Exotic closed aspherical 4-manifolds

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A space which has the homotopy type of a CW complex is *aspherical*, if its universal cover is contractible (or equivalently, if all of its higher homotopy groups vanish).

Conjecture (The Borel Conjecture)

Suppose two closed aspherical manifolds have isomorphic fundamental groups. Then they are homeomorphic.

Question

Can we replace "homeomorphic" by "diffeomorphic" in the above conjecture?

1. No, when the dimension $n \geq 5$: there exists different smooth structures on the n-torus (Wall, Hsiang and Shaneson in the late 1960s)
2. Yes, when $n \leq 3$: this is classical for $n = 2$, in dimension 3 it uses Perelman's proof of Thurston's Geometrization Conjecture.
3. What about $n = 4$? In fact, the answer is also "no" for $n = 4$.

Exotic closed aspherical 4-manifolds

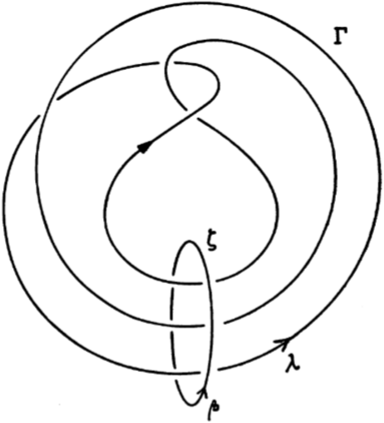
Theorem (D.-Hayden-Huang-Ruberman-Sunukjian)

There exists a pair of smooth, closed aspherical 4-manifolds which are homeomorphic but not diffeomorphic. (In fact, infinitely many such pairs.)

A version for manifolds with boundary was known (even for $\pi_1 = 1$): There is a pair of compact contractible smooth 4-manifolds with boundary that are homeomorphic but not diffeomorphic (Akbulut and Ruberman 2016).

The *Akbulut cork* (or *Mazur manifold*) is a certain smooth compact contractible 4-manifold C with boundary (its boundary being a nonsimply connected homology sphere). Its boundary admits a smooth involution $f : \partial C \to \partial C$, called a *cork twist*, which does not extend to any self-diffeomorphism $C \to C$ (although it does extend to a self-homeomorphism). All previously known examples of simply connected smooth 4-manifolds M^4 which are h-cobordant but not diffeomorphic are constructed by removing a copy of C from M^4 and then glueing it back by a cork twist.

The Mazur manifold



(From Barry Mazur's 1960 paper)

Rough plan: Start with a pair of 4-dimensional compact aspherical 4-manifolds with boundary, X and X', that are homeomorphic but not diffeomorphic (as constructed by Hayden and Piccirillo), and then apply a "doubling procedure" (called the "Reflection Group Trick") to obtain a pair of closed aspherical manifolds, $Q(X)$ and $Q(X')$, with the same property. First idea: For X, X' use Mazur manifold and its twisted version. (I don't know whether or not this works.)

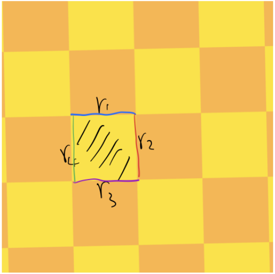
1. The *Reflection Group Trick*: given a compact smooth aspherical manifold X with boundary, one can build a closed aspherical manifold which retracts onto X by "doubling" copies of X along parts of ∂X.
2. First, I explain the basic building blocks X, X' (of Hayden and Piccirillo).
3. Then I explain why the closed aspherical manifolds obtained by this doubling process applied to X and X' are homeomorphic but not diffeomorphic.

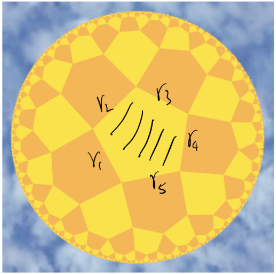
Ingredient I: The Reflection Group Trick

Let Γ be a finite simplicial graph with vertex set $\mathrm{Vert}(\Gamma) = \{v_1, v_2, \ldots, v_m\}$. The associated *right-angled Coxeter group* G_Γ is the group with generating set $\{v_i\}_{i=1}^m$ and the following two types of relators:

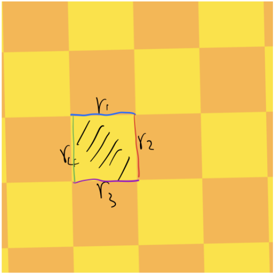
1. $v_i^2 = 1$ for $1 \leq i \leq m$;
2. $v_i v_j = v_j v_i$ whenever $\{v_i, v_j\} \in \mathrm{Edge}(\Gamma)$.

