$

where \((m_{ij})\) is a symmetric matrix of posiive integers with 1’s on the diagonal and of-diagonal entries \(\geq 2\).

\((m_{ij})\) is called the Coxeter matrix while the matrix \(\cos(\pi/m_{ij})\) is the associated cosine matrix.

The formula above says: Gram matrix = cosine matrix.
Suppose $M = (m_{ij})$ is a Coxeter matrix (a symmetric $(n + 1) \times (n + 1)$ matrix with 1’s on diagonal and off-diagonals $\geq 2$). Sometimes we allow the off-diagonal $m_{ij}$ to $= \infty$, but not here.

**Theorem**

*Let $M$ be a Coxeter matrix as above and $C$ its associated cosine matrix (ie, $c_{ij} = -\cos(\pi / m_{ij})$. Then there is a geometric refl gp with fund simplex $\sigma^n \subset X^n_\varepsilon \iff C$ is type $\varepsilon$.***
So, the problem of determining the geometric reflection groups with fund polytope a simplex in $X^n_\varepsilon$ becomes the problem of determining the Coxeter matrices $M$ whose cosine matrix is type $\varepsilon$. This was done by Coxeter for $\varepsilon = 1$ or 0 and by Lannér for $\varepsilon = -1$. The information in a Coxeter diagram is best encoded by a “Coxeter diagram.”
Associated to \((W, S)\), there is a labeled graph \(\Gamma\) called its “Coxeter diagram.”

\[
\text{Vert}(\Gamma) := S.
\]

Connect distinct elements \(s, t\) by an edge iff \(m(s, t) \neq 2\). Label the edge by \(m(s, t)\) if this is \(> 3\) or \(= \infty\) and leave it unlabeled if it is \(= 3\). \((W, S)\) is **irreducible** if \(\Gamma\) is connected. (The components of \(\Gamma\) give the irreducible factors of \(W\).)

The next slide shows Coxeter’s classification of irreducible spherical and cocompact Euclidean reflection gps.
Euler characteristics
Classification of 2-orbifolds
Spaces of constant curvature
Geometric reflection groups

Spherical Diagrams

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<th>Group</th>
<th>Diagram</th>
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<tbody>
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<td>$A_n$</td>
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<td>$B_n$</td>
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<td>$D_n$</td>
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<td>$I_2(p)$</td>
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<td>$H_3$</td>
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<td>$F_4$</td>
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Euclidean Diagrams

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<tbody>
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<td>$\tilde{A}_n$</td>
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<td>$\tilde{E}_8$</td>
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The next slide shows Lannér’s classification of hyperbolic reflection gps with fund polytope a simplex in $\mathbb{H}^n$.

Exercise

Derive Lannér’s list on the next slide from Coxeter’s lists on the previous slide.
Hyperbolic Simplicial Diagrams

\( n = 2 \)

\[
\begin{array}{c}
p \\ \downarrow \\ q \\ \downarrow \\ r
\end{array}
\]

with \((p^{-1} + q^{-1} + r^{-1}) < 1\)

\( n = 3 \)

\[
\begin{array}{c}
5 \\ 5 \quad 4 \\ 5 \quad 5 \\ 5
\end{array}
\]

\[
\begin{array}{c}
4 \\ 5 \\ 4
\end{array}
\]

\[
\begin{array}{c}
5 \\ 5 \\ 5 \\ 5
\end{array}
\]

\( n = 4 \)

\[
\begin{array}{c}
5 \\ 5 \quad 4 \\ 5 \quad 5 \\ 5
\end{array}
\]

\[
\begin{array}{c}
4 \\ 5 \\ 4
\end{array}
\]

\[
\begin{array}{c}
5
\end{array}
\]