Orbifolds 2

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https://people.math.osu.edu/davis.12/slides.html

- We know what is meant by the Euler characteristic of a closed mfld or a finite CW complex (the alternating sum of the number of cells). A key property is that it is multiplicative under finite covers: if $M' \to M$ is an m-fold cover, then
$$\chi(M') = m\,\chi(M).$$
- The Euler characteristic of an orbfld should be a rational number with same multiplicative property, ie, if $M \to Q$ is an m-fold cover and M is a mfld, then it should have $\chi(M) = m\,\chi^{orb}(Q)$. That is,
$$\chi^{orb}(Q) = \frac{1}{m}\,\chi(M).$$
("m-fold cover" means $\mathrm{Card}(p^{-1}(\text{generic pt})) = m$.)

The Euler characteristic of an orbifold

Suppose Q is an orbfld which cellulated as a CW complex so that the local gp is constant on each open cell c. Let $G(c)$ be the local gp at c and $|G(c)|$ its order.

$$\chi^{orb}(Q) := \sum_{\text{cells } c} \frac{(-1)^{\dim c}}{|G(c)|}$$

Exercise

Suppose $\Gamma \curvearrowright M$ properly, cocompactly, locally linearly and $\Gamma' \subset \Gamma$ is a subgp of index m. Show

$$\chi^{orb}(M/\!/\Gamma') = m\,\chi^{orb}(M/\!/\Gamma).$$

Alternate formula

Each stratum S of a compact orbifold Q is the interior of a compact mfld with bdry $\hat{S}$. Define $e(S) := \chi(\hat{S}) - \chi(\partial\hat{S})$.

$$\chi^{orb}(Q) = \sum_{\text{strata } S} \frac{e(S)}{|G(S)|}$$

Example

Suppose $|Q| = D^2$ and Q has k mirrors and k corner reflectors labeled $m_1, \ldots, m_k$. Then

$$\chi^{orb}(Q) = 1 - \frac{k}{2} + \left(\frac{1}{2m_1} + \cdots + \frac{1}{2m_k}\right) = 1 - \frac{1}{2}\sum_i \left(1 - \frac{1}{m_i}\right)$$

Example

Suppose $|Q| = S^2$ and Q has l cone points labeled $n_1, \dots, n_l$. Then

$$\chi^{orb}(Q) = 2 - l + \left(\frac{1}{n_1} + \cdots + \frac{1}{n_l}\right) = 2 - \sum_i \left(1 - \frac{1}{n_i}\right)$$

(This is twice the previous example, as it should be.)

Example (The general formula)

Suppose $|Q|$ is a surface with bdry, Q has k corner reflectors labeled $m_1, \dots, m_k$ and l cone points labeled $n_1, \dots, n_l$. Then

$$\chi^{orb}(Q) = \chi(|Q|) - \frac{1}{2}\sum_{i=1}^{k}\left(1 - \frac{1}{m_i}\right) - \sum_{i=1}^{l}\left(1 - \frac{1}{n_i}\right).$$

Remark

$\chi^{orb}(Q) \leq \chi(|Q|)$ with equality iff there are no cone points or corner reflectors.

Notation

If a 2-dim orbfld has k corner reflectors labeled $m_1, \dots, m_k$ and l cone points labeled $n_1, \dots, n_l$, we will denote this by

$$(n_1, \dots, n_l; m_1, \dots, m_k).$$

If $\partial|Q| = \emptyset$, then there can be no mirrors or corner reflectors and we simply write $(n_1, \dots, n_l)$.

Recall that closed surfaces are classified by orientability and Euler characteristic:

- $\chi(M^2) > 0 \implies M^2 = S^2$ or $\mathbf{R}P^2$ (positive curvature).
- $\chi(M^2) = 0 \implies M^2 = T^2$ or the Klein bottle (flat).
- $\chi(M^2) < 0 \implies$ arbitrary genus > 1 (negative curvature).

