Coxeter groups, Artin groups, buildings

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1. Introduction
   - Geometric reflection groups
   - Coxeter systems
   - Artin groups
   - Buildings

2. The $K(\pi, 1)$-question
   - Nonpositive curvature
   - The $K(\pi, 1)$-problem

The basic object is a Coxeter system. To one of these, we can associate various cell complexes: the Davis-Moussong complex, the Deligne complex of an Artin group, and the "standard realization" of any building whose type is the Coxeter system. These are described in

Chapter 4 of my new book

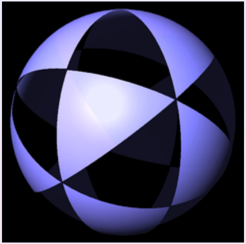
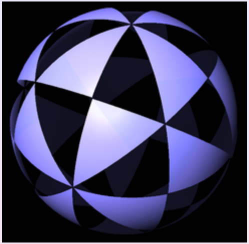
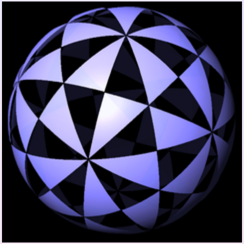
Infinite group actions on polyhedra, to appear in Springer, 2024. (Most recent information for publication date: July 22, 2024).

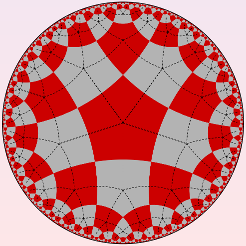
Geometric reflection groups

A classical topic is: groups generated by reflections on spaces of constant curvature, $\mathbb{S}^n$, $\mathbb{E}^n$, $\mathbb{H}^n$. Let W be such a reflection group.

There is a strict fundamental domain K for the W-action on the manifold M, where $M = \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$. K is the closure of a connected component of complement of fixed sets of the reflections,

$$K = \text{closure of } \left(M - \bigcup_{\text{reflections } r} M_r \right)$$



Properties

Let $S = \{$reflections across the “walls” of K$\}$.
Then S is a set of generators for W.
For each $(s,t) \in S \times S$, let $m(s,t) =$ order of st. All relations in W are consequences of relations of the form $(st)^{m(s,t)} = 1$. In other words, W has a presentation of the form $W = \langle S \mid \mathcal{R} \rangle$, where $\mathcal{R} = \{(st)^{m(s,t)}\}_{(s,t) \in S \times S}\}$. The pair (W, S) is called a “Coxeter system.”

We will show how to reconstruct the fundamental chamber K from this presentation. K has a codimension-one face (or “mirror”),K_s, for each $s \in S$ and a codimension-two face (or “corner”), $K_{s,t} = K_s \cap K_t$, whenever $s \neq t$ and $m(s,t) \neq \infty$.

Coxeter systems

$S =$ a set (of generators).
$M = m(s,t)_{(s,t)\in S\times S}$ is a *Coxeter matrix*, ie, an $(S\times S)$-symmetric matrix with entries in $\mathbb{N}\cup\{\infty\}$, 1 s on the diagonal and off-diagonal entries ≥ 2. The *Coxeter group W* is defined by the presentation $\langle S \mid \mathcal{R}\rangle$, where

$$\mathcal{R} = \{(st)^{m(s,t)}\}_{(s,t)\in S\times S}.$$

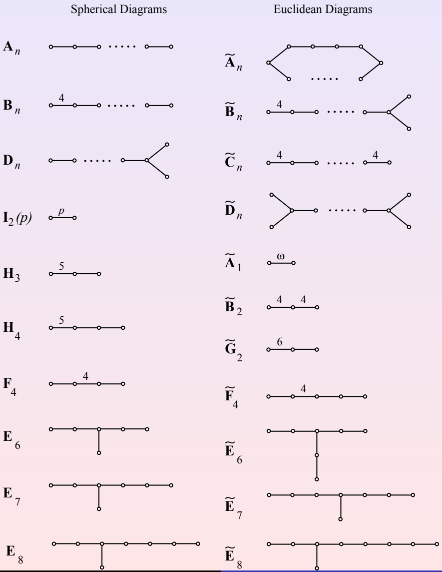
In other words, $W = F(S)/N(\mathcal{R}))$, where $F(S)$ is free group on S and $N(\mathcal{R})$ is the normal subgroup generated by $\mathcal{R}$.
(W, S) is called a *Coxeter system*.