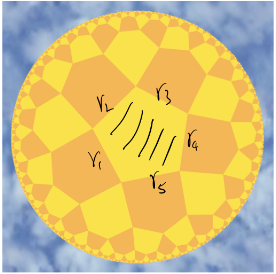
Γ is the *defining graph* of G_Γ.

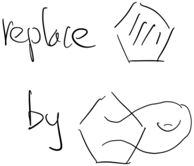




$W = \langle r_1, r_2, r_3, r_4 \mid r_i^2 = 1,\ r_i r_{i+1} = r_{i+1} r_i,\ 1 \leq i \leq 4 \}$







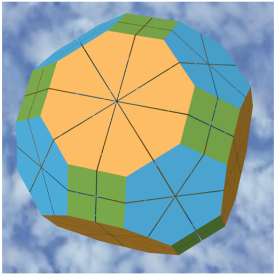
Step 1: Start with a compact manifold X with boundary, $= \partial X$, and a "mirror structure" on the boundary.

Step 2: There is an associated *right-angled Coxeter group* G whose generators $\{s_i\}_{i=1}^m$ are in 1-1 correspondence with the mirrors of ∂X. The relators are $s_i^2 = 1$ for each i, and $s_i s_j = s_j s_i$ whenever the associated mirrors have nonempty intersection of codimension 1 in ∂X.
Glue together copies of X to obtain an open manifold $D(X)$ and a proper action $G \curvearrowright D(X)$ with fundamental chamber X.

Step 3: Take G' to be a finite index, torsion free subgroup of G. Then $Q(X) = D(X)/G'$ is a closed manifold. (We can take G' to be the commutator subgroup of G.)

The mirror structure on X

Each triangulation $\mathcal{T}$ of ∂X determines a "mirror structure" on X, whose mirrors are top-dimensional "dual cells" of the triangulation.

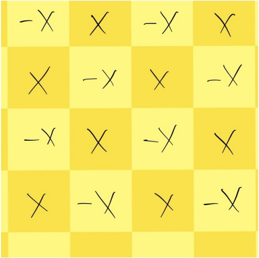


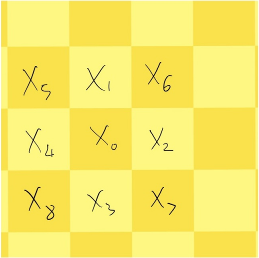
Mirrors of ∂X are in 1-1 correspondence with $\text{Vert}(\mathcal{T})$.

The defining graph of the associated right-angled Coxeter group G is equal to the 1-skeleton of $\mathcal{T}$.

More about $D(X)$

$D(X)$ is tiled by copies of X, i.e. $D(X) = \bigcup_{g \in G} gX$.

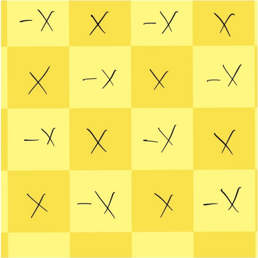


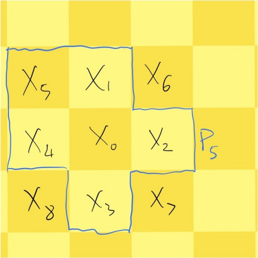


Given two copies of X, denoted g_1X and g_2X in $D(X)$, the distance between them is the word distance $d(g_1, g_2)$.

We enumerate copies of X in $D(X)$ as $X_0, X_1, X_2, \ldots$ so that $d(X_i, X_0) \leq d(X_{i+1}, X_0)$.

Properties of the construction





$D(X)$ admits a filtration by the $P_n = \bigcup_{i=0}^{n} X_i$.

Definition

A simplicial complex Z is a *flag complex*, if whenever Z has a subcomplex isomorphic to the 1-skeleton of a k-simplex, $k \geq 2$, then the subcomplex actually spans a k-simplex in Z.

