Right angularity, flag complexes, asphericity

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I want to discuss three related constructions of spaces and give conditions for each of them to be aspherical:

- polyhedral product construction,
- reflection trick applied to a “corner” of spaces,
- the pulback of a “corner” via a coloring of a simplicial complex.

All three are related to the construction of the Davis complex for a RACG

Notation

Given a simplicial complex L with vertex set I, put

- $\mathcal{S}(L) := \{\text{simplices in } L\}$ (including $\emptyset$)
- Given $\sigma \in \mathcal{S}(L)$, $I(\sigma)$ is its vertex set.
- $[-1, 1]^I$ is the $|I|$-dim'l cube.
- $\mathbf{C}_2$ $(= \{\pm 1\})$ is the cyclic group of order 2. It acts on $[-1, 1]$. Hence, $(\mathbf{C}_2)^I \curvearrowright [-1, 1]^I$.
- For $\mathbf{x} := (x_i)_{i \in I}$, a point in $[-1, 1]^I$, put

$$\operatorname{Supp}(\mathbf{x}) := \{i \in I \mid x_i \in (-1, 1)\}.$$

Define a subcx $\mathcal{Z}_L$ (or $\mathcal{Z}_L([-1,1],\{\pm 1\})$) of $[-1,1]^I$ by

$$\mathcal{Z}_L := \{\mathbf{x} \in [-1,1]^I \mid \operatorname{Supp}(\mathbf{x}) \in \mathcal{S}(L)\}$$

Alternatively, suppose $\mathcal{Z}_\sigma$ is the union of all faces parallel to $[-1,1]^{I(\sigma)}$, ie, $\mathcal{Z}_\sigma = \{\mathbf{x} \in [-1,1]^I \mid x_i = \pm 1 \text{ if } i \notin I(\sigma)\}$. Then

$$\mathcal{Z}_L = \bigcup_{\sigma \in \mathcal{S}(L)} \mathcal{Z}_\sigma .$$

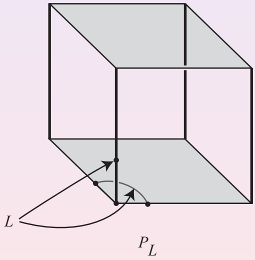
$\mathcal{Z}_L$ is stable under the $(\mathbf{C}_2)^I$-action on $[-1,1]^I$.

Example

If $L = \Delta$, the simplex on I, then $\mathcal{S}(L)$ is the power set of I and $\mathcal{Z}_\Delta = [-1,1]^I$.

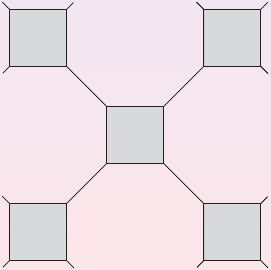
Example

Suppose the vertex set I consists of 3 points and L is the disjoint union of 1 edge and 1 point.



- Let $\widetilde{\mathcal{Z}}_L$ be the universal cover of $\mathcal{Z}_L$.
- The group of all lifts of the $(\mathbf{C}_2)^I$-action is W_{L^1}, the RACG associated to L^1, ie,

$$1 \to \pi_1(\mathcal{Z}_L) \to W_{L^1} \to (\mathbf{C}_2)^I \to 1$$



$\tilde{P}_L$

- $\widetilde{\mathcal{Z}}_L$ is CAT(0) iff L is a flag cx. (Pf: the link of each vertex is L. Then use Gromov's Lemma.)
- (Recall L is a *flag cx* if it is obtained from L^1 by filling in every complete subgraph with simplex.)
- If this is the case, $\widetilde{\mathcal{Z}}_L$ is the *Davis complex* associated to W_L.
- Next, we want to extend this to arbitrary pairs of spaces rather than $([-1,1],\{\pm 1\})$.

Data

- A simplicial complex L with vertex set I.
- A family of pairs of spaces $(\underline{A}, \underline{B}) = \{(A(i), B(i))\}_{i \in I}$, indexed by I.