Alternate method for encoding this data

Graph L^1 with $\text{Vert } L^1 = S$ and edges corresponding to unordered pairs $\{s, t\}$ with $m(s, t) \neq \infty$ and with labeling of edges $m : \text{Edge } L^1 \to \{2, 3, \dots\}$, where edge $\{s, t\}$ is labeled $m(s, t)$.

A third method: Coxeter diagrams

Leave out edges labeled 2 but put in edges labeled ∞.


Spherical Diagrams
Euclidean Diagrams
$\mathbf{A}_n$
$\mathbf{B}_n$
4
$\mathbf{D}_n$
$\mathbf{I}_2(p)$
p
$\mathbf{H}_3$
5
$\mathbf{H}_4$
5
$\mathbf{F}_4$
4
$\mathbf{E}_6$
$\mathbf{E}_7$
$\mathbf{E}_8$
$\widetilde{\mathbf{A}}_n$
$\widetilde{\mathbf{B}}_n$
4
$\widetilde{\mathbf{C}}_n$
4
4
$\widetilde{\mathbf{D}}_n$
$\widetilde{\mathbf{A}}_1$
ω
$\widetilde{\mathbf{B}}_2$
4
4
$\widetilde{\mathbf{G}}_2$
6
$\widetilde{\mathbf{F}}_4$
4
$\widetilde{\mathbf{E}}_6$
$\widetilde{\mathbf{E}}_7$
$\widetilde{\mathbf{E}}_8$

The following definition is due to Tits.

Coxeter groups

Any group with a presentation of this form is called a *Coxeter group*. The graph L^1 and the labeling $m : \mathrm{Edge}\, L^1 \to \{2, 3, \dots\}$ are arbitrary. (The point is that if W is the group defined by the presentation, then $m(s, t) =$ order of st.) W could also be called a "abstract reflection group."

Special subgroups

For any $T \leq S$, put $W_T = \langle T \rangle$. When W_T is finite, it is called a *spherical subgroup* and T is a *spherical subset*.

Poset of spherical subsets

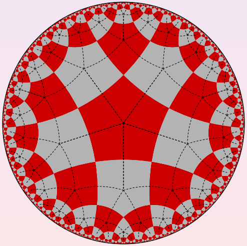
$\mathcal{S} = \{$spherical subsets of $S\}$. $\mathcal{S}^{op}$ is the opposite poset.
$L(W, S)$ is the simplicial complex with vertex set S and $\{$nonempty simplices$\} = \{T \in \mathcal{S} \mid T \neq \emptyset\}$. $L(W, S)$ is called *nerve* of (W, S).
$K(W, S) = |\mathcal{S}|$, the cone on the barycentric subdivision of L.
Called the standard *fundamental chamber*.

Geometric realization of a poset $\mathcal{P}$

This means the simpicial complex $|\mathcal{P}|$ with one k-simplex for each chain $p_0 < p_1 < \cdots < p_k$.

Example

W is generated by reflections across the edges of a hyperbolic pentagon. Then the midpoint of each edge is labeled by an element of S. The center of the pentagon is labeled by $\emptyset$ and each vertex is labeled by an unordered pair $\{s, t\}$.



Strict fundamental domains

There are two possibilities:

A simplex Δ

The codimension-one faces of Δ are indexed by S. Faces are indexed by proper subsets $T < S$.The face $\Delta_T = \bigcap_{s \in T} \Delta_s$ has codimension $\mathrm{Card}\, T$.

The chamber $K(W, S)$

$K(W, S)$ is the geometric realization of $\mathcal{S}$ (or $\mathcal{S}^{op}$). Its k-simplices are chains $T_0 < \cdots < T_k$. These can be assembled into faces: $K_T = |\mathcal{S}_{\geq T}| = \bigcap_{s \in T} K_s$.

For $x \in K$, put $S(x)$ be the smallest T where $x \in K_T - \partial K_T$
(Here $\partial K_T = |\mathcal{S}_{>T}|$.)

Davis-Moussong complex $\Sigma(W, S)$

$\Sigma(W, S) = (W \times K)/\sim$, also denoted by $D(W, K)$, where

$$(w, x) \sim (w', x') \iff x = x' \text{ and } wW_{S(x)} = w'W_{S(x)}.$$

The Coxeter complex

$D(W, \Delta) = (W \times \Delta)/\sim$, where $\sim$ is defined as above.

Definition of simple complex of groups

This means a poset of groups, that is, a poset $\mathcal{P}$ and a system of groups $\{G_p\}_{p\in\mathcal{P}}$ so that whenever $p < q$ we are given an injective homorphism $G_q \to G_p$. Moreover, these inclusions are compatible with one another.

Direct limits

If $\{G_p\}_{p\in\mathcal{P}}$ is simple complex of groups, then one can form the direct limit: $G = \lim G_p$. (Universal property: If $\{\varphi_p : G_p \to H\}$ is a compatible family of homomorphisms, then $\exists!$ homomorphism $\varphi : G \to H$ compatible with the φ_p.)

Associated to any group action with a strict fundamental domain; namely, $\mathcal{P}$ is strata of fundamental domain and G_p is the stabilizer of a generic point in the stratum.

Basic example

$W\mathcal{S}^{op} = (W_T)_{T\in\mathcal{S}}$. This is the *simple complex of spherical subgroups* of W.

Poset of spherical cosets of W

$\mathrm{Coset}(W) = \coprod_{T\in\mathcal{S}} W/W_T$, called the *development* of $W\mathcal{S}^{op}$

The basic construction

$D(W,K) = (W \times K)/\sim$, where $K = K(W,S) = |\mathcal{S}|$.
$(D(W,K) \cong |\mathrm{Coset}(W)|)$

The point of this talk is that there are several other important simple complexes of groups and cell complexes associated to a Coxeter system (W, S), namely,

- Artin groups
- Buildings (with a chamber transitive group of automorphisms)

In each case, the underlying poset is the same, namely, $\mathcal{S}$, and the strict fundamental domain is K $(= |\mathcal{S}|)$.

	Coxeter system	Artin group	building
Notation	(W, S)	A	$\mathcal{C}$
spherical subsets	$\mathcal{S} = \{T < S \mid W_T \text{ is finite}\}$	*same*	*same*
fund. chamber	$K(W, S) = \lvert\mathcal{S}\rvert$	same	same
cell complex	Davis-Moussong cx $\Sigma(W, S)$	Deligne cx Λ	realization $\lvert\mathcal{C}\rvert$
simple cx gps	$(W_T)_{T\in\mathcal{S}}$	$(A_T)_{T\in\mathcal{S}}$	$(G_T)_{T\in\mathcal{S}}$
spherical cosets	$\coprod_{T\in\mathcal{S}} W/W_T$	$\coprod_{T\in\mathcal{S}} A/A_T$	$\coprod_{T\in\mathcal{S}} \mathcal{R}(T)$
CAT(0)?	yes	?	yes
contractible?	yes	?	yes
$K(\pi, 1)$ ques?	yes	?	yes

Motivation

Geometric representations

If W is a finite Coxeter group, then W has a "geometric representation" as a group generated by orthogonal reflections on $\mathbb{R}^n$, where $n = \text{Card}\, S$. So, W also acts on the unit sphere, $\mathbb{S}^{n-1}$ (hence, the name "spherical"). For example, the symmetric group S_n of degree n is a reflection group on $\mathbb{R}^{n-1}$ with reflecting hyperplanes H_{ij} defined by $x_i = x_j$.

Braid groups

$M = \mathbb{C}^{n-1} - \bigcup H_{ij} \otimes \mathbb{C}$. Then S_n acts freely on M. The *braid group* B_n can be defined as $B_n = \pi_1(M/S_n)$. (Braid groups were defined by E. Artin in 1947.)

Spherical Artin group = generalized braid group

(W, S) a spherical Coxeter group. W acts on $\mathbb{C}^n = \mathbb{R}^n \otimes \mathbb{C}$. Put

$$M = \mathbb{C}^n - \bigcup (H_r \otimes \mathbb{C}).$$

The group W acts freely on M. Then $A = \pi_1(M/W)$ is one definition of the Artin group associated to (W, S).

