Complements of hyperplane arrangements as posets of spaces

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Notation

- $\mathcal{A} = \{H\}$, a hyperplane arrangement in $\mathbf{C}^n$.
- $L(\mathcal{A}) =$ the intersection poset, partially ordered by reverse inclusion. (So, the minimum element is $\mathbf{C}^n$.)
- $\Sigma(\mathcal{A}) = \bigcup_{H \in \mathcal{A}} H, \qquad \mathcal{M}(\mathcal{A}) = \mathbf{C}^n - \Sigma(\mathcal{A})$.
- $\forall G \in L(\mathcal{A})$, put

$$\mathcal{A}_G := \{H \in \mathcal{A} \mid G \subseteq H\}$$
$$\mathcal{A}^G := \{H \cap G \mid H \cap G \neq \emptyset, G \not\subseteq H\}$$

Goal

Give $\mathcal{M}(\mathcal{A})$ the structure of a poset of spaces, where the indexing poset is $L(\mathcal{A})$ and

$$M_G := \mathcal{M}(\mathcal{A})_G \sim \mathcal{M}(\mathcal{A}_G)$$

Motivation

- In 3 papers, with 1) Januszkiewicz-Leary, 2) Januszkiewicz - Leary - Okun, and 3) Settepanella, we computed $H^*(\mathcal{M}(\mathcal{A}); A)$ with local coefficients in A, where $A = \mathcal{N}_{\mathbf{q}}(\pi_1)$, $\mathbf{Z}\pi_1$ or a generic flat line bundle. (Here $\pi_1 = \pi_1(\mathcal{M}(\mathcal{A}))$.)
- Key fact: if $\mathcal{A}$ is a central arrangement $H^*(\mathcal{M}(\mathcal{A}); A)$ is nonzero in at most one degree.
- Original method: a Mayer-Vietoris spectral sequence. $\mathcal{U} = \{U\}$ a cover of $\mathbf{C}^n$ by convex neighborhoods of central arrangements. $\hat{\mathcal{U}} = \{U - \Sigma\}$ is a cover of $\mathcal{M}(\mathcal{A})$.

Method

$\exists$ spectral sequence $\implies H^*(\mathcal{M}(\mathcal{A}); A)$ with

$$E_2^{i,j} = \bigoplus_{G \in L(\mathcal{A})} H^i(N(\mathcal{U}|_G), N(\mathcal{U}|_{\Sigma(\mathcal{A}^G)}); H^j(\mathcal{M}(\mathcal{A}_G))$$

with locally constant coefficients in each summand and with $H^j(N(\mathcal{U}|_G), N(\mathcal{U}|_{\Sigma(\mathcal{A}^G)}) = H^j(G, \Sigma(\mathcal{A}^G))$

We claimed the coefficients in each summand were constant; however, Graham Denham pointed out to us that in some related situations this wasn't true.

Goal: fix this.

- $\mathcal{P}$: a poset $|\mathcal{P}|$: its order complex
- A *poset of spaces* is a functor $Y : \mathcal{P} \to$ **Top**, ie, $p \to Y_p$, and if $p < q$ a map $f_{pq} : Y_p \to Y_q$.
- Its *homotopy pushout* (or "homotopy colimit") is generalization of mapping cylinder:

$$\Delta Y = \left(\coprod_{\sigma \in |\mathcal{P}_{\geq p}|} \sigma \times Y_p \right) / \sim$$

- $\exists$ projection $\pi : \Delta Y \to |\mathcal{P}|$
- Put $\Delta Y_{\leq p} = \pi^{-1}(|\mathcal{P}_{\leq p}|)$.

Alternate Definition (D - Okun)

A *poset of spaces in Y over* $\mathcal{P}$ is a cover $\mathcal{V} = \{Y_p\}_{p\in\mathcal{P}}$ of Y by open subsets (or by subcomplexes) so that the elements of the cover are indexed by $\mathcal{P}$ and so that

- $p < q \implies Y_p \subset Y_q$, and
- the vertex set $\mathrm{Vert}(\sigma)$ of any simplex $\sigma \in N(\mathcal{V})$ has a greatest lower bound $\wedge\sigma$ in $\mathcal{P}$, and
- $\mathcal{V}$ is closed under taking finite nonempty intersections, ie, for any simplex σ of $N(\mathcal{V})$,

$$\bigcap_{p\in\sigma} Y_p = Y_{\wedge\sigma}.$$

Condition (Z′)

Let A be a system of local coeffiicients on Y. In D - Okun we have condition:

(Z′) if $p < q$, then $\forall j$ (including $j = 0$), then

$$H^j(Y_q; A) \to H^J(Y_p; A)$$

is the 0-map.

Theorem (D - O)

Suppose (Z') holds. Then $\exists$ a spectral sequence $\implies$ $H^*(Y;A)$ which decomposes as a direct sum:

$$E_2^{i,j} = \bigoplus_{p\in\mathcal{P}} H^i(\mathcal{P}_{\geq p}, \mathcal{P}_{>p}; H^j(Y_p; A)).$$

Moreover, in each summand the coefficients are constant.

Sketch of proof.

We have a poset of coefficients $p \to \mathcal{H}^j(Y_p; A)$. In general:

$$E_1^{i,j} = \bigoplus_{p\in\mathcal{P}} C^i(\mathcal{P}_{\geq p}, \mathcal{P}_{>p}; H^j(Y_p; A))$$

□

Usually horizontal differentials don't respect the direct sum decomposition; however, (Z′) $\implies$ they do. □

- $b \in \mathcal{M}(\mathcal{A})$ is a “generic” base point.
- $R_b(H)$ is the real $(2n-1)$-dim affine space spanned by b and H.
  $E_b(H)$ is the half-space in $R_b(H)$ bounded by H on opposite side from b. $E_b(H)$ is called a *slit*.
- $$M_{\mathbf{C}^n,b} = \mathcal{M}(\mathcal{A}) - \bigcup_{H\in\mathcal{A}} E_b(H)$$
- For $G \in L(\mathcal{A})$,
  $$M_{G,b} = \mathcal{M}(\mathcal{A}) - \bigcup_{H\in\mathcal{A}-\mathcal{A}_G} E_b(H)$$

Lemma

$\{M_{G,b}\}_{G\in L(\mathcal{A})}$ is a poset of spaces in $\mathcal{M}(\mathcal{A})$, ie,

$$\bigcap_{G\in \mathrm{Vert}(\sigma)} M_G = M_{\wedge\sigma}.$$

Suppose, b is a generic base point, $a \in G$, and D is a small convex neigborhood (say an ellipsoid) of $[a, b]$. Let $\rho_b : \mathbf{C}^n \to D$ be radial deformation retraction in direction towards b onto D.

Lemma

$\rho_b|_{M_G}$ is a deformation retraction onto $D - \Sigma(\mathcal{A}_G)$ $(\sim \mathcal{M}(\mathcal{A}_G))$.

Suppose $\mathcal{P} = L(\mathcal{A})$. Then

- $|\mathcal{P}|_{\geq G} = |\mathcal{P}^{op}|_{\leq G}$.
- Folkman's Theorem: $|\mathcal{P}^{op}|_{<\mathbf{C}^n} \sim \Sigma(\mathcal{A})$ and $|\mathcal{P}^{op}|_{<G} \sim \Sigma(\mathcal{A}^G)$. So,

$$H^i((\mathcal{P}^{op})_{\leq G}, (\mathcal{P}^{op})_{<G}) = H^i(G, \Sigma(\mathcal{A}^G)) = \overline{H}^{i-1}(\Sigma(\mathcal{A}^G)).$$

  Moreover, $\Sigma(\mathcal{A}^G)$ is homotopy equivalent to a wedge of spheres.
- Let $\pi_1 = \pi_1(\mathcal{M}(\mathcal{A}))$

Theorem (DJLO)

Suppose $\mathcal{A}$ is an affine arrangement of rank n. Then $H^*(\mathcal{M}(\mathcal{A}); \mathbf{Z}\pi_1)$ is free abelian and concentrated in degree n.

Sketch of Proof.

Any central arrangement is $\mathbf{C}^*$-bundle over an affine arrangement, so by induction on rank we can assume result is true for each central arrangement of form $\mathcal{M}(\mathcal{A}_G)$. We have spectral sequence:

$$E_2^{i,j} = \bigoplus_{G \in \mathcal{P}} H^i(\mathcal{P}_{\geq G}, \mathcal{P}_{>G}; H^j(M_G; \mathbf{Z}\pi_1))$$

Also,
$H^i(\mathcal{P}_{\geq G}, \mathcal{P}_{>G}) = H^i((\mathcal{P}^{op})_{\leq G}, (\mathcal{P}^{op})_{<G}) = \overline{H}^{i-1}(\Sigma(\mathcal{A}^G)).$ □

So, $H^*(\mathcal{P}_{\geq G}, \mathcal{P}_{>G})$ is free abelian and concentrated in degree $i = \dim G$ (actually $= \mathrm{rk}(\mathcal{A}^G)$) and $H^*(M_G; \mathbf{Z}\pi_1)$ is concentrated in degree $j = \operatorname{codim} G$ $(= \mathrm{rk}(\mathcal{A}_G))$. Therefore, $E_2^{i,j} \neq 0$ only for $i + j = n$. □

Notation

- T is the torus $(\mathbf{C}^*)^n$. Universal cover: $\pi : \mathbf{C}^n \to T$. The group of deck transformations is $\Gamma = 2\pi i\mathbf{Z}^n \subset \mathbf{C}^n$.
- $\mathcal{T}$ an arrangement of codim 1 subtori in T (a *toric hyperplane arrangement* in T).

$$\Sigma(\mathcal{T}) = \bigcup_{H\in\mathcal{T}} H \quad \text{and} \quad \mathcal{R}(\mathcal{T}) = T - \Sigma(\mathcal{T}).$$

- The inverse images of the toric hyperplanes gives an arrangement $\mathcal{A}$ of affine hyperplanes in $\mathbf{C}^n$.
  $L(\mathcal{A})$ and $L(\mathcal{T})$ are the respective intersection posets.

General Set-up

- Suppose $\{Y_p\}_{p\in\mathcal{P}}$ is a poset of spaces over $\mathcal{P}$ in a space Y. Let $\pi : \tilde{Y} \to Y$ be a regular covering space with group of covering transformations Γ. Then $\{\pi_0(\pi^{-1}(Y_p)\}_{p\in\mathcal{P}}$ gives a poset $\tilde{\mathcal{P}}$ with Γ-action, with $\tilde{\mathcal{P}}/\Gamma = \mathcal{P}$. The quotient projection $\tilde{\mathcal{P}} \to \mathcal{P}$ is denoted by the same letter π.
- We also get a poset of spaces in $\tilde{Y}$ over $\tilde{\mathcal{P}}$: if $\tilde{p} \in \tilde{\mathcal{P}}$, then $\tilde{Y}_{\tilde{p}}$ is the corresponding component of $\pi^{-1}(Y_{\pi(\tilde{p})})$.
- The structure of a poset of spaces for $\tilde{Y}$ gives an equivariant map $\tilde{Y} \to |\tilde{\mathcal{P}}|$ and hence, a map $E\Gamma \times_\Gamma \tilde{Y} \to E\Gamma \times_\Gamma |\tilde{\mathcal{P}}|)$. We consider the Leray-Serre spectral sequence of this map.
- If $H^*(\tilde{Y}; A)$ is a local coefficient system, then there is a version of (Z′).

Theorem

Suppose (Z') holds. There is a spectral sequence converging to $H^*(E\Gamma \times_\Gamma \tilde{Y}; A)$ whose E_2-term decomposes as a direct sum:

$$E_2^{i,j} = \bigoplus_{p \in \mathcal{P}} H^i(E\Gamma_{\tilde{p}} \times_{\Gamma_{\tilde{p}}} (|\tilde{\mathcal{P}}_{\geq \tilde{p}}|, |\tilde{\mathcal{P}}_{> \tilde{p}}|); H^j(Y_p; A))$$

The coefficients in each summand are locally constant.

For toric arrangements this gives:

Theorem

$$E_2^{i,j} = \bigoplus_{G \in L(\mathcal{T})} H^i(G, \Sigma(\mathcal{T}^G); H^j(\mathcal{M}(\mathcal{T}_G); A)$$

If we knew the coefficients were untwisted we would recover the vanishing results in D - Settepanella on cohomology with coefficients in a generic local system, von Neumann algebra or $\mathbf{Z}\pi_1$.