Definition

The *polyhedral product* $\mathcal{Z}_L(\underline{A}, \underline{B})$ is the subset of $\prod_{i \in I} A(i)$ consisting of those $\mathbf{x}$ such that $\mathrm{Supp}(\mathbf{x}) \in \mathcal{S}(L)$.

Equivalently, if $\mathcal{Z}_\sigma(\underline{A}, \underline{B}) := \{\mathbf{x} \in \prod A(i) \mid x_i \in B(i) \text{ if } i \notin \sigma\}$,

$$\mathcal{Z}_L(\underline{A}, \underline{B}) = \bigcup_{\sigma \in \mathcal{S}(L)} \mathcal{Z}(\underline{A}, \underline{B}).$$

If all $(A(i), B(i))$ are the same, say (A, B), then we omit underlining and write $\mathcal{Z}_L(A, B)$ instead of $\mathcal{Z}_L(\underline{A}, \underline{B})$.

Example

- If L is a flag cx, then$\mathcal{Z}_L(S^1, *)$ is the standard $K(\pi, 1)$ for the RAAG associated to L^1.
- The space $\mathcal{Z}_L(D^2, S^1)$ has been called the *moment angle complex*. It is simply connected and admits an $(S^1)^I$-action.

There has been a great deal of work lately on computing cohomology and stable homotopy type of polyhedral products, by Bahri-Bendersky-Cohen-Gitler, Denham-Suciu, Buchstaber-Panov and others.

Data

- A simplicial graph L^1 with vertex set I.
- A family of discrete gps $\underline{G} = \{G_i\}_{i \in I}$

Definition

The *graph product* of the G_i is the group Γ formed quotienting the free product of the G_i by the normal subgroup generated by all commutators of the form $[g_i, g_j]$ where $\{i, j\} \in \text{Edge}(L^1)$, $g_i \in G_i$ and $g_j \in G_j$.

Example

- If all $G_i = \mathbf{C}_2$, then Γ is the RACG determined by L^1.
- If all $G_i = \mathbb{Z}$, then Γ is the RAAG determined by L^1.

If each $A(i)$ is path connected & $B(i)$ is a basepoint, $*_i$, then $\pi_1(\mathcal{Z}_L(\underline{A}, \underline{B}))$ is the graph product of the $G_i := \pi_1(A(i), *_i)$. (Pf: van Kampen's Theorem).

Relative graph products

More data

For each $i \in I$, suppose given a gp G_i & a G_i-set $E(i)$.
Put $(\text{Cone}\,\underline{E}, \underline{E}) := \{(\text{Cone}\,E(i), E(i))\}_{i \in I}$.

- Form the polyhedral product $\mathcal{Z}_L(\text{Cone}\,\underline{E}, \underline{E})$. It is not simply connected if at least 1 $E(i)$ has more than 1 element. Let

$$\widetilde{\mathcal{Z}}_L(\text{Cone}\,\underline{E}, \underline{E}) := \text{ the univ cover of } \mathcal{Z}_L(\text{Cone}\,\underline{E}, \underline{E}).$$

- $G = \prod_{i \in I} G_i \curvearrowright \mathcal{Z}_L(\text{Cone}\,\underline{E}, \underline{E})$. Let Γ be the gp of all lifts of G-action to $\widetilde{\mathcal{Z}}_L(\text{Cone}\,\underline{E}, \underline{E})$. Γ is the *graph product of the G_i relative to the $E(i)$*. (Only the 1-skeleton, L^1, matters in this defn.) (This defn needs to be tweaked if G does not act effectively on $\prod E(i)$.)

Example

If each $G_i = \mathbf{C}_2$, then $\mathcal{Z}_L(\text{Cone}\,\mathbf{C}_2, \mathbf{C}_2)$ is the space $\mathcal{Z}_L([-1,1], \{\pm 1\})$ considered previously.