The idea is to classify orbflds Q^2 by their Euler characteristics. Since $\chi^{orb}(\ )$ is multiplicative under finite covers, this will tell us which mflds can finitely cover a given orbfld. For example, if $Q = S^2 /\!/ G$, with G finite, then $\chi^{orb}(S^2 /\!/ G) > 0$. Conversely, if Q is developable and $\chi^{orb}(Q) > 0$, then its universal cover is S^2.

Exercise

List the 2-dim orbflds Q with $\chi^{orb}(Q) \geq 0$.

Sample calculation

Suppose $|Q| = D^2$ with $(\,; m_1, \dots, m_k)$. Recall

$$\chi^{orb}(Q) = 1 - \frac{1}{2}\sum_{i=1}^{k}(1 - (m_i)^{-1})$$

Since $1 - (m_i)^{-1} \geq 1/2$, we see that if $k \geq 4$, then $\chi^{orb}(Q) \leq 0$ with equality iff $k = 4$ and all $m_i = 2$. Hence, if $\chi^{orb}(Q) > 0$ then $k \leq 3$.

More calculations

Suppose $|Q| = D^2$ and $k = 3$ (st Q is a triangle). Then

$$\chi^{orb}(Q) = \frac{1}{2}(-1 + (m_1)^{-1} + (m_2)^{-1} + (m_3)^{-1})$$

So, as $(\pi/m_1 + \pi/m_2 + \pi/m_3)$ is $>$, $=$ or $< \pi$, $\chi^{orb}(Q)$ is, respectively, $>$, $=$ or < 0. For $\chi^{orb} > 0$ we see the only possibilities are: $(; 2, 2, m)$, $(; 2, 3, 3)$, $(; 2, 3, 4)$, $(; 2, 3, 5)$. The last 3 correspond to the symmetry gps of the Platonic solids. For $\chi^{orb} = 0$, the only possibilities are: $(; 2, 3, 6)$, $(; 2, 4, 4)$ $(; 3, 3, 3)$.

$\chi^{orb}(Q) > 0$

- Nondevelopable orbifolds:
  - $|Q| = D^2$: $(\, ; m)$, $(\, ; m_1, m_2)$ with $m_1 \neq m_2$.
  - $|Q| = S^2$: (n), (n_1, n_2) with $n_1 \neq n_2$.
- Spherical orbifolds:
  - $|Q| = D^2$: $(\, ; \,)$, $(\, ; m, m)$, $(\, ; 2, 2, m)$, $(\, ; 2, 3, 3)$, $(\, ; 2, 3, 4)$, $(\, ; 2, 3, 5)$, $(2; m)$, $(3; 2)$.
  - $|Q| = S^2$: $(\,)$, (n, n), $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$.
  - $|Q| = \mathbf{R}P^2$: $(\,)$, (n)

Implications for 3-dim orbflds

- The list of 2-dim spherical orbflds is the list of finite subgps of $O(3)$.
- Every 3-dim orbfld is locally isomorphic to the cone on one of the spherical 2-orbflds.
- For example, if $|Q| = S^2$ with 3 cone points, (n_1, n_2, n_3), then Cone(Q) has underlying space an open 3-disk. The 3 cone points yield 3 codim 2 strata labeled m_1, m_2, m_3 and the origin is labeled by the corresponding fintie subgp of $O(3)$.

The 17 wallpaper groups

Flat orbifolds: $\chi^{orb}(Q) = 0$

- $|Q| = D^2$: $(;2,3,6)$, $(;2,4,4)$, $(;3,3,3)$, $(;2,2,2,2)$, $(2;2,2)$, $(3;3)$, $(4;2)$, $(2,2;)$.
- $|Q| = S^2$: $(2,3,6)$, $(2,4,4)$, $(3,3,3)$, $(2,2,2,2)$.
- $|Q| = \mathbf{R}P^2$: $(2,2)$,
- $|Q| = T^2$: $( )$.
- $|Q| =$ Klein bottle: $( )$.
- $|Q| =$ annulus: $(;)$.
- $|Q| =$ Möbius band: $(;)$.

$\chi^{orb}(Q) < 0$

It turns out that all remaining 2-dim orbflds are developable and can be given a hyperbolic structure.
The triangular orbifolds (ie, $|Q| = D^2$; $(\,; m_1, m_2, m_3)$ have a unique hyperbolic structure. The others have a positive dimensional moduli space.

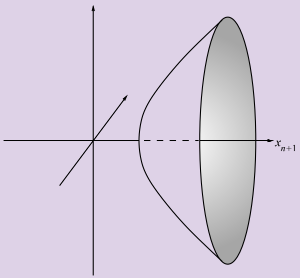
In each dimension n, there are 3 simply connected spaces of constant curvature: $\mathbb{S}^n$ (the sphere), $\mathbb{E}^n$ (Euclidean space) and $\mathbb{H}^n$ (hyperbolic space).

Minkowski space

Let $\mathbf{R}^{n,1}$ denote $\mathbf{R}^{n+1}$ equipped with the indefinite symmetric bilinear form:

$$\langle x, y \rangle := x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}.$$

The hypersurface defined by $\langle x, x \rangle = -1$ is a hyperboloid of two sheets. The component with $x_{n+1} > 0$ is $\mathbb{H}^n$.



Riemannian metric on $\mathbb{H}^n$

As in the case of a sphere, given $x \in \mathbb{H}^n$, $T_x\mathbb{H}^n = x^\perp$. Since $\langle x, x\rangle < 0$, the restriction of $\langle\ ,\ \rangle$ to T_x is positive definite. So this defines a Riem metric on $\mathbb{H}^n$. It turns out this metric has constant curvature -1.

Geometric structures on orbifolds

- Suppose G is a gp of isometries acting real analytically on a mfld $\mathbb{X}$. (The only examples we will be concerned with are $\mathbb{X}^n = \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ and G the full isometry group.)
- By a $(G, \mathbb{X})$*-structure* we mean that each of the charts $(\widetilde{U}, H, U, \pi)$ has $\widetilde{U} \subset \mathbb{X}$, that H is a finite subgp of G and the overlap maps ($=$ compatibility maps) are required to be restrictions of isometries in G.

Convex polytopes in $\mathbb{X}^n$

- A *hyperplane* or *half-space* in $\mathbb{S}^n$ or $\mathbb{H}^n$ is the intersection of a linear hyperplane or half-space with the hypersurface. The *unit normal vector u* to a hyperplane means that the hyperplane is the orthogonal complement, $u^\perp$, of u (orthogonal wrt the standard bilinear form, in the case of $\mathbb{S}^n$, or the form $\langle \, , \, \rangle$, in the case of $\mathbb{H}^n$).
- A *half-space* in $\mathbb{H}^n$ bounded by the hyperplane $u^\perp$ is a set of the form $\{x \in \mathbb{H}^n \mid \langle u, x \rangle \geq 0\}$ and similarly, for $\mathbb{S}^n$.
- A convex polytope in $\mathbb{S}^n$ or $\mathbb{H}^n$ is a compact intersection of a finite number of half-spaces.

Reflections in $\mathbb{S}^n$ and $\mathbb{H}^n$

Suppose u is unit vector in $\mathbf{R}^{n+1}$. Reflection across the hyperplane $u^\perp$ (either in $\mathbf{R}^{n+1}$ or $\mathbb{S}^n$) is given by

$$x \mapsto x - 2(x \cdot u)u.$$

Similarly, suppose $u \in \mathbf{R}^{n,1}$ satisfies $\langle u, u \rangle = 1$. Reflection across the hyperplane $u^\perp$ in $\mathbb{H}^n$ is given by

$$x \mapsto x - 2\langle x, u \rangle u.$$

Suppose K is a convex polytope in $\mathbb{X}^n$ ($= \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$) such that if two codim 1 faces have nonempty intersection, then the dihedral angle between them has form π/m for some integer $m \geq 2$. (This condition is familiar: it means that each codim 2 face has the structure of a codim 2 corner reflector.) Let W be the subgp of $\text{Isom}(\mathbb{X}^n)$ generated by reflections across the codim 1 faces of K.

Some basic facts

- W is discrete and acts properly on $\mathbb{X}^n$
- K is a strict fundamental domain in the sense that the restriction to K of the orbit map, $p : \mathbb{X}^n \to \mathbb{X}^n/W$, is a homeomorphism. It follows that $\mathbb{X}^n /\!\!/ W \cong K$ and hence, K is an orbifold with an $\mathbb{X}^n$-structure.

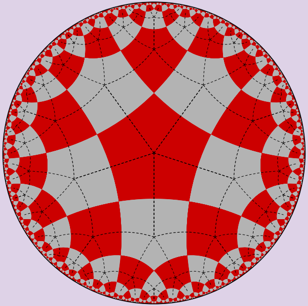
(Neither fact is obvious.)

- In 1852 Möbius determined the finite subgroups of $O(3)$ generated by isometric reflections on the 2-sphere.
- The fundamental domain for such a group on the 2-sphere was a spherical triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, $\frac{\pi}{r}$, with p, q, r integers ≥ 2.
- Since the sum of the angles is $> \pi$, we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.
- For $p \geq q \geq r$, the only possibilities are: $(p, 2, 2)$ for any $p \geq 2$ and $(p, 3, 2)$ with $p = 3$, 4 or 5. The last three cases are the symmetry groups of the Platonic solids.
- Later work by Riemann and Schwarz showed there were discrete gps of isometries of $\mathbb{E}^2$ or $\mathbb{H}^2$ generated by reflections across the edges of triangles with angles integral submultiples of π. Poincaré and Klein: a similar result for polygons in $\mathbb{H}^2$.

In 2^{nd} half of the 19^{th} century work began on finite reflection gps on $\mathbb{S}^n$, $n > 2$, generalizing Möbius' results for $n = 2$. It developed along two lines.

- Around 1850, Schläfli classified regular polytopes in $\mathbf{R}^{n+1}$, $n > 2$. The symmetry group of such a polytope was a finite gp generated by reflections and as in Möbius' case, the projection of a fundamental domain to S^n was a spherical simplex with dihedral angles integral submultiples of π.
- Around 1890, Killing and E. Cartan classified complex semisimple Lie algebras in terms of their root systems. In 1925, Weyl showed the symmetry gp of such a root system was a finite reflection gp.
- These two lines were united by Coxeter in the 1930's. He classified discrete groups reflection gps on $\mathbb{S}^n$ or $\mathbb{E}^n$.

Let K be a fundamental polytope for a geometric reflection gp. For $\mathbb{S}^n$, K is a simplex. For $\mathbb{E}^n$, K is a product of simplices. For $\mathbb{H}^n$ there are other possibilities, eg, a right-angled pentagon in $\mathbb{H}^2$ or a right-angled dodecahedron in $\mathbb{H}^3$.



- Conversely, given a convex polytope K in $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ st all dihedral angles have form π/integer, there is a discrete gp W generated by isometric reflections across the codim 1 faces of K.
- Let S be the set of reflections across the codim 1 faces of K. For $s, t \in S$, let $m(s,t)$ be the order of st. Then S generates W, the faces corresponding to s and t intersect in a codim 2 face iff $m(s,t) \neq \infty$, and for $s \neq t$, the dihedral angle along that face is $\pi/m(s,t)$. Moreover,
- $$\langle S \mid (st)^{m(s,t)}, \ \text{ where } (s,t) \in S \times S \rangle$$

  is a presentation for W.

Polytopes with nonobtuse dihedral angles

Lemma (Coxeter)

Suppose $K \subset \mathbb{S}^n$ is an n-dim convex polytope which is "proper" (meaning that it does not contain any pair of antipodal points). Further suppose that whenever two codim 1 faces intersect along a codim 2 face, the dihedral angle is $\leq \pi/2$. Then K is a simplex.

A similar result holds for a polytope $K \subset \mathbb{E}^n$ which is not a product.

Corollary

The fundamental polytope for a spherical reflection gp is a simplex.

Proof.

For m an integer ≥ 2, we have $\pi/m \leq \pi/2$. □

Corollary

The fund domain for a finite linear reflection gp on $\mathbf{R}^n$ *is a simplicial cone.*

Corollary

Suppose that a convex polytope $K \subset \mathbb{X}^n$ is fund domain for reflection gp in $\mathrm{Isom}(\mathbb{X}^n)$ *(where $\mathbb{X}^n = \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$). Then K is a simple polytope. (This means that exactly n facets meet at each vertex.)*

Corollary

Suppose Q is an n-orbifold with all the local groups $\cong$ finite reflection gps on $\mathbf{R}^n$. Then the underlying space of Q is naturally a mfld with corners (meaning that it is locally modeled on the simplicial cone $[0, \infty)^n$).

Gram matrix of a simplex in $\mathbb{X}^n$

Suppose σ^n is a simplex in $\mathbb{X}^n$. Let $u_0, \dots u_n$ be its inward pointing unit normal vectors. (The u_i lie in $\mathbf{R}^{n+1}$, $\mathbf{R}^n$ or $\mathbf{R}^{n,1}$ as $\mathbb{X}^n = \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$.) The *Gram matrix*, G, of σ is the symmetric $(n+1) \times (n+1)$ matrix (g_{ij}) defined by $g_{ij} = u_i \cdot u_j$.

A symmetric matrix G with 1's on the diagonal is type

(1) if $G > 0$,

(0) if G is positive semidefinite with 1-dim kernel, each principal submatrix is > 0, and $\exists$ a vector $v \in \operatorname{Ker} G$ with all its coordinates > 0.

(-1) if G has signature $(n, 1)$ and each principal submx is > 0.

Linear algebra fact

The extra condition in type 0 (that $\operatorname{Ker} G$ is spanned by a vector with positive coordinates) is automatic when G is indecomposable and $g_{ij} \leq 0 \ \forall i \neq j$ (ie, when all dihedral angles are nonobtuse).

Theorem

Suppose G is a symmetric $(n+1) \times (n+1)$ *matrix with* 1*'s on the diagonal. Let* $\varepsilon \in \{+1, 0, -1\}$*. Then G is the Gram matrix of a simplex* $\sigma^n \subset \mathbb{X}^n_\varepsilon \iff G$ *is type* ε*.*

Recall $\mathbb{X}^n_\varepsilon$ is $\mathbb{S}^n$, $\mathbb{E}^n$, $\mathbb{H}^n$ as $\varepsilon = +1$, 0, -1.

Proof.

For $\mathbb{S}^n$: we can find basis vectors $u_0, \dots u_n$ in $\mathbf{R}^{n+1}$, well-defined up to isometry, st $(u_i \cdot u_j) = G$. (This is because $G > 0$.) Since the u_i form a basis, the half-spaces, $u_i \cdot x \geq 0$, intersect in a simplicial cone and the intersection of this with $\mathbb{S}^n$ is σ^n.

Proof, continued.

The proof for $\mathbb{H}^n$ is similar. For $\mathbb{E}^n$, the argument has some additional complications. □

Suppose $\sigma^n \subset \mathbb{X}^n$ is fund simplex for a geometric reflecton gp. Let $\{u_0, \dots u_n\}$ be the inward-pointing unit normal vectors. Then

$$u_i \cdot u_j = -\cos(\pi/m_{ij})$$

where (m_{ij}) is a symmetric matrix of posiive integers with 1's on the diagonal and of-diagonal enries ≥ 2.
((m_{ij}) is called the *Coxeter matrix* while the matrix $(\cos(\pi/m_{ij}))$ is the associated *cosine matrix*.)
The formula above says: Gram matrix $=$ cosine matrix.

Suppose $M = (m_{ij})$ is a Coxeter matrix (a symmetric $(n+1) \times (n+1)$ matrix with 1's on diagonal and off-diagonals ≥ 2). Sometimes we allow the off-diagonal m_{ij} to $= \infty$, but not here.

Theorem

Let M be a Coxeter matrix as above and C its associated cosine matrix (ie, $c_{ij} = -\cos(\pi/m_{ij})$. Then there is a geometric refl gp with fund simplex $\sigma^n \subset \mathbb{X}^n_\varepsilon \iff C$ is type ε.

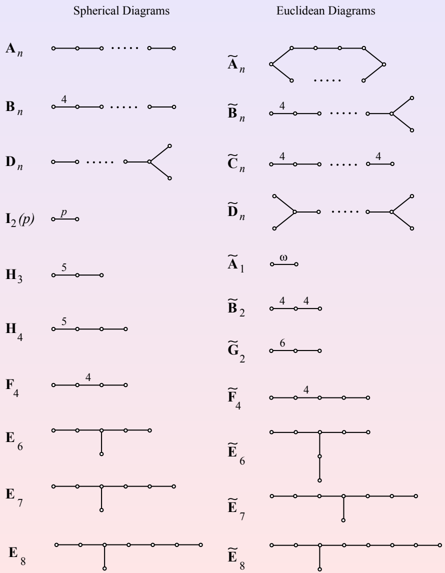
So, the problem of determining the geometric reflection gps with fund polytope a simplex in $\mathbb{X}^n_\varepsilon$ becomes the problem of determining the Coxeter matrices M whose cosine matrix is type ε. This was done by Coxeter for $\varepsilon = 1$ or 0 and by Lannér for $\varepsilon = -1$. The information in a Coxeter diagram is best encoded by a "Coxeter diagram."

Coxeter diagrams

Associated to (W, S), there is a labeled graph Γ called its "Coxeter diagram."

$$\mathrm{Vert}(\Gamma) := S.$$

Connect distinct elements s, t by an edge iff $m(s,t) \neq 2$. Label the edge by $m(s,t)$ if this is > 3 or $= \infty$ and leave it unlabeled if it is $= 3$. (W, S) is *irreducible* if Γ is connected. (The components of Γ give the irreducible factors of W.)
The next slide shows Coxeter's classification of irreducible spherical and cocompact Euclidean reflection gps.


Spherical Diagrams
Euclidean Diagrams
A n
B n
4
D n
I 2 (p)
p
H 3
5
H 4
5
F 4
4
E 6
E 7
E 8
Ã n
B̃ n
4
C̃ n
4
4
D̃ n
Ã 1
ω
B̃ 2
4
4
G̃ 2
6
F̃ 4
4
Ẽ 6
Ẽ 7
Ẽ 8

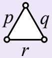
The next slide shows Lannér's classification of hyperbolic reflection gps with fund polytope a simplex in $\mathbb{H}^n$.

Exercise

Derive Lannér's list on the next slide from Coxeter's lists on the previous slide.

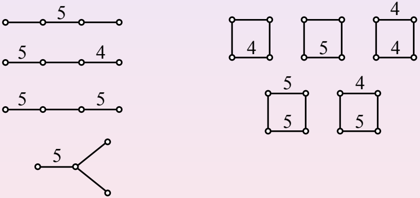
Hyperbolic Simplicial Diagrams

$n = 2$



with $(p^{-1} + q^{-1} + r^{-1}) < 1$

$n = 3$



$n = 4$