Reflection groups 4

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3 The canonical cell complex $\Sigma(W, S)$
Let \((W, S)\) be a Coxeter system. \(S\) is the poset of spherical subsets of \(S\). We want to see that \(S\) is the poset of simplicices of a simplicial cx, \(L(W, S)\), (or simply \(L\)), called the nerve of \((W, S)\).

- The vertex set of \(L\) is \(S\).
- A subset \(T \subseteq S\) spans a simplex \(\iff T\) is spherical (ie \(W_T\) is finite).
Question

Given a simplicial complex $L'$, when is $L' = L(W, S)$ for some Coxeter system $(W, S)$?

This should remind us of the dual form of Andreev’s Theorem. It says that the simplicies of $L'$ are completely determined by $(W, S)$, ie, by the $m(s, t)$. 
Definition (Flag complexes)

A simplicial complex $L'$ is a *flag cx* if given any finite collection of vertices which are pairwise connected by edges span a simplex. In other words, any complete subgraph of $L'$ is the 1-skeleton of a simplex in $L'$. (There are no empty triangles in $L'$. Gromov calls this the “no $\Delta$ condition.”)

Example

The barycentric subdivision of any cell cx is a flag cx.
Proposition

Given any flag cx $L$, there is a right-angled Coxeter system $(W, S)$ with $L(W, S) = L$.

“Right-angled” means all $m(s, t)$, for $s \neq t$, are $= 2$ or $\infty$.

Define the $m(s, t)$ and hence, $(W, S)$ by

$$m(s, t) = \begin{cases} 2, & \text{if } \{s, t\} \text{ is an edge of } L; \\ \infty, & \text{if not.} \end{cases}$$
X is a space with mirror structure. \( X_T = \bigcap_{s \in T} X_s \).

\( \mathcal{U}(W, X) = (W \times X) / \sim \). Recall \( W \) acts properly on \( \mathcal{U} \) \( \iff \) the mirror structure is proper, i.e., \( \iff X_T = \emptyset \) whenever \( T \notin S \).

If \( \mathcal{U} \) is contractible, then \( X_T \neq \emptyset \) (and \( X_T \) is acyclic) whenever \( T \in S \).

So, if \( \mathcal{U} \) is contractible and the action is proper, the pattern of nonempty intersections is completely determined by the nerve \( L(W, S) \).
Since $\mathcal{U} (= \mathcal{U}(W, X))$ is a mfld with locally linear $W$-action, $X$ must be an orbifold, ie, $X$ is a mfld with corners. 

Since $\mathcal{U}$ is contractible, $X$ is a contractible mfld with bdry. 

A stratum of codimension $k$ has the form $X_T$, where $T \in S$ and $\# T = k$. Each such stratum is an acyclic manifold with boundary (contained in $\partial X$).

In other words, $X$ looks a like a simple convex polytope up to homology. ($X$ is a contractible “manifold with faces” and each face is acyclic.)
Suppose, as above, $X$ is a compact manifold with corners, $\mathcal{U}(W, X)$ is a contractible $n$-manifold.

**Theorem**

Let $L = L(W, S)$, where $(W, S)$ is the Coxeter system associated to $X$. Then

- $H_*(L) \cong H_*(S^{n-1})$.
- $L$ is a polyhedral homology $(n-1)$-manifold, i.e., for each $\sigma \in L$,
  $$H_*(Lk(\sigma)) \cong H_*(S^{\text{codim } \sigma - 1}).$$

**Definition**

A simplicial complex with the above two properties is a *generalized homology $(n-1)$-sphere* (abbreviated a $\text{GHS}^{n-1}$).
Sketch of Proof.

$\partial X$ is covered by the faces $X_s$. The nerve of the cover is $L$. Since each face and each intersection of faces is acyclic, it follows that $L$ and $\partial X$ have the same homology, ie, the homology of $S^{n-1}$.

Similarly, if $\sigma = T$ is a simplex of $L$, then $\partial X_T$ has the homology of a sphere of dim $= \text{codim}\sigma - 1$; it is covered by faces of the form $X_{T \cup \{s\}}$ and the nerve of this cover is $\text{Lk}(\sigma)$. This proves the first statement.
If $L$ is a GHS$^{n-1}$ then it is possible to “dualize” it (or resolve it) to a contractible mfld with faces. Because of 4-dim problems it may only be possible to do this topologically; however, if each 3-dim link smoothly bounds a contractible 4 mfld, then one can find a smooth contractible $n$-mfld dual to $L$. 
For example suppose that $L$ is a PL manifold with the same homology as $S^{n-1}$ (ie, $L$ is a homology sphere). We can find a contractible manifold $X$ with $\partial X = L$. Triangulate $L$ and use the dual cells to give $X$ the structure of a manifold with faces in which each face is a disk. Thus, $X$ looks like a simple polytope except that its boundary need not be homeo to $S^{n-1}$, eg, when $n \geq 2$, it need not be simply connected.

Continuing along this line, after replacing $L$ by its barycentric subdivision, we can assume $L$ is a flag complex. Let $(W, S)$ be the associated right-angled Coxeter system and $\mathcal{U}(W, X)$ the associated contractible $n$-manifold.
Theorem

Suppose $L$ and $X$ are as on the previous page and that

\[ \pi_1(L) \neq 1. \]

Then $U(W, X)$ is a contractible mfld not homeo to $R^n \ (n \geq 4)$.

The reason is that $U(W, X)$ is not simply connected at $\infty$.

Remark

Suppose $\Gamma \subset W$ is a torsion-free subgp of finite index (such $\Gamma$ exist). Then $\Gamma$ acts freely on $U$ and $M = U/\Gamma$ is a closed aspherical mfld not covered by $R^n$. 
Definition

An $n$-dim orbifold $Q$ is a *reflectofold* if it is locally modeled on finite linear reflection groups $\sim \mathbb{R}^n$.

If $\mathcal{W} \sim \mathbb{R}^n$ as a finite reflection group, then $\mathbb{R}^n/\mathcal{W}$ is a simplicial cone, i.e., up to linear isomorphism it looks like $[0, \infty)^n$. It follows that the underlying space of a reflectofold $Q$ is a manifold with corners. Conversely, to give a manifold with corners the structure of a reflectofold, essentially all we need to do is label its codim 2 strata by integers $\geq 2$ in such a way that the strata of higher codim correspond to *finite* Coxeter groups.
If $|Q|$ is simply connected and $Q$ is developable, then any codim 2 stratum is contained in the closures of 2 distinct codim 1 strata. Otherwise we would have a nondevelopable suborbifold pictured to the right.

Similarly, developable $\implies$ if intersection of 2 codim 1 strata contains 2 distinct codim 1 strata, then they are labeled by the same integer.
Definition

An orbifold is *aspherical* if its universal cover is a contractible manifold.

Question

*Is it true that a contractible orbifold is automatically a manifold?*

This was recently answered by Lytchak.
Remark

A 2-dim orbifold $Q^2$ is aspherical $\iff \chi^{orb}(Q^2) \leq 0$.

Conjecture (Hopf, Chern, Thurston)

Suppose $Q^{2n}$ is a closed aspherical orbifold. Then $(-1)^n \chi^{orb}(Q^{2n}) \geq 0$. 

My favorite conjecture
Idea

Let $X$ be a compact aspherical manifold with boundary. Proceed as before: triangulate $\partial X$ as a flag complex $L$ and then take the dual cellulation to give $X$ the structure of a reflectofold. For example, $X$ could be a 2-dimensional orbifold with mirrors and corner reflectors on $\partial X$, but no cone points. Then $\mathcal{U}(W, X)$ should be aspherical and $X$ should be an aspherical reflectofold.

We will prove this later.
For a long time there we have known examples of groups $\pi$ with $B\pi$ a finite complex – such a $\pi$ is said to be type $F$. For example, many finite 2-complexes are known to be aspherical. On the other hand, some years ago must examples of aspherical mflds came from differential geometry or Lie gps. For example, the Cartan-Hadmard Thm asserts that the universal cover of a complete Riem mfld of nonpositive curvature is diffeo to $\mathbb{R}^n$.

The reflection gp trick gives us a method for producing many more examples of aspherical mflds. Given a gp $\pi$ of type $F$ we thicken $B\pi$ a mfld with bdry $X$ and then apply the reflection gp trick to get an aspherical mfld, whose $\pi_1$ retracts onto $\pi$. 
Notation

\[ U = U(W, X), \quad \tilde{U} \text{ its universal cover.} \]
Let \( \Gamma \) be a torsion-free subgp of finite index in \( W \).
Put \( M = U/\Gamma \) and \( \tilde{\Gamma} = \pi_1(M) \).
The quotient map \( U \to X \) induces a map \( r : M \to X \), which is a retraction.

Hence, \( r_\ast : \tilde{\Gamma} = \pi_1(M) \to \pi_1(X) \) is a retraction from the fundamental gp of a closed aspherical mfld onto the gp \( \pi_1(X) \).
∃ examples of closed aspherical manifolds $M$ such that

- $\pi_1(M)$ is not residually finite.
- $\pi_1(M)$ contains not finitely generated abelian subgroup $A$, e.g., $A = \mathbb{Z}[1/2]$.
- $\pi_1(M)$ has unsolvable word problem.
Theorem (D - Hausmann)

∃ examples of aspherical mflds M which not homotopy equivalent to any smooth mfld.

Sketch of Proof.

∃ example of topological aspherical $n$-mfld with bdry $X^n$ st that its Spivak normal fiber space (ie its homotopy normal bundle) does not admit a reduction to a linear vector bundle. For example, $X^n$ could be a thickening of a $m$-torus, eg, with $n = 13$ and $m = 4$.
Remark

With different techniques this can be improved to \( n \geq 4 \).
There are two constructions of $\Sigma$.

**Geometric realization of a poset**

Given a poset $\mathcal{P}$, let $\text{Flag}(\mathcal{P})$ denote the abstract simplicial complex with vertex set $\mathcal{P}$ and with simplices all finite, totally ordered subsets of $\mathcal{P}$. The geometric realization of $\text{Flag}(\mathcal{P})$ is denoted $|\mathcal{P}|$.

**First construction of $\Sigma$**

- Recall $\mathcal{S}$ is the poset of spherical subsets of $S$. The *fundamental chamber* $K$ is defined by $K := |\mathcal{S}|$. ($K$ is the cone on the barycentric subdivision of $L$.)
- Mirror structure: $K_s := |\mathcal{S}_{\geq \{s\}}|$.
- $\Sigma := \mathcal{U}(W, K)$. 
Second construction

Let $WS$ denote the disjoint union of all spherical cosets (partially ordered by inclusion):

$$WS := \bigsqcup_{T \in S} W/W_T$$

and $\Sigma := |WS|$.

Coxeter polytopes

Suppose $W_T$ is finite reflection gp on $\mathbb{R}^T$. Choose a point $x$ in the interior of fundamental simplicial cone and let $P_T$ be convex hull of $W_Tx$. $P_T$ is determined up to isometry once we specify the distance of $x$ from each bounding hyperplane.
There is a cell structure on $\Sigma$ with $\{\text{cells}\} = WS$.

This follows from the fact that the poset of cells in $P_T$ is $\cong W_TS_{\leq T}$. The cells of $\Sigma$ are defined as follows: the geometric realization of the subposet of cosets $\leq wW_T$ is $\cong$ the barycentric subdiv of $P_T$. 
Properties of this cell structure on $\Sigma$

- $W$ acts cellularly on $\Sigma$.
- $\Sigma$ has one $W$-orbit of cells for each spherical subset $T \in S$ and $\dim(\text{cell}) = \text{Card}(T)$.
- The 0-skeleton of $\Sigma$ is $W$.
- The 1-skeleton of $\Sigma$ is $\text{Cay}(W, S)$.
- The 2-skeleton of $\Sigma$ is the Cayley 2 complex of the presentation.
- If $W$ is right-angled, then each Coxeter cell is a cube and we have the cubical cell structure on $\tilde{P}_L$ discussed in the last lecture.
- Moussong: the induced piecewise Euclidean metric on $\Sigma$ is $\text{CAT}(0)$. 
More properties

- $\Sigma$ is contractible. (This follows from the fact it is CAT(0).
- The $W$-action is proper (by construction each isotropy subgp is conjugate to some finite $W_T$).
- $\Sigma/W = K$, which is compact (so the action is cocompact)
- If $W$ is finite, then $\Sigma$ is a Coxeter polytope

If $W$ is a geometric reflection gp on $\mathbb{X}^n = \mathbb{E}^n$ or $\mathbb{H}^n$, then $K$ can be identified with the fundamental polytope, $\Sigma$ with $\mathbb{X}^n$ and the cell structure is dual to the tessellation of $\Sigma$ by translates of $K$. 