Remarks

- If each $E(i) = G_i$, then the group of lifts, Γ, agrees with the first definition of graph product.
- The inverse image of $\prod_{i \in I} E(i)$ in $\widetilde{\mathcal{Z}}_L(\text{Cone}\,\underline{E}, \underline{E})$ is the set of (centers of) chambers in a "right-angled building" (a RAB).
- If L is a flag complex, then $\widetilde{\mathcal{Z}}_L([-1,1], \{\pm 1\})$ is the Davis complex for the RACG W and $\widetilde{\mathcal{Z}}_L(\text{Cone}\,\underline{E}, \underline{E})$ is the standard realization of the RAB.

- Let $(\underline{A}, \underline{B}) = (A(i), B(i))_{i \in I}$. Suppose each $A(i)$ is path connected. Let $p_i : \widetilde{A}(i) \to A(i)$ be the univ cover.
- Put $G_i = \pi_1(A(i))$ and let $E(i)$ be the set of path components of $p_i^{-1}(B(i))$ in $\widetilde{A}(i)$. So, $E(i)$ is a G_i-set.

Proposition

$\pi_1(\mathcal{Z}_L(\underline{A}, \underline{B})) = \Gamma$, *where* Γ *is the relative graph product of the* $(G_i, E(i))$.

Remember: $G = \prod G_i$ acts on $\mathcal{Z}_L(\text{Cone}\,\underline{E}, \underline{E})$ and Γ is gp of lifts of G-action to $\widetilde{\mathcal{Z}}_L(\text{Cone}\,\underline{E}, \underline{E})$.

Proof of Proposition.

$(\underline{\widetilde{A}}, \underline{\widetilde{B}}) := \{(\widetilde{A}(i), p_i^{-1}(B(i))\}_{i \in I}$. $\mathcal{Z}_L(\underline{\widetilde{A}}, \underline{\widetilde{B}}) \to \mathcal{Z}_L(\underline{A}, \underline{B})$ is an intermediate covering space and G is the gp of deck transformations. There is a G-equivariant map $\mathcal{Z}_L(\underline{\widetilde{A}}, \underline{\widetilde{B}}) \to \mathcal{Z}_L(\text{Cone}\,\underline{E}, \underline{E})$ inducing an iso on π_1. The univ cover $\widetilde{\mathcal{Z}}_L(\underline{A}, \underline{B}) \to \mathcal{Z}_L(\underline{\widetilde{A}}, \underline{\widetilde{B}})$ is induced from $\widetilde{\mathcal{Z}}_L(\text{Cone}\,\underline{E}, \underline{E}) \to \mathcal{Z}_L(\text{Cone}\,\underline{E}, \underline{E})$. □

When is a polyhedral product aspherical?

Suppose L is a flag cx. $\mathcal{Z}_L(A, B)$ is aspherical in the following cases:

- $\mathcal{Z}_L(S^1, *) = BA_L$, where A_L is the associated RAAG.
- $\mathcal{Z}_L([-1, 1], \{\pm 1\}) = B\pi$, where $\pi = \mathrm{Ker}(W_L \to (\mathbf{C}_2)^m)$.

What is the common generalization?

Definition

A pair of CW complexes (A, B) is *aspherical*, if A is aspherical, each path component of B is aspherical and the fundamental gp of any such component injects into $\pi_1(A)$.

Definition

A vertex i of a simplicial cx L is *conelike* if it is connected by an edge to every other vertex.

