Colloquium - Univ. of Warsaw

April 26, 2018

I. Examples of BG

II. Dimension(s) of G

III Conjectures

IV Computing the action dimension

I. $G =$ a group

$\exists$ CW complex BG $(K(G,1))$

s.t. $\pi_1(BG) = G$ and

$\widetilde{BG} = EG$ is contractible.

Such a space is <u>aspherical</u>

If G has torsion, then BG

is ∞-dim'l. Interested when

BG is a finite dim'l

- finite cx (type G)
- mfld (usually compact mfld with boundary)

<u>Examples</u>

- dim 1: BG a connected graph

  $G = F_n$

- dim 2: $BG = \Sigma_g^2$ $g \geq 1$
- dim 3: $BG = M^3$ irreducible 3-mfld with ∞ π_1.
- dim n: $BG = M^n$, closed sect. curv ≤ 0
- G = lattice in a semi-simple

Lie gp $\mathfrak{g}$

$BG = K \backslash \mathfrak{g} / G$

- $G = MCG(\Sigma_g)$

$M = BG =$ Moduli-space

$\tilde{M} = EG =$ Teichmuller space

Examples from Reflection gp trick.

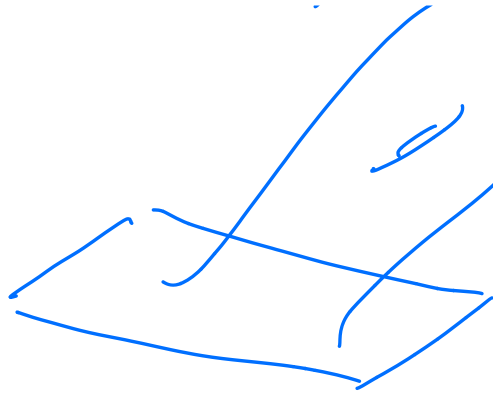
Method for converting mfld with ∂ into closed mfld.

$M =$ aspherical mfld w ∂

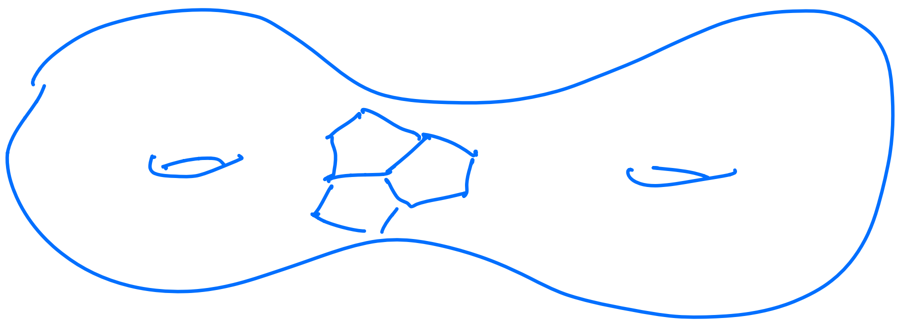
$\pi_1 \partial M \hookrightarrow \pi_1 M$ is injective

$DM = M \cup_{\partial M} M$ is BG

$G = \pi_1 M *_{\pi_1(\partial M)} \pi_1 M$



3 mfld : M^3 = handlebody



Produce a aspherical closed mfld M' which retracts onto M.

II Dimension of G

$\mathrm{gdim}\, G = \min\{\dim X \mid X \sim BG\}$

$\mathrm{cdim}\, G$ = cohomological version

$\mathrm{actdim}\, G = \min\{\dim M \mid M \sim BG\}$

Eilenberg - Ganea (1957):

$\operatorname{cd} G = \operatorname{gdim} G$, except possibly when $\operatorname{cd} G = 2$ & $\operatorname{gdim} G = 3$

$\operatorname{gdim} G \leq \operatorname{actdim} G \leq 2 \operatorname{gdim} G$

$=$ iff BG is closed mfld

thickening

III Conjectures

Remark M closed mfld

$\chi(M^{odd}) = 0$

Euler Char Conj: $BG \sim M^{2k}$ closed

Then $(-1)^k \chi(M^{2k}) \geq 0$

Evidence $\chi(\Sigma_g^2) \le 0$ So TRUE FOR PRODUCT of surfaces

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ℓ^2-Betti numbers

$$\ell^2 b_i(G) = \dim_{\mathcal{N}(G)} \ell^2 H^i(BG)$$

Singer Conj: $BG \sim M^n$ closed

Then $\ell^2 b_i(G) = 0$, $i \neq \frac{n}{2}$.

Note: Singer Conj $\Rightarrow$ Euler Char Conj

Pf $\chi(BG) = \sum (-1)^i \ell^2 b_i(G)$

If only $\ell^2 b_k(G) \neq 0$. Then

$$(-1)^k \chi(M^{2k}) = (-1)^k \ell^2 b_k(G)$$

$\square$

Action Dim Conj. For any gp G of type F, $\ell^2 b_i(G) = 0$ if $i > \frac{1}{2}$ act dim G.

Prop ADC $\iff$ Singer Conj

Remarks ADC makes sense for any G of type F

Pf ($\Rightarrow$) Suppose $BG \sim M^n$, closed

by ADC $\ell^2 b_i(M^n) = 0$ $i > n/2$

Poincaré duality $\ell^2 b_i(M^n) = 0$ $i < n/2$

($\Leftarrow$) (Okun - Schreve)

Use Reflection gp trick. $\square$

IV Computing actdim(G)

Method of Bestvina-Kapovich-Kleiner, based on

classical van Kampen obstruction

for embedding $K^d \subset \mathbb{R}^m$

Recall always have $K^d \subset \mathbb{R}^{2d+1}$.

$vk^m(K) = \mathbb{Z}/2$ cohomology class

= obstruction to embedding in $\mathbb{R}^m$

Idea of BKK: If $vk^m(K) \neq 0$

then Cone K $(= [0,\infty) \times K)/\sim$

does not coarsely embed in $\mathbb{R}^{m+1}$

(or any contractible mfld)

(K is an "obstructor")

So if Cone $K \hookrightarrow G$ or EG

then $G \not\hookrightarrow \mathbb{R}^{m+1}$ c.e, i.e

$\text{actdim}\, G \geq m+2$.

$$\boxed{\text{How to define coarse embeddings of } \operatorname{Cone} K \hookrightarrow G\,?}$$

In practice, Cone K comes from a configuration of subgroups.

If $H \subset G$ fin gen., the inclusion is coarse embeddin

Ex If $H = \mathbb{Z}^k$, $K = \partial H = S^{k-1}$

$\operatorname{Cone} K = \mathbb{R}^k$

More generally Cone K could

be some configuration of

subgps & $K = \bigcup$ their bdr's