Note: The barycentric subdivision of any simplicial complex is a flag complex.

More properties

Assume ∂X has a PL structure. Triangulate ∂X as a flag complex $\mathcal{T}$. Since each mirror is the dual cell to a vertex and since $\mathcal{T}$ is a flag complex, $P_n \cap X_{n+1}$ is a top-dimensional PL closed disk in ∂X_{n+1} (and in ∂P_n).

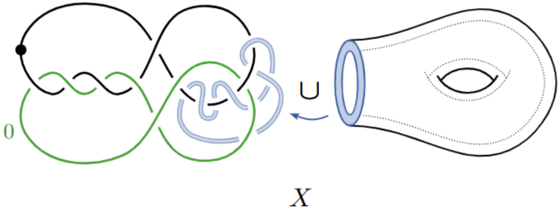
So, P_n is a boundary sum of the chambers X_i for $i \leq n$: $P_n \overset{PL}{\cong} X_0 \natural \cdots \natural X_n$.
(Since $P_n \cong P_{n-1} \natural X_n$.)
So, if the input manifold X is aspherical, then $D(X)$ is aspherical; and hence, $Q(X)$ is also aspherical.

$\pi_1(D(X))$ is the free product of countably many copies of $\pi_1(X)$.

Also, if X is a smooth manifold with boundary, then $D(X)$ is a smooth manifold.

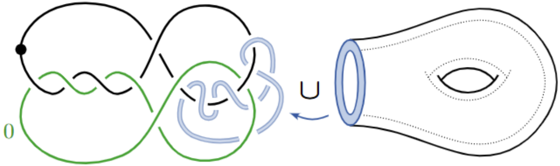
Ingredient II: Hayden-Piccirillo manifolds

X is obtained from the contractible manifold C (the *Akbulut cork*) by attaching a "genus-1 handle" along a certain knot in ∂C. (A *genus-1 handle* is a copy of $F \times \mathbb{D}^2$ where F is a genus-1 surface with one boundary component.) We identify $\partial F \times \mathbb{D}^2$ with a tubular neighborhood of a knot K in ∂C.
So, $X = C \cup (F \times \mathbb{D}^2)$.



X is formed by glueing $F \times \mathbb{D}^2$ onto C along the blue tube.
X' is obtained from X by removing the interior of C and reglueing it by the cork twist.

The Hayden-Piccirillo manifolds X and X'



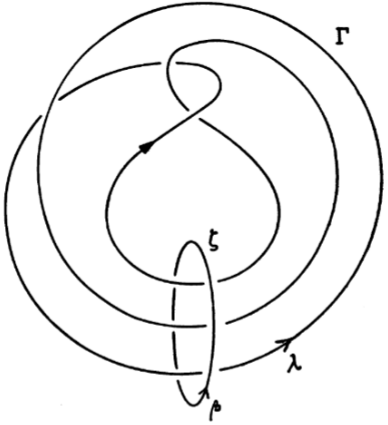
X

The cork twist $f : \partial C \to \partial C$ interchanges the green and black curves. X' is obtained from X by removing the interior of C and reglueing it by the cork twist.

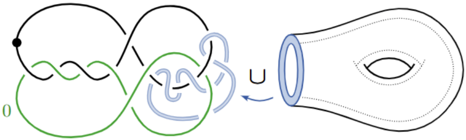
Remark

In what follows almost everything works, if we use a genus g-handle, with $g \geq 1$, instead of a genus-1 handle.

The Mazur manifold



Key properties of Hayden-Piccirillo manifolds



X

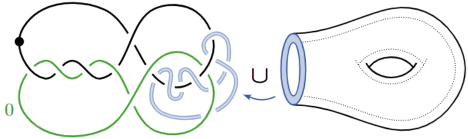
X' is obtained from X by removing the interior of C and reglueing it by the cork twist.

Theorem

1. *X and X' are homeomorphic (the homeomorphism is Id outside $\mathring{C}$);*
2. *X is homotopy equivalent to the 2-torus;*
3. *X embeds smoothly in B^4;*
4. *every homologically essential, smoothly embedded surface in X has genus ≥ 2;*
5. *$H_2(X')$ is generated by a smoothly embedded torus in X'.*

X is a Stein domain

The proof of (4) uses the fact that the interior of X can be given the structure of a Stein manifold together with the Adjunction Formula and a computation showing that $c_1(X) = 2\alpha$, where α is the generator of $H^2(X)$.