Theorem

$\mathcal{Z}_L(\underline{A}, \underline{B})$ is aspherical $\iff$

1. *Each $A(i)$ is aspherical.*
2. *For each non-conelike vertex $i \in I$, $(A(i), B(i))$ is aspherical.*
3. *L is a flag cx.*

Corollary

If $(A(i), B(i)) = (BG_i, *)$ *and* L *is a flag cx, then* $\mathcal{Z}_L(\underline{A}, \underline{B}) = B\Gamma$*, the classifying space for the graph product* Γ*.*

Corollary

Suppose each $(A(i), B(i)) = (M_i, \partial M_i)$ *is a mfld with bdry and an aspherical pair. Also suppose* L *is a flag triangulation of a sphere. Then* $\mathcal{Z}_L(\underline{A}, \underline{B}) \subset \prod M_i$ *is a closed aspherical mfld.*

Ingredients for the proof

Retraction Lemma

Suppose $L' \subset L$ is a full subcx on vertex set I'. Then the map $r : \mathcal{Z}_L(\underline{A}, \underline{B}) \to Z_{L'}(\underline{A}, \underline{B})$ induced by $\prod_{i \in I} A(i) \to \prod_{i \in I'} A(i)$ is a retraction.

RAB Lemma

Suppose $\underline{E} = (E(i))_{i \in I}$ is a collection of sets (each with the discrete topology). Then $\widetilde{\mathcal{Z}}_L(\text{Cone}\,\underline{E}, \underline{E})$ is contractible $\iff$ L is a flag complex. Moreover, if this is the case, then $\widetilde{\mathcal{Z}}_L(\text{Cone}\,\underline{E}, \underline{E})$ is the “standard realization” of a RAB of type W_L.

Restatement of Theorem

$\mathcal{Z}_L(\underline{A}, \underline{B})$ is aspherical $\iff$

1. Each $A(i)$ is aspherical.
2. For each non-conelike vertex $i \in I$, $(A(i), B(i))$ is aspherical.
3. L is a flag cx.

Comment

What is the point of Condition (ii)? If L is flag, then the set of conelike vertices spans a simplex Δ and L decomposes as a join, $L = L' * \Delta$, and $\mathcal{Z}_L$ as a product:

$$\mathcal{Z}_L(\underline{A}, \underline{B}) = Z_{L'}(\underline{A}, \underline{B}) \times \prod_{i \in \operatorname{Vert} \Delta} A(i),$$

so the $B(i)$ for conelike vertices do not enter the picture.

Restatement of Theorem

$\mathcal{Z}_L(\underline{A}, \underline{B})$ is aspherical $\iff$

1. Each $A(i)$ is aspherical.
2. For each non-conelike vertex $i \in I$, $(A(i), B(i))$ is aspherical.
3. L is a flag cx.

Sketch of proof in $\implies$ direction

- Retraction Lemma $\implies$ (i) and (ii).
- RAB Lemma $\implies$ (iii)

Definition

A *mirror structure* $\mathcal{M}$ on a space X is a collection $\{X_i\}_{i \in I}$ of closed subspaces. Its *nerve* is the simplicial cx, $N(\mathcal{M})$, with vertex set I and $\sigma \leq I$ is a simplex iff $X_\sigma \neq \emptyset$, where $X_\sigma := \bigcap_{i \in \sigma} X_i$.

$$[n] = \{1, \dots, n\}.$$

Definition

The mirror structure $\mathcal{M}$ is a *corner* if $I = [n]$ and $X_{[n]} \neq \emptyset$, ie, if $N(\mathcal{M}) = \Delta^{n-1}$.

- Let W be a finite Coxeter group of rank n with fund set of generators indexed by $[n]$, eg, $W = (\mathbf{C}_2)^n$.
- Define equiv. relation $\sim$ on $W \times X$ by $(w, x) \sim (w', x')$ iff $x = x'$ and $w\, W_{I(x)} = w'\, W_{I(x)}$, where $I(x) := \{i \in [n] \mid x \in X_i\}$. The *basic construction* is the W-space,
- $\mathcal{U}(W, X) := (W \times X)/\sim$.
- **Question:** When is $\mathcal{U}(W, X)$ aspherical?

