Reflection groups 5

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1. The canonical cell complex $\Sigma(W, S)$
2. $\mathrm{CAT}(0)$-spaces

There are two constructions of Σ.

Geometric realization of a poset

Given a poset $\mathcal{P}$, let $\mathrm{Flag}(\mathcal{P})$ denote the abstract simplicial complex with vertex set $\mathcal{P}$ and with simplices all finite, totally ordered subsets of $\mathcal{P}$. The geometric realization of $\mathrm{Flag}(\mathcal{P})$ is denoted $|\mathcal{P}|$.

First construction of Σ

- Recall $\mathcal{S}$ is the poset of spherical subsets of S. The *fundamental chamber* K is defined by $K := |\mathcal{S}|$. (K is the cone on the barycentric subdivision of L.)
- Mirror structure: $K_s := |\mathcal{S}_{\geq \{s\}}|$.
- $\Sigma := \mathcal{U}(W, K)$.

Another way to understand K

Start with a simplex Δ, with one facet Δ_s, for each $s \in S$. Let $\Delta_T = \bigcap_{s \in T} \Delta_s$. Take the barycetric subdivision of Δ. Then K is the full subcomplex spanned by the barycenters of the Δ_T, for all spherical T spherical (ie for all T with W_T finite). The barycenter of Δ corresponds to $T = \emptyset$. (This description works whenever W is infinite.)

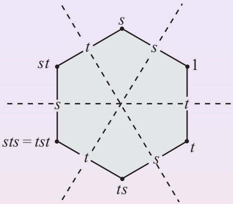
Second construction

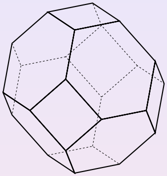
Let $W\mathcal{S}$ denote the disjoint union of all spherical cosets (partially ordered by inclusion:

$$W\mathcal{S} := \coprod_{T \in \mathcal{S}} W/W_T \quad \text{and} \quad \Sigma := |W\mathcal{S}|.$$

Coxeter polytopes

Suppose W_T is finite reflection gp on $\mathbf{R}^T$. Choose a point x in the interior of fundamental simplicial cone and let P_T be convex hull of $W_T x$. P_T is determined up to isometry once we specify the distance of x from each bounding hyperplane.





There is a cell structure on Σ with $\{\text{cells}\} = W\mathcal{S}$.

This follows from fact that poset of cells in P_T is $\cong W_T\mathcal{S}_{\leq T}$.
The cells of Σ are defined as follows: the geometric realization of subposet of cosets $\leq wW_T$ is $\cong$ barycentric subdiv of P_T.

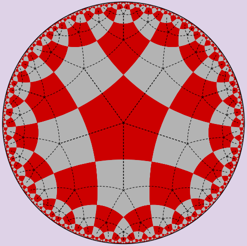
Properties of this cell structure on Σ

- W acts cellularly on Σ.
- Σ has one W-orbit of cells for each spherical subset $T \in \mathcal{S}$ and $\dim(\text{cell}) = \mathrm{Card}(T)$.
- The 0-skeleton of Σ is W
- The 1-skeleton of Σ is $\mathrm{Cay}(W, S)$.
- The 2-skeleton of Σ is the Cayley 2 complex of the presentation.
- If W is right-angled, then each Coxeter cell is a cube and we have the cubical cell structure on $\widetilde{P}_L$ discussed in the last lecture.
- Moussong: the induced piecewise Euclidean metric on Σ is $\mathrm{CAT}(0)$.

More properties

- Σ is contractible. (This follows from the fact it is $\mathrm{CAT}(0)$).
- The W-action is proper (by construction each isotropy subgp is conjugate to some finite W_T).
- $\Sigma/W = K$, which is compact (so the action is cocompact)
- If W is finite, then Σ is a Coxeter polytope

If W is a geometric reflection gp on $\mathbb{X}^n = \mathbb{E}^n$ or $\mathbb{H}^n$, then K can be identified with the fundamental polytope, Σ with $\mathbb{X}^n$ and the cell structure is dual to the tessellation of Σ by translates of K.



"$\mathrm{CAT}(0)$-space" is a term invented by Gromov.
Also, called "Hadamard space." Roughly, a space which is "nonpositively curved" and simply connected.

$$\begin{aligned} \text{C} &= \text{“Comparison” or “Cartan”} \\ \text{A} &= \text{“Aleksandrov”} \\ \text{T} &= \text{“Toponogov”} \end{aligned}$$

In the 1940's and 50's Aleksandrov introduced the notion of a "length space" and the idea of curvature bounds on length space. He was primarily interested in lower curvature bounds (defined by reversing the $\mathrm{CAT}(\kappa)$ inequality). He proved that a length metric on S^2 has nonnegative curvature iff it is isometric to the boundary of a convex body in $\mathbb{E}^3$. First Aleksandrov proved this result for nonnegatively curved piecewise Euclidean metrics on S^2, i.e., any such metric was isometric to the boundary of a convex polytope. By using approximation techniques, he then deduced the general result (including the smooth case) from this.

One of the first papers on nonpositively curved spaces was a 1948 paper of Busemann. The recent surge of interest in nonpositively curved polyhedral metrics was initiated by Gromov's seminal 1987 paper.

Some definitions

Let (X, d) be a metric space. A path $c : [a, b] \to X$ is a *geodesic* (or a *geodesic segment*) if $d(c(s), c(t)) = |s - t|$ for all $s, t \in [a, b]$. (X, d) is a *geodesic space* if any two points can be connected by a geodesic segment.

Given a path $c : [a, b] \to X$, its *length*, $l(c)$, is defined by

$$l(c) := \sup\{\sum_{i=1}^{n} d(c(t_{i-1}, t_i)\},$$

where $a = t_0 < t_1 < \cdots t_n = b$ runs over all possible subdivisions. The metric space (X, d) is a *length space* if

$$d(x, y) = \inf\{l(c) \mid c \text{ is a path from } x \text{ to } y\}.$$

(Here we allow ∞ as a possible value of d.) Thus, a length space is a geodesic space iff the above infimum is always realized and is $\neq \infty$.

The $\mathrm{CAT}(\kappa)$-inequality

For $\kappa \in \mathbf{R}$, $\mathbb{X}^2_\kappa$ is the simply connected, complete, Riemannian 2-manifold of constant curvature κ:

- $\mathbb{X}^2_0$ is the Euclidean plane $\mathbb{E}^2$.
- If $\kappa > 0$, then $\mathbb{X}^2_\kappa = \mathbb{S}^2$ with its metric rescaled so that its curvature is κ (i.e., it is the sphere of radius $1/\sqrt{\kappa}$).
- If $\kappa < 0$, then $\mathbb{X}^2_\kappa = \mathbb{H}^2$, the hyperbolic plane, with its metric rescaled.

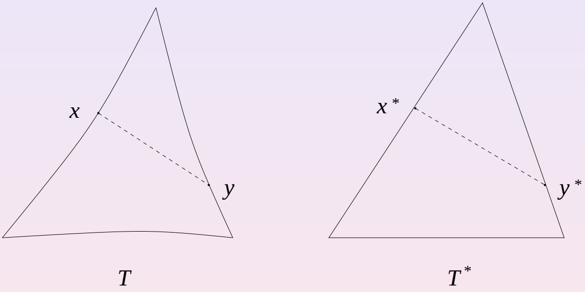
A *triangle* T in a metric space X is a configuration of three geodesic segments (the "edges") connecting three points (the "vertices") in pairs. A *comparison triangle* for T is a triangle T^* in $\mathbb{X}^2_\kappa$ with the same edge lengths. When $\kappa \leq 0$, a comparison triangle always exists. When $\kappa > 0$, a comparison triangle exists $\iff$ $l(T) \leq 2\pi/\sqrt{\kappa}$, where $l(T)$ denotes the sum of the lengths of the edges. (The number $2\pi/\sqrt{\kappa}$ is the length of the equator in a 2-sphere of curvature κ.)

If T^* is a comparison triangle for T, then for each edge of T there is a well-defined isometry, denoted $x \to x^*$, which takes the given edge of T onto the corresponding edge of T^*. A metric space X *satisfies* $\mathrm{CAT}(\kappa)$ (or is a $\mathrm{CAT}(\kappa)$*-space*) if the following two conditions hold:

- If $\kappa \leq 0$, then X is a geodesic space, while if $\kappa > 0$, it is required there be a geodesic segment between any two points $< \pi/\sqrt{\kappa}$ apart.
- (*The* $\mathrm{CAT}(\kappa)$ *inequality*). For any triangle T (with $l(T) < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$) and any two points $x, y \in T$, we have

$$d(x, y) \leq d^*(x^*, y^*),$$

  where x^*, y^* are the corresponding points in the comparison triangle T^* and d^* is distance in $\mathbb{X}^2_\kappa$.


x
y
T
x^*
y^*
T^*

Definition

A metric space X *has curvature* $\leq \kappa$ if the $\mathrm{CAT}(\kappa)$ inequality holds locally.

Observations

- If $\kappa' < \kappa$, then $\mathrm{CAT}(\kappa') \implies \mathrm{CAT}(\kappa)$.
- $\mathrm{CAT}(0) \implies$ contractible.
- curvature $\leq 0 \implies$ aspherical.

Theorem (Aleksandrov and Toponogov)

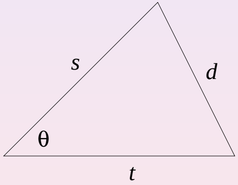
Riemannian mfld has sectional curvature $\leq \kappa$ iff $\mathrm{CAT}(\kappa)$ holds locally.

The cone on a $\mathrm{CAT}(1)$-space

The *cone on* X, denoted $\mathrm{Cone}(X)$, is the quotient space of $X \times [0, \infty)$ by the equivalence relation $\sim$ defined by $(x, s) \sim (y, t)$ if and only if $(x, s) = (y, t)$ or $s = t = 0$. The image of (x, s) in $\mathrm{Cone}(X)$ is denoted $[x, s]$. The *cone of radius* r, denoted $\mathrm{Cone}(X, r)$, is the image of $X \times [0, r]$.

Given a metric space X and $\kappa \in \mathbf{R}$, we will define a metric d_κ on Cone(X). (When $\kappa > 0$, the definition will only make sense on the open cone of radius $\pi/\sqrt{\kappa}$.) The idea: when $X = \mathbb{S}^{n-1}$, by using "polar coordinates" and the exponential map, Cone($\mathbb{S}^{n-1}$) can be identified with (an open subset of) $\mathbb{X}^n_\kappa$. Transporting the constant curvature metric on $\mathbb{X}^n_\kappa$ to Cone($\mathbb{S}^{n-1}$), gives a formula for d_κ on Cone($\mathbb{S}^{n-1}$). The same formula defines a metric on Cone(X) for any metric space X.

To write this formula recall the Law of Cosines in $\mathbb{X}^2_\kappa$. Suppose we have a triangle in $\mathbb{X}^2_\kappa$ with edge lengths s, t and d and angle θ between the first two sides



The Law of Cosines

- in $\mathbb{E}^2$:

$$d^2 = s^2 + t^2 - 2st\cos\theta$$

- in $\mathbb{S}^2_\kappa$:

$$\cos\sqrt{\kappa}d = \cos\sqrt{\kappa}s\cos\sqrt{\kappa}t + \sin\sqrt{\kappa}s\sin\sqrt{\kappa}t\cos\theta$$

- in $\mathbb{H}^2_\kappa$:

$$\cosh\sqrt{-\kappa}d = \cosh\sqrt{-\kappa}s\cosh\sqrt{-\kappa}t + \sinh\sqrt{-\kappa}s\sinh\sqrt{-\kappa}t\cos\theta$$

Given $x, y \in X$, put $\theta(x, y) := \min\{\pi, d(x, y)\}$. Define the metric d_0 on Cone(X) by

$$d_0([x, s], [y, t]) := (s^2 + t^2 - 2st \cos \theta(x, y))^{1/2}.$$

Metrics d_κ, $\kappa \neq 0$ are defined similarly. Denote Cone(X) equipped with the metric d_κ by Cone$_\kappa(X)$.

Remark

If X is a $(n-1)$-dimensional spherical polytope, then Cone$_\kappa(X)$ is isometric to a convex polyhedral cone in $\mathbb{X}^n_\kappa$.

Proposition

Suppose X is a complete and that any two points of distance $\leq \pi$ can be joined by a geodesic. Then

- $\mathrm{Cone}_\kappa(X)$ is a complete geodesic space.
- $\mathrm{Cone}_\kappa(X)$ is $\mathrm{CAT}(\kappa)$ if and only if X is $\mathrm{CAT}(1)$.

Polyhedra of piecewise constant curvature

We call a convex polytope in $\mathbb{X}^n_\kappa$ an $\mathbb{X}_\kappa$*-polytope* when we don't want to specify n.

Definition

Suppose $\mathcal{F}$ is the poset of faces of a cell complex. An $\mathbb{X}_\kappa$*-cell structure* on $\mathcal{F}$ is a family $(C_F)_{F\in\mathcal{F}}$ of $\mathbb{X}_\kappa$-polytopes s.t. whenever $F' < F$, $C_{F'}$ is isometric to the corresponding face of C_F.

Example

Piecewise Euclidean cell complexes Suppose a collection of convex polytopes in $\mathbb{E}^n$ is a convex cell cx in the classical sense. Then the union Λ of these polytopes is a $\mathbb{X}_0$-polyhedral complex.

Piecewise spherical ($=$ *PS*) polyhedra play a distinguished role in this theory. In any $\mathbb{X}_\kappa$-polyhedral cx each “link” naturally has a *PS* structure.

Geometric links

Suppose P is an n-dimensional $\mathbb{X}_\kappa$-polytope and $x \in P$. The *geometric link*, $\mathrm{Lk}(x, P)$, (or "space of directions") of x in P is the set of all inward-pointing unit tangent vectors at x. It is an intersection of a finite number of half-spaces in $\mathbb{S}^{n-1}$. If x lies in the interior of P, then $\mathrm{Lk}(x, P) \cong \mathbb{S}^{n-1}$, while if x is a vertex of P, then $\mathrm{Lk}(x, P)$ is a spherical polytope. Similarly, if $F \subset P$ is a face of P, then $\mathrm{Lk}(F, P)$ is the set of inward-pointing unit vectors in the normal space to F (in the tangent space of P). If Λ is an $\mathbb{X}_\kappa$-polyhedral complex and $x \in \Lambda$, define

$$\mathrm{Lk}(x, \Lambda) := \bigcup_{x \in P} \mathrm{Lk}(x, P).$$

$\mathrm{Lk}(x, P)$ is a PS length space. Similarly, $\mathrm{Lk}(F, \Lambda) := \bigcup \mathrm{Lk}(F, P)$.

Theorem (Gromov)

Let Λ be an $\mathbb{X}_\kappa$-polyhedral complex. TFAE

- $\mathrm{curv}(\Lambda) \leq \kappa$.
- $\forall x \in \Lambda$, $\mathrm{Lk}(x, \Lambda)$ *is* $\mathrm{CAT}(1)$.
- $\forall$*cells* P *of* Λ, $\mathrm{Lk}(P, \Lambda)$ *is* $\mathrm{CAT}(1)$.
- $\forall v \in \mathrm{Vert}\,\Lambda$, $\mathrm{Lk}(v, \Lambda)$ *is* $\mathrm{CAT}(1)$.

Proof.

Any $x \in \Lambda$ has a nbhd isometric to $\mathrm{Cone}_\kappa(\mathrm{Lk}(x, \Lambda), \varepsilon)$. $\square$

Lemma (Gromov)

Let L be an all right PS simplicial cx. Then

$$L \text{ is } \mathrm{CAT}(1) \iff \text{flag cx.}$$

Corollary

A cubical cx is $\mathrm{CAT}(0) \iff$ *the link of each vertex is a flag cx.*

Corollary

For any right-angled Coxeter system (W, S), $\Sigma(W, S)$ *is* $\mathrm{CAT}(0)$.

Metric flag complexes

Suppose L is a PS simplicial complex st each edge has length $\geq \pi/2$ (eg. the edge lengths might have the form $\pi - \pi/m$, where $m = m(s, t)$. L is a *metric flag complex* if every time the edge lengths of a complete subgraph are the edge length of a spherical simplex, then this simplex is filled in (ie is in L). For example, with its natural PS metric, $L(W, S)$ is a metric flag cx.

Lemma (Moussong)

A metric flag cx is $\mathrm{CAT}(1)$.

Corollary (Moussong)

For any Coxeter system (W, S), $\Sigma(W, S)$ *is* $\mathrm{CAT}(0)$.