Reflection groups 3

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1. More reflection groups
   - Generalities on abstract reflection groups
   - Geometric reflection groups, again
2. Andreev's Theorem
   - Vinberg's Theorem
   - A dimension count
3. Contractibility
   - When is $\mathcal{U}(W, X)$ contractible?
   - When is $\mathcal{U}$ a contractible mfld?

Recall $\mathbb{X}^n$ stands for $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$. Let $K \subset \mathbb{X}^n$ be a convex polytope with dihedral angles between codim 1 faces of the form π/m, where m is an integer ≥ 2 or the symbol ∞ (where π/∞ means the faces do not intersect). W the gp generated by reflections across the codim 1 faces of K.

Goal

Show W is discrete, acts properly on $\mathbb{X}^n$ and that K is an orbfld with geometric structure of an $\mathbb{X}^n$-orbfld.

Coxeter systems

A Coxeter group is an "abstract reflection group."

Suppose W is a group and S a set of involutions which generate it. For each $s, t \in S$, let $m(s,t)$ denote the order of st. (W, S) is a *Coxeter system* (and W is a *Coxeter group*) if the group defined by the presentation:

$$\{\text{generators}\} = S$$
$$\{\text{relations}\} = \{(st)^{m(s,t)}\} \quad \text{where } (s,t) \in S \times S,\ m(s,t) \neq \infty$$

is isomorphic to W (via the natural map).

Equivalent definition

Let $\text{Cay}(W, S)$ be the Cayley graph of (W, S). Then (W, S) is a Coxeter system iff for each $s \in S$, the fixed point set of s separates $\text{Cay}(W, S)$.

Given a subset $T \subset S$, the subgp $W_T := \langle T \rangle$ is a *special subgroup*. T is a *spherical subset* if W_T is finite. Let $\mathcal{S}(W, S)$ (or simply $\mathcal{S}$ denote the poset of spherical subsets of S.

For each $T \subset S$, let W_T denote the subgp generated by T.

The subset $T \subset S$ is *spherical* if $|W_T| < \infty$. Let $\mathcal{S}$ denote the poset of spherical subsets of S.

Mirror structures

A *mirror structure* on a space X, indexed by a set S, is a family of closed subspaces $\{X_s\}_{s\in S}$. The X_s are called *mirrors*. For each $x \in X$, put $S(x) := \{s \in S \mid x \in X_s\}$.

Example

Suppose K is a convex polytope with its codim 1 faces indexed by S and K_s denotes the face corresponding to s. This is a mirror structure on K. $S(x)$ is the set of faces which contain x. (In particular, if x is in the interior of K, then $S(x) = \emptyset$.

The basic construction

Starting with a Coxeter system (W, S) and a mirror structure $\{X_s\}_{s \in S}$ we are going to define a new space $\mathcal{U}(W, X)$ with W-action. The idea is to paste together copies of X one for each element in W. Each copy of X will be a fundamental domain and will be called a "chamber."

Define an equivalence relation $\sim$ on $W \times X$ by

$$(w, x) \sim (w', x') \iff x = x' \text{ and } wW_{S(x)} = w'W_{S(x)}.$$

(Here W has the discrete topology.) Put

$$\mathcal{U}(W, X) = (W \times X)/\sim .$$

Let $\mathcal{U} = \mathcal{U}(W, X)$. Denote the image of (w, x) in $\mathcal{U}$ by $[w, x]$.

Some properties of the construction

- $W \curvearrowright \mathcal{U}$ via $u[w, x] = [uw, x]$. The isotropy subgp at $[w, x]$ is $wW_{S(x)}w^{-1}$.
- We can identify X with the image of $1 \times X$ in $\mathcal{U}$. X is a strict fundamental domain for the W-action in the sense that the restriction of the orbit map $\mathcal{U} \to \mathcal{U}/W$ to X is a homeomorphism (ie, $\mathcal{U}/W = X$).
- $W \curvearrowright \mathcal{U}$ properly $\iff$ X is Hausdorff and each $W_{S(x)}$ is finite (ie, if $\bigcap_{s \in T} X_s = \emptyset$, whenever $|W_T| = \infty$).

Universal property

Suppose $W \curvearrowright Z$ and $f : X \to Z$ is a map st $\forall s \in S$, $f(X_s) \subset Z^s$. (Z^s denotes the fixe set of s on Z.)
Then there is a unique extension to a W-equivariant map $\tilde{f} : \mathcal{U}(W, X) \to Z$. In fact, $\tilde{f}$ is defined by $\tilde{f}([w, x]) = wf(x)$.

Exercise

Prove the above property and those on the previous slide.

The set up:

- K is a convex polytope in $\mathbb{X}^n$ ($= \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$). S is the set of reflections across the codim 1 faces of K. The face corresponding to s is denoted K_s.
- If $K_s \cap K_t \neq \emptyset$, then it is a codim 2 stratum and the dihedral angle is $\pi/m(s,t)$, where $m(s,t)$ is some integer ≥ 2. (We know this implies K is a simple polytope.) If $K_s \cap K_t = \emptyset$, then put $m(s,t) = \infty$.
- Let $\overline{W} \subset \mathrm{Isom}(\mathbb{X}^n)$ be the subgp generated by S.
- Let W be the gp defined by the presentation corresponding to the ($S \times S$) Coxeter matrix, $(m(s,t))$, It turns out that (W,S) is a Coxeter system. (There is something to prove here.) Let $p : W \to \overline{W}$ be the natural surjection.

By the universal property, the inclusion $\iota : K \hookrightarrow \mathbb{X}^n$ induces a W-equivariant map $\tilde{\iota} : \mathcal{U}(W, X) \to \mathbb{X}^n$.

Theorem

$\tilde{\iota} : \mathcal{U}(W, K) \to \mathbb{X}^n$ is a W-equivariant homeomorphism.

Some consequences

- $p : W \to \overline{W}$ is an isomorphism
- $\overline{W}$ is discrete and acts properly on $\mathbb{X}^n$
- K is a strict fund domain for the action on $\mathbb{X}^n$ (ie, $\mathbb{X}^n/W = K$).

More consequences

- $\mathcal{U}(W, K)$ is a mfld (because $\mathbb{X}^n$ is a mfld).
- K is an $\mathbb{X}^n$-orbfld (because it is $= \mathbb{X}^n /\!/ W$).
- If $W' \curvearrowright \mathbf{R}^n$ as a finite linear gp, then $\mathbf{R}^n /\!/ W'$ is isomorphic to the fundamental simplicial cone.

Sketch of proof of the theorem.

By induction on the dimension n. A nbhd of a point in K looks like cone over the suspension of a spherical simplex which has an $\mathbb{S}^{n-1}$-structure by induction. It follows that $\tilde{\imath} : \mathcal{U}(W, K) \to \mathbb{X}^n$ is a local homeo and a covering projection. Since $\mathbb{X}^n$ is simply connected, we are done. (The case $\mathbb{X}^n = \mathbb{S}^1$ is handled separately.) $\square$

A geometric reflection group on $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ is determined by its fundamental polytope. In the spherical case the fundamental polytope must be a simplex and in the Euclidean case it must be a a product of simplices. There is nothing more to said in the spherical and Euclidean cases.

In the hyperbolic case we know what happens in dim 2: the fundamental polygon can be a k-gon for any $k \geq 3$ and almost any assignment of angles can be realized by a hyperbolic polygon (there are a few exceptions when $k = 3$ or 4). What happens in dim 3?

There is a beautiful theorem due to Andreev, which gives a complete answer.

Roughly, it says given a simple polytope K, for it to be the fund polytope of a hyperbolic reflection gp,

- there is no restriction on its combinatorial type
- subject to the condition that the gp at each vertex be finite, almost any assignment of dihedral angles to the edges of K can be realized (provided a few simple inequalities hold).

In contrast to dim 2, the 3-dim hyperbolic polytope is uniquely determined, up to isometry, by its dihedral angles – the moduli space is a point (because of the Mostow Rigidity Theorem).

Theorem (Thurston's Conjecture, Perelman's Theorem)

A closed 3-orbfld Q^3 with infinite π_1^{orb} admits a hyperbolic structure iff it satisfies the following two conditions:

- *Q^3 is developable.*
- *Every embedded 2-dim spherical suborbfld bounds a quotient of a 3-ball in Q^3 ($\Longrightarrow$ Q^3 is aspherical).*
- *There is no incompressible 2-dim Euclidean suborbfld in Q^3 (i.e., Q^3 is "atoroidal").*

("Incompressible" means induces an injection on $\pi_1^{orb}(\ )$.)

Theorem (Andreev ∼1967)

Suppose K is (the combinatorial type of) a simple 3-dim polytope, different from a tetrahedron. E is its edge set and $\theta : E \to (0, \pi/2]$ *any function. Then* (K, θ) *can be realized as a convex polytope in* $\mathbb{H}^3$ *with dihedral angles as prescribed by* θ *if and only if the following conditions hold:*

- *At each vertex, the angles at the three edges* e_1, e_2, e_3 *which meet there satisfy* $\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi$.
- *If three faces intersect pairwise but do not have a common vertex, then the angles at the three edges of intersection satisfy* $\theta(e_1) + \theta(e_2) + \theta(e_3) < \pi$.
- *Four faces cannot intersect cyclically with all four angles* $= \pi/2$ *unless two of the opposite faces also intersect.*
- *If K is a triangular prism the angles along base and top cannot all be* $\pi/2$.

Moreover, when (K, θ) is realizable, it is unique up to an isometry of $\mathbb{H}^3$.

Corollary

Suppose K is (the combinatorial type of) a simple 3-polytope, different from a tetrahedron, that $\{F_s\}_{s \in S}$ is its set of codim 1 faces and that e_{st} is the edge $F_s \cap F_t$ (when $F_s \cap F_t \neq \emptyset$). Given an angle assignment $\theta : E \to (0, \pi/2]$, with $\theta(e_{st}) = \pi/m(s,t)$ and $m(s,t)$ an integer ≥ 2, then (K, θ) is a hyperbolic orbfld iff the $\theta(e_{st})$ satisfy Andreev's Conditions. Moreover, the geometric reflection gp W is unique up to conjugation in $\text{Isom}(\mathbb{H}^3)$.

Examples

- K is a dodecahedron with all dihedral angles $= \pi/2$.
- K is a cube with disjoint edges in different directions labeled by integers > 2 and all other edges labeled 2

Exercise

Make up your own examples.

Theorem (Vinberg)

Suppose $K \subset \mathbb{H}^n$ is a compact, convex polytope with all dihedral angles integral submultiples of π. Then $n \leq 28$. (So,cocompact reflection gps do not exist on hyperbolic spaces of dimension ≥ 29.)

Remark

Examples of cocompact hyperbolic reflection gps are known only when $n \leq 8$. (I'm not sure that the highest known dimension is 8.)

Summary of cocompact geometric reflection gps

They are completely classified in the spherical and Euclidean cases: the irreducible gps are always simplicial reflection gps. In the Euclidean case, the fundamental polytope is a product of simplices. In the case of $\mathbb{H}^n$, there is a rich theory and complete classification for $n = 2$ or 3; there are relatively few known examples for $4 \leq n \leq 8$ and hyperbolic reflection gps do not exist for $n \geq 29$.

Dual form of Andreev's Theorem

Let L be the triangulation of S^2 dual to ∂K.

$$\begin{aligned} \text{Vert}(L) &\longleftrightarrow \text{Face}(K) \\ \text{Edge}(L) &\longleftrightarrow \text{Edge}(K) \\ \{\text{2-simplices in } L\} &\longleftrightarrow \text{Vert}(K) \end{aligned}$$

Input data

$\theta : \text{Edge}(L) \to (0, \pi/2]$

The condition that K have a spherical link at each vertex: if e_1, e_2, e_3 are the edges of a triangle, then $\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi$.

Theorem (Dual form of Andreev's Thm)

Suppose L is a triangulation of S^2 and $L \neq \partial\Delta^3$.
$\theta : \mathrm{Edge}(L) \to (0, \pi/2]$. *Then dual polytope K can be realized as convex polytope in* $\mathbb{H}^3$ *with prescribed dihedral angles* $\iff$

- *If e_1, e_2, e_3 are the edges of any triangle, then* $\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi$.
- *If e_1, e_2, e_3 are the edges of a 3-circuit $\neq \partial\Delta^2$, then* $\theta(e_1) + \theta(e_2) + \theta(e_3) < \pi$.
- *If e_1, e_2, e_3, e_4 are the edges of a 4-circuit $\neq$ to bdry of union of 2 adjacent triangles, then not all 4 $\theta(e_i)$ can* $= \pi/2$.
- *If L is suspension of $\partial\Delta^2$, then all "vertical" edges cannot have $\theta(e_i) = \pi/2$.*

Given a convex 3-dim polytope K, Andreev's Theorem asserts that a certain map θ from the space $C(K)$ of isometry classes convex polyhedra of the same combinatorial type as K to a certain open subset of $\mathbf{R}^E$ (where $E := \text{Edge}(K)$) is a homeomorphism. Let's compute $\dim C(K)$.

For each $F \in \text{Face}(K)$, let $u_F \in \mathbb{S}^{2,1}$ be the inward-pointing unit normal vector to F (Here $\mathbb{S}^{2,1} := \{x \in \mathbf{R}^{2,1} \mid \langle x, x \rangle = 1\}$). The $(u_F)_{F \in \text{Face}(K)}$ determine K (since K is the intersection of the half-spaces determined by the u_F). The assumption that K is simple means that the hyperbolic hyperplanes normal to the u_F intersect in general position. So, a slight perturbation of the u_F will not change the combinatorial type of K. That is to say, the set of $\mathcal{F}$-tuples (u_F) which define a polytope combinatorially equivalent to K is an open subset Y of $(\mathbb{S}^{2,1})^{\text{Face}(K)}$.

dim $C(K)$

- $f = \#(\text{Face}(K))$, $e = \#(\text{Edge}(K)$, $v = \#(vertex(K))$
- $\text{Isom}(\mathbb{H}^3) = O(3,1)$, $\dim(O(3,1) = 6$, and $dim\mathbb{S}^{3,1} = 3$
- Previous page $\implies \dim C(K) = 3f - 6$

Since $f - e + v = 2$, $3f - 6 = 3e - 3v$. Since 3 edges meet at each vertex, $3v = 2e$.
$\therefore\ 3f - 6 = 3e - 3v = e$.

So, $\theta : C(K) \to \mathbf{R}^E$ is a map between mflds (with bdry) of the same dimension.

Recall X is a space with a mirror structure $\{X_s\}_{s \in S}$ and

$$\mathcal{U}(W, X) = (W \times X)/\sim .$$

When Is X an orbifold?

X must be a mfld with corners and each mirror must be a codimension-one stratum. (The point being that a fundamental chamber for a spherical reflection gp is a simplex and so, on $\mathbf{R}^n$, the fundametal chamber of a finite reflection gp is a simplicial cone. By definition, a mfld with corners is a space which is locally differentiably modeled on a simplicial cone.) Moreover, the mirror structure must be proper.

Theorem

$\mathcal{U}(W, X)$ is simply connected $\iff$

- *X is simply connected.*
- *For each $s \in S$, X_s is nonempty and path connected.*
- *For each $\{s, t\} \in \mathcal{S}^{(2)}$ (ie for $m(s, t) \neq \infty$), we have $X_{\{s,t\}} \neq \emptyset$.*

Let $\mathcal{S}$ be the set of spherical subsets of S (ie, $\mathcal{S} = \{T \subset S \mid W_T \text{ is finite}\}$. Recall that for each $T \subset S$, $X_T := \bigcap_{s \in T} X_s$.

Theorem

$\mathcal{U}(W, X)$ is contractible $\iff$

- *X is contractible.*
- *For each $T \in \mathcal{S}$, X_T is acyclic.*

- Since $\mathcal{U}$ $(= \mathcal{U}(W, X)$ is a mfld with locally linear W-action, X must be an orbifold, ie, X is a mfld with corners.
- Since $\mathcal{U}$ is contractible, X is a contractible mfld with bdry.
- A stratum of codimension k has the form X_T, where $T \in \mathcal{S}$ and $\#T = k$. Each such stratum is an acylic manifold with boundary (contained in ∂X).
- In other words, X looks a like a simple convex polytope up to homology. (X is a contractible "manifold with faces" and each face is acyclic.)