- Let $p : \widetilde{X} \to X$ be the univ cover. For $i \in [n]$, let E_i be the set of path components of $p^{-1}(X_i)$ and let $E = \coprod_{i \in [n]} E_i$.
- There is an induced mirror structure, $\widetilde{\mathcal{M}} = \{\widetilde{X}_e\}_{e \in E}$, where $\widetilde{X}_e := e$. Let N $(= N(\widetilde{\mathcal{M}}))$ denote its nerve.
- There is a"coloring" $f : N \to \Delta^{n-1}$ defined by $E_i \to i$ and an induced Coxeter group W_N with set of fund generators indexed by E.

Three conditions

1. X is aspherical (ie, $\widetilde{X}$ is contractible),
2. For each $\sigma \in N(\widetilde{\mathcal{M}})$, $\widetilde{X}_\sigma$ is acyclic.
3. $N(\widetilde{\mathcal{M}})$ is a flag cx.

Theorem

$\mathcal{U}(W, X)$ is aspherical iff conditions 1, 2, 3.

Sketch.

Univ. cover is $\mathcal{U}(W_N, \widetilde{X})$. Three conditions $\Longrightarrow$ it is contractible. □

When do these conditions hold?

Example (Products)

Suppose $(\underline{A}, \underline{B}) = (A(i), B(i))_{i \in [n]}$, where $A(i)$ is aspherical and each component of $B(i)$ is aspherical and π_1-injective. Put $X = \prod_{i \in [n]} A(i)$. Define a corner structure on X by $X_i = \{\mathbf{x} \in \prod A(i) \mid x_i \in B(i)\}$. Let E_i be the set of path components of $p^{-1}(X_i)$ in $\widetilde{X}$. Since $N(\widetilde{\mathcal{M}})$ is the n-fold join, $* E_i$, it is a flag cx.

Example (Borel-Serre compactifications)

Suppose Γ is a torsion-free arithmetic lattice in the real points of an algebraic Lie group G of $\mathbb{Q}$-rank $n > 0$. The quotient of the symmetric space by Γ can be compactified to a manifold with corners X (which is a corner) so that each stratum is aspherical, π_1-injective. The nerve $N(\widetilde{\mathcal{M}})$ is the spherical bldg associated to $G(\mathbb{Q})$; hence, a flag cx.

The idea of applying the reflection gp trick to these examples was explained to me by Tam Phan, who has recently written a preprint on the subject.

- $f : L \to \Delta^{n-1}$ a nondegenerate simplicial surjection (a "coloring). It induces $f : \mathcal{S}(L) \to \mathcal{S}(\Delta^{n-1}) = \mathcal{P}([n])$. For $\tau \le [n]$, let $\tau^{\vee} := [n] - \tau$
- $\mathcal{M} = \{X_i\}_{i \in [n]}$, a corner structure on X.
- Want to define a new space $f^*(X)$. For each $\sigma \in \mathcal{S}(L)$, define $Q(\sigma) \le \mathcal{S}(L) \times X$, by $Q(\sigma) := (\sigma, X_{f(\sigma)^{\vee}})$.
There is an obvious equiv relation on $Q := \coprod_{\sigma \in \mathcal{S}(L)} Q(\sigma)$ which identifies $(\sigma', X_{f(\sigma')^{\vee}})$ with the corresponding face of $(\sigma, X_{f(\sigma)^{\vee}})$, whenever $\sigma' \le \sigma$. Put

$$f^*(X) = Q/\sim .$$

When is $f^*(X)$ aspherical?

Recall the three conditions

1. X is aspherical (ie, $\widetilde{X}$ is contractible),
2. For each $\sigma \in N(\widetilde{\mathcal{M}})$, $\widetilde{X}_\sigma$ is acyclic.
3. $N(\widetilde{\mathcal{M}})$ is a flag cx.

Theorem

Suppose conditions 1,2,3 $+$

- *L is a flag cx.*

Then $f^*(X)$ is aspherical.

Sketch.

Let Y be cubical cx associated to $N(\widetilde{\mathcal{M}})$. