Complements of hyperplane arrangements as posets of spaces

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https://people.math.osu.edu/davis.12/slides.html
\[ \mathcal{A} = \{H\}, \text{ a hyperplane arrangement in } \mathbb{C}^n. \]

\[ L(\mathcal{A}) = \text{the intersection poset, partially ordered by reverse inclusion. (So, the minimum element is } \mathbb{C}^n). \]

\[ \Sigma(\mathcal{A}) = \bigcup_{H \in \mathcal{A}} H, \quad \mathcal{M}(\mathcal{A}) = \mathbb{C}^n - \Sigma(\mathcal{A}). \]

\[ \forall G \in L(\mathcal{A}), \text{ put} \]

\[ \mathcal{A}_G := \{H \in \mathcal{A} \mid G \subseteq H\} \]

\[ \mathcal{A}^G := \{H \cap G \mid H \cap G \neq \emptyset, G \not\subseteq H\} \]
Goal

Give $\mathcal{M}(\mathcal{A})$ the structure of a poset of spaces, where the indexing poset is $L(\mathcal{A})$ and

$$M_G := \mathcal{M}(\mathcal{A})_G \sim \mathcal{M}(\mathcal{A}_G)$$
In 3 papers, with 1) Januszkiewicz-Leary, 2) Januszkiewicz - Leary - Okun, and 3) Settepanella, we computed $H^*(\mathcal{M}(A); A)$ with local coefficients in $A$, where $A = \mathcal{N}_q(\pi_1), \mathbb{Z}\pi_1$ or a generic flat line bundle. (Here $\pi_1 = \pi_1(\mathcal{M}(A)).$)

Key fact: if $\mathcal{A}$ is a central arrangement $H^*(\mathcal{M}(A); A)$ is nonzero in at most one degree.

Original method: a Mayer-Vietoris spectral sequence. $\mathcal{U} = \{U\}$ a cover of $\mathbb{C}^n$ by convex neighborhoods of central arrangements. $\hat{\mathcal{U}} = \{U - \Sigma\}$ is a cover of $\mathcal{M}(A)$. 

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Method

∃ spectral sequence \[ \Longrightarrow H^*(\mathcal{M}(A); A) \] with

\[ E_2^{i,j} = \bigoplus_{G \in L(A)} H^i(N(U|_G), N(U|_{\Sigma(A^G)}); H^j(\mathcal{M}(A_G))) \]

with locally constant coefficients in each summand and with

\[ H^j(N(U|_G), N(U|_{\Sigma(A^G)})) = H^j(G, \Sigma(A^G)) \]

We claimed the coefficients in each summand were constant; however, Graham Denham pointed out to us that in some related situations this wasn’t true.

Goal: fix this.
1. Introduction

2. Posets of spaces
   - A spectral sequence
   - Subspaces of $\mathcal{M}(\mathcal{A})$

3. Cohomology with group ring coefficients

4. Toric arrangements
• \( \mathcal{P} \): a poset  \(|\mathcal{P}|\): its order complex

• A poset of spaces is a functor \( Y : \mathcal{P} \to \text{Top} \), ie, \( p \to Y_p \), and if \( p < q \) a map \( f_{pq} : Y_p \to Y_q \).

• Its homotopy pushout (or “homotopy colimit”) is a generalization of mapping cylinder:

\[
\Delta Y = \left( \coprod_{\sigma \in |\mathcal{P}| \geq p} \sigma \times Y_p \right) / \sim
\]

• \( \exists \) projection \( \pi : \Delta Y \to |\mathcal{P}| \)

• Put \( \Delta Y_{\leq p} = \pi^{-1}(|\mathcal{P}|_{\leq p}) \).
Alternate Definition (D - Okun)

A poset of spaces in $Y$ over $\mathcal{P}$ is a cover $\mathcal{V} = \{ Y_p \}_{p \in \mathcal{P}}$ of $Y$ by open subsets (or by subcomplexes) so that the elements of the cover are indexed by $\mathcal{P}$ and so that

1. $p < q \implies Y_p \subset Y_q$, and
2. the vertex set $\text{Vert}(\sigma)$ of any simplex $\sigma \in N(\mathcal{V})$ has a greatest lower bound $\wedge \sigma$ in $\mathcal{P}$, and
3. $\mathcal{V}$ is closed under taking finite nonempty intersections, ie, for any simplex $\sigma$ of $N(\mathcal{V})$,

$$\bigcap_{p \in \sigma} Y_p = Y_{\wedge \sigma}. $$
Condition \((Z')\)

Let \(A\) be a system of local coefficients on \(Y\). In D - Okun we have condition:

\((Z')\) if \(p < q\), then \(\forall j\) (including \(j = 0\)), then

\[
H^i(Y_q; A) \to H^j(Y_p; A)
\]

is the 0-map.
Theorem (D - O)

Suppose \((Z')\) holds. Then \(\exists\) a spectral sequence \(\Rightarrow\) \(H^*(Y; A)\) which decomposes as a direct sum:

\[
E_{2}^{i,j} = \bigoplus_{p \in \mathcal{P}} H^i(\mathcal{P}_{\geq p}, \mathcal{P}_{> p}; H^j(Y_p; A)).
\]

Moreover, in each summand the coefficients are constant.

Sketch of proof.

We have a poset of coefficients \(p \rightarrow H^j(Y_p; A)\). In general:

\[
E_{1}^{i,j} = \bigoplus_{p \in \mathcal{P}} C^i(\mathcal{P}_{\geq p}, \mathcal{P}_{> p}; H^j(Y_p; A))
\]
Usually horizontal differentials don’t respect the direct sum decomposition; however, $(Z') \implies$ they do.
• $b \in \mathcal{M}(\mathcal{A})$ is a “generic” base point.

• $R_b(H)$ is the real $(2n - 1)$-dim affine space spanned by $b$ and $H$.

  $E_b(H)$ is the half-space in $R_b(H)$ bounded by $H$ on opposite side from $b$. $E_b(H)$ is called a slit.

  $$M_{C^n,b} = \mathcal{M}(\mathcal{A}) - \bigcup_{H \in \mathcal{A}} E_b(H)$$

• For $G \in L(\mathcal{A})$,

  $$M_{G,b} = \mathcal{M}(\mathcal{A}) - \bigcup_{H \in \mathcal{A} - \mathcal{A}_G} E_b(H)$$
Lemma

\{ M_{G,b} \}_{G \in L(A)} is a poset of spaces in \( \mathcal{M}(A) \), ie,

\[ \bigcap_{G \in \text{Vert}(\sigma)} M_G = M_{\wedge \sigma}. \]

Suppose, \( b \) is a generic base point, \( a \in G \), and \( D \) is a small convex neighborhood (say an ellipsoid) of \( [a, b] \). Let \( \rho_b : \mathbb{C}^n \to D \) be radial deformation retraction in direction towards \( b \) onto \( D \).

Lemma

\( \rho_b |_{M_G} \) is a deformation retraction onto \( D - \Sigma(A_G) \) (\( \sim \mathcal{M}(A_G) \)).
Suppose $\mathcal{P} = L(\mathcal{A})$. Then

- $|\mathcal{P}| \geq_G = |\mathcal{P}^{\text{op}}| \leq_G$.
- Folkman’s Theorem: $|\mathcal{P}^{\text{op}}|_{< c^n} \sim \Sigma(\mathcal{A})$ and $|\mathcal{P}^{\text{op}}|_{< G} \sim \Sigma(\mathcal{A}^G)$. So,

$$H^i((\mathcal{P}^{\text{op}})_{\leq G}, (\mathcal{P}^{\text{op}})_{< G}) = H^i(G, \Sigma(\mathcal{A}^G)) = H^{i-1} (\Sigma(\mathcal{A}^G)).$$

Moreover, $\Sigma(\mathcal{A}^G)$ is homotopy equivalent to a wedge of spheres.
- Let $\pi_1 = \pi_1(\mathcal{M}(\mathcal{A}))$
Theorem (DJLO)

Suppose $\mathcal{A}$ is an affine arrangement of rank $n$. Then $H^*(\mathcal{M}(\mathcal{A}); \mathbb{Z}\pi_1)$ is free abelian and concentrated in degree $n$.

Sketch of Proof.

Any central arrangement is $\mathbb{C}^*$-bundle over an affine arrangement, so by induction on rank we can assume result is true for each central arrangement of form $\mathcal{M}(\mathcal{A}_G)$. We have spectral sequence:

$$E_2^{i,j} = \bigoplus_{G \in \mathcal{P}} H^i(\mathcal{P}_{\geq G}, \mathcal{P}_{> G}; H^j(M_G; \mathbb{Z}\pi_1))$$

Also,

$$H^i(\mathcal{P}_{\geq G}, \mathcal{P}_{> G}) = H^i((\mathcal{P}^{op})_{\leq G}, (\mathcal{P}^{op})_{< G}) = \overline{H}^{i-1}(\Sigma(\mathcal{A}_G))$$
So, $H^*(\mathcal{P}_{\geq G}, \mathcal{P}_{> G})$ is free abelian and concentrated in degree $i = \dim G$ (actually $= \text{rk}(\mathcal{A}^G)$) and $H^*(M_G; \mathbb{Z}^{\pi_1})$ is concentrated in degree $j = \text{codim } G$ ($= \text{rk}(\mathcal{A}_G)$). Therefore, $E_2^{i,j} \neq 0$ only for $i + j = n$. 
$T$ is the torus $(\mathbb{C}^*)^n$. Universal cover: $\pi: \mathbb{C}^n \to T$. The group of deck transformations is $\Gamma = 2\pi i \mathbb{Z}^n \subset \mathbb{C}^n$.

$\mathcal{T}$ an arrangement of codim 1 subtori in $T$ (a toric hyperplane arrangement in $T$).

$$\Sigma(\mathcal{T}) = \bigcup_{H \in \mathcal{T}} H \quad \text{and} \quad \mathcal{R}(\mathcal{T}) = T - \Sigma(\mathcal{T}).$$

The inverse images of the toric hyperplanes gives an arrangement $\mathcal{A}$ of affine hyperplanes in $\mathbb{C}^n$. $L(\mathcal{A})$ and $L(\mathcal{T})$ are the respective intersection posets.
General Set-up

Suppose \( \{ Y_p \}_{p \in \mathcal{P}} \) is a poset of spaces over \( \mathcal{P} \) in a space \( Y \). Let \( \pi : \tilde{Y} \to Y \) be a regular covering space with group of covering transformations \( \Gamma \). Then \( \{ \pi_0(\pi^{-1}(Y_p)) \}_{p \in \mathcal{P}} \) gives a poset \( \tilde{\mathcal{P}} \) with \( \Gamma \)-action, with \( \tilde{\mathcal{P}} / \Gamma = \mathcal{P} \). The quotient projection \( \tilde{\mathcal{P}} \to \mathcal{P} \) is denoted by the same letter \( \pi \).

We also get a poset of spaces in \( \tilde{Y} \) over \( \tilde{\mathcal{P}} \): if \( \tilde{p} \in \tilde{\mathcal{P}} \), then \( \tilde{Y}_{\tilde{p}} \) is the corresponding component of \( \pi^{-1}(Y_{\pi(\tilde{p})}) \).

The structure of a poset of spaces for \( \tilde{Y} \) gives an equivariant map \( \tilde{Y} \to |\tilde{\mathcal{P}}| \) and hence, a map \( E\Gamma \times_{\Gamma} \tilde{Y} \to E\Gamma \times_{\Gamma} |\tilde{\mathcal{P}}| \). We consider the Leray-Serre spectral sequence of this map.

If \( H^*(\tilde{Y}; A) \) is a local coefficient system, then there is a version of \((\mathbb{Z}')\).
Theorem

Suppose \((Z')\) holds. There is a spectral sequence converging to \(H^*(E\Gamma \times \Gamma \tilde{Y}, A)\) whose \(E_2\)-term decomposes as a direct sum:

\[
E_2^{i,j} = \bigoplus_{p \in \mathcal{P}} H^i(E\Gamma \tilde{p} \times \Gamma \tilde{p} (|\tilde{P}_{\geq \tilde{p}}|, |\tilde{P}_{> \tilde{p}}|); H^j(Y_p; A))
\]

The coefficients in each summand are locally constant.
For toric arrangements this gives:

**Theorem**

\[ E_{2}^{i,j} = \bigoplus_{G \in L(T)} H^{i}(G, \Sigma(T^{G}); H^{j}(\mathcal{M}(T_{G}); A) \]

If we knew the coefficients were untwisted we would recover the vanishing results in D - Settepanella on cohomology with coefficients in a generic local system, von Neumann algebra or \( \mathbb{Z}_{\pi_{1}} \).