Definition of Artin group

(W, S) as before. For letters a, b and $m \in \{2, 3, \dots\}$, put

$$\mathrm{prod}(a, b; m) = \underbrace{ab\cdots}_{m \text{ terms}}$$

Let $\{a_s\}_{s\in S}$ be new symbols for generators. Define

$$A = A(W, S) = \langle \{a_s\} \mid \mathrm{prod}(a_s, a_t; m) = \mathrm{prod}(a_t, a_s; m)\rangle,$$

where $s \in S$ and $\{s, t\} \in \mathrm{Edge}\, L^1$. For $T \subset S$, put $A_T = \langle \{a_s\}_{s\in T}\rangle$.

The simple complex of groups

$A\mathcal{S}^{op} = \{A_T\}_{T\in\mathcal{S}}$, the *complex of spherical subgroups*. Other possibility: the underlying poset is the set of proper subsets of S, that is, the poset is $\{\text{faces of } \Delta\}$.

Poset of spherical cosets of A

$\text{Coset}(A) = \coprod_{T\in\mathcal{S}(W,S)} A/A_T$, is the *development* of $A\mathcal{S}^{op}$. The corresponding cell complex is the *Deligne complex*. If we use the proper subsets, the corresponding poset of cosets is called the *Artin complex*.

Deligne complex

$\Lambda(W, S) = D(A, K) = (A \times K)/\sim$, as before

$$(a, x) \sim (a', x') \iff x = x' \text{ and } aA_{S(x)} = a'A_{S(x)}.$$

When fundamental chamber is simplex Δ, define the *Artin complex* to be $D(A, \Delta)$. When A is spherical, Deligne proved that $D(A, \Delta)$ is homotopy equivalent to a wedge of spheres.

The Deligne cx is similar to Davis-Moussong cx except that along each codimension 1 face, instead of 2 chambers meeting, we have an infinite cyclic group worth of chambers.

Buildings

Combinatorially, a “building” is a set $\mathcal{C}$ of “chambers” with extra structure. In particular, each building will have an associated Coxeter system (W, S).

Chamber systems

A *chamber system* over S is a set $\mathcal{C}$ together with a family of equivalence relations in indexed by S. Each s-equivalence class must have at least 2 elements. Two s-equivalent chambers are *s-adjacent* if they are not equal. A chamber system is *thick* if each s-equivalence class has a least 3 elements.

Example

The Coxeter group W is a chamber system over S. Two elements w, w' are *s-equivalent* if they belong to the same coset in $W/W_{\{s\}}$, ie, if $w' = ws$. The set of chambers is the the group W. (This is the "thin building" of type (W, S).)

Example

The Artin group $A = A(W, S)$ is a chamber system over S; it is usually not a building.

Example

Suppose G is a group, $B = G_\emptyset$ is a subgroup and $(G_s)_{s \in S}$ is a family of subgroups with $B < G_s$ indexed by S. This defines a chamber system $\mathcal{C} = G/B$ over S. Two elements of G/B are s-adjacent if they determine the same coset in G/G_s.

People who work on buildings like to use the following terminology.

Galleries

A *gallery* in $\mathcal{C}$ is a sequence of adjacent chambers $C_0, C_1, \dots, C_k$. If C_{i-1} is s_i-adjacent to C_i, then the gallery has *type* $(s_1, s_2, \dots, s_k)$. If each $s_i \in T \subset S$, then the gallery is a *T-gallery*.

Residues

A *T-residue* is a T-gallery connected component. For example, the $\{s\}$-residue containing a chamber C is the s-equivalence class containing C (analogous to a coset).

Examples of rank 2 buildings, $S = \{s, t\}$

Trees

The set of edges in a tree (without a terminal vertex) is a chamber system over $S = \{s, t\}$ and a building of type (D_∞, S).

Generalized *m*-gons

Given $m \in \mathbb{N}$, $m \geq 2$, a finite bipartite graph Γ is called a *generalized m-gon* if it has girth $2m$ and diameter m. $\mathcal{C} = \operatorname{Edge} \Gamma$ is a chamber system over S and a building of type (D_m, S). A generalized 2-gon is a complete bipartite graph such as $K_{m,n}$. A generalized 3-gon is a *projective plane*.

Chamber systems of type (W, S)

Let $\mathcal{C}$ be a (gallery connected) chamber system over S and $M = (m(s,t))$ a Coxeter matrix. Then $\mathcal{C}$ has *type* M (or type (W, S)) if each $\{s, t\}$ residue is a generalized $m(s,t)$-gon. The chamber system is *thick* if each s-residue has more than 2 elements.

Feit-Higman Theorem

Finite, thick generalized m-gons exist only for $m \in \{2, 3, 4, 6, 8\}$.

W-distance

Define $\delta : \mathcal{C} \times \mathcal{C} \to W$ as follows. Suppose $C, D \in \mathcal{C}$ and $C = C_0, \cdots, C_k = D$ is a minimal gallery between them. Let $(s_1, \ldots, s_k)$ be its type and let $w = s_1 \cdots s_k$ be the associated element of W. Then $\delta(C, D) = w$.

Definition of building

A chamber system $\mathcal{C}$ of type (W, S) equipped with a W-distance function $\delta : \mathcal{C} \to \mathcal{C}$ is a *building* if δ satisfies certain axioms (which we won't state).

Geometric realization

This is a space $|\mathcal{C}|$ where there is a copy of $K(W, S)$ for each chamber in $\mathcal{C}$. In other words, $|\mathcal{C}| = (\mathcal{C} \times K(W, S))/\sim$, where as before,

$$(C, x) \sim (C', x') \iff x = x' \text{ and } C, C' \in \text{same } S(x)\text{-residue.}$$

Example (Thick spherical buildings)

Suppose G is an algebraic group over finite field $\mathbb{F}$. Then G acts on a building $\mathcal{C} = G/B$ with chamber stabilizer B, eg, $G = PGL(n, \mathbb{F})$, $B = \{\text{upper triangular}\}$.

Theorem (Tits' Theorem, rough version)

If $\mathcal{C}$ is a spherical building of rank ≥ 3, then $\mathcal{C}$ comes from an algebraic group such as the above.

Chamber-transitive group actions

Suppose G is a chamber-transitive group of automorphisms of $\mathcal{C}$. Fix $C \in \mathcal{C}$ and let B (or $G_\emptyset$) denote the stabilizer of C. For $T \subset S$, let $G_T =$ stabilizer of T-residue containing C.
Then $G\mathcal{S}^{op} = \{G_T\}_{T \in \mathcal{S}}$ is a simple complex of groups.
Moreover, $G = \lim G_T$.

Recovering the building

$\mathcal{C} = G/B$. $\quad \text{Coset}(G) = \coprod_{T \in \mathcal{S}} G/G_T$ is the poset of spherical cosets in $G\mathcal{S}^{op}$. A coset of G_T is the same thing as a T-residue. We can recover the building from the simple complex of groups: $D(G, K) = (G \times K)/\sim$. $(= |\mathcal{C}|)$.

Right-angled buildings

RABs

Suppose (W, S) is right-angled. Let $(G_s)_{s \in S}$ be any family of groups indexed by S. For each $T \in \mathcal{S}$, let G_T be the direct product of the G_s, $s \in T$. The direct limit G of the G_T is the *graph product* and $G\mathcal{S}^{op} = \{G_T\}_{T \in \mathcal{S}}$ defines a *right-angled building* with $D(G, K) = (G \times K)/\sim$.

Example (RAAGs)

If $G_s \cong \mathbb{Z}$, then the graph product is a *right-angled Artin group* (or RAAG).

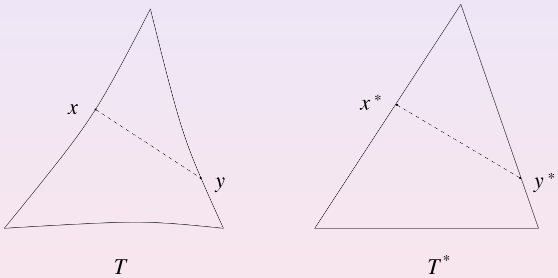
	Coxeter system	Artin group	building
Notation	(W, S)	A	$\mathcal{C}$
spherical subsets	$\mathcal{S} = \{T < S \mid W_T \text{ is finite}\}$	*same*	*same*
fund. chamber	$K(W, S) = \lvert\mathcal{S}\rvert$	same	same
cell cx	Davis-Moussong cx $\Sigma(W, S)$	Deligne cx Λ	realization $\lvert\mathcal{C}\rvert$
simple cx gps	$(W_T)_{T\in\mathcal{S}}$	$(A_T)_{T\in\mathcal{S}}$	$(G_T)_{T\in\mathcal{S}}$
spherical cosets	$\coprod_{T\in\mathcal{S}} W/W_T$	$\coprod_{T\in\mathcal{S}} A/A_T$	$\coprod_{T\in\mathcal{S}} \mathcal{R}(T)$
CAT(0)?	yes	?	yes
contractible?	yes	?	yes
$K(\pi, 1)$ ques?	yes	?	yes

CAT(0) spaces

Gromov defined what it means for a complete geodesic metric space to be "CAT(0)" by comparing its triangles with triangles in $\mathbb{R}^2$. A space is "nonpositively curved" (abbreviated NPC) if it is locally CAT(0).

Basic facts

1. Simply connected and NPC $\implies$ CAT(0).
2. CAT(0) $\implies$ contractible.
3. A piecewise euclidean polyhedron is NPC if the link of each of each cell (a piecewise spherical polyhedron) is CAT(1).


x
y
T
x^*
y^*
T^*

Theorem (Moussong 1988)

$\Sigma(W, S)$ is CAT(0).

Corollary (D.)

If $\mathcal{C}$ is a building of type (W, S), then $|\mathcal{C}|$ is CAT(0)*. If $\mathcal{C}$ is a spherical building, then $D(\mathcal{C}, \Delta^n)$ (the link of the cone point) is* CAT(1)*.*

Spherical Coxeter groups

Suppose W is finite and acts as a reflection group on S^n with fundamental chamber a spherical simplex Δ^n. Then the Coxeter complex $D(W, \Delta^n) \cong S^n$; hence, is CAT(1).

Conjecture (Charney-Davis)

When (W, S) is not spherical, the Deligne complex, $D(A, K)$, is CAT(0)*.*

Conjecture (Charney-Davis)

When (W, S) is spherical, the Artin complex $D(A, \Delta^n)$ is CAT(1)*.*

This implies the previous conjecture for general Artin groups. (Since the link of a cell in Λ corresponds to a spherical Artin subgroup.)

Suppose $GQ = \{G_T\}_{T \in Q}$ is a simple complex of groups over a poset Q. Each group G_T has a classifying space BG_T which is aspherical, i.e., is a $K(G_T, 1)$

Using the injections $G_T \to G_{T'}$ we can glue together the BG_T to form a new space BGQ, called the *aspherical realization* of GQ. Its homotopy type is well-defined. Its fundamental group is G.

$K(\pi, 1)$-problem

Is $BGQ = BG$, i.e., is BGQ aspherical?

Theorem

If $D(G, |Q|)$ is contractible, then the $K(\pi, 1)$-question for GQ has a positive answer.

Corollary

If G is a Coxeter group or a chamber transitive group on a building, then the $K(\pi, 1)$-question for $G\mathcal{S}^{op}$ has a positive answer.

Theorem [Charney-D]

The answer is also positive for RAAGs and for Artin groups with $\dim K \leq 2$

The $K(\pi, 1)$-question for general Artin groups is an important open question in geometric group theory.

	Coxeter system	Artin group	building
Notation	(W, S)	A	$\mathcal{C}$
spherical subsets	$\mathcal{S} = \{T < S \mid W_T \text{ is finite}\}$	*same*	*same*
fund. chamber	$K(W, S) = \lvert\mathcal{S}\rvert$	same	same
cell cx	Davis-Moussong cx $\Sigma(W, S)$	Deligne cx Λ	realization $\lvert\mathcal{C}\rvert$
simple cx gps	$(W_T)_{T\in\mathcal{S}}$	$(A_T)_{T\in\mathcal{S}}$	$(G_T)_{T\in\mathcal{S}}$
spherical cosets	$\coprod_{T\in\mathcal{S}} W/W_T$	$\coprod_{T\in\mathcal{S}} A/A_T$	$\coprod_{T\in\mathcal{S}} \mathcal{R}(T)$
CAT(0)?	yes	?	yes
contractible?	yes	?	yes
$K(\pi, 1)$ ques?	yes	?	yes

Thank you