X

The proof of (5) uses the fact that for X', the blue attaching curve for the genus-1 handle is a "slice knot" in that it bounds a 2-disk in C. The same is not true for X (since the blue curve links the black curve).

$D(X)$ vs $D(X')$

Observation: Since X and X' are homeomorphic, $D(X)$ and $D(X')$ are G-equivariantly homeomorphic; hence, $Q(X)$ and $Q(X')$ are homeo.

Theorem

$D(X)$ and $D(X')$ are not diffeomorphic.

Suppose there is a diffeo $f : D(X') \to D(X)$.
Take a chamber X' in $D(X')$, and a smoothly embedded torus $T^2 \hookrightarrow X'$ representing the generator of $H_2(X')$.

$f(X') \subset P_n(X)$ for some n. So, $H_2(X')$ has nonzero image in $H_2(X_k)$ for some $k \leq n$. Since each X_i embeds in B^4,

$$P_n \overset{PL}{\cong} X_0 \natural \cdots \natural X_k \natural \cdots \natural X_n \ \hookrightarrow \ B^4 \natural \cdots \natural X_k \natural \cdots \natural B^4 \cong X_k = X$$

$T^2 \hookrightarrow f(X') \overset{PL}{\hookrightarrow} P_n \hookrightarrow X$, contradiction (using Wall's smoothing theorem for locally flat codimension-2 PL submanifolds of smooth manifolds).

$Q(X)$ vs $Q(X')$

Theorem

For a particular choice of mirror structure on $\partial X = \partial X'$, $Q(X)$ and $Q(X')$ are not diffeomorphic.

Recall that $Q(X) = D(X)/G'$, hence
$1 \to \pi_1(D(X)) \to \pi_1(Q(X)) \to G' \to 1$.

Observation: if $\pi_1(D(X))$ is a characteristic subgroup of $\pi_1(Q(X))$, then any diffeomorphism $Q(X) \to Q(X')$ lifts to a diffeomorphism $D(X) \to D(X')$.

Note: $\pi_1(D(X))$ is a free product of $\mathbb{Z}^2$'s, one for each chamber of $D(X)$.

Lemma

For some choice of mirror structure on $\partial X = \partial X'$, any $\mathbb{Z}^2$ subgroup of $\pi_1(Q(X))$ is contained in $\pi_1(D(X))$.
As a consequence, $\pi_1(D(X))$ is a characteristic subgroup of $\pi_1(Q(X))$.

Existence of $\mathbb{Z}^2$-subgroups

$1 \to \pi_1(D(X)) \to \pi_1(Q(X)) \to G' \to 1$

We want G' not to contain any $\mathbb{Z}^2$-subgroup, which is equivalent to saying the Coxeter group G has no $\mathbb{Z}^2$-subgroups.

Recall: generators of G are in 1-1 correspondence with vertices of $\mathcal{T}$, and two generators commute if the associated vertices are adjacent in $\mathcal{T}$.

An *empty square* in $\mathcal{T}$ is an embedded 4-cycle $x_1x_2x_3x_4$ in $\mathcal{T}^1$ such that x_1 and x_3 are not joined by an edge, and neither are x_2 and x_4.

Observation: if $\mathcal{T}$ has an empty square, then G has a $\mathbb{Z}^2$-subgroup.

Theorem (Moussong 1988)

If G has a $\mathbb{Z}^2$-subgroup, then $\mathcal{T}$ has an empty square.

Flag, no-square triangulations of 3-manifolds

Corollary

If ∂X admits a flag, no-square triangulation $\mathcal{T}$, then the resulting RACG G has no $\mathbb{Z}^2$-subgroup.

Theorem (Przytycki-Swiatkowski 2009)

Any 3-dimensional polyhedron admits a flag, no-square triangulation.

Remark (Vinberg 1985)

No triangulation of a sphere of dimension ≥ 4 is flag, no-square.

Theorem

Suppose X and X' are a pair of Hayden-Piccirillo manifolds (with genus-1 handles), with a flag, no-square triangulation of $\partial X = \partial X'$. Then $Q(X)$ and $Q(X')$ are closed aspherical manifolds that are homeomorphic, but not diffeomorphic.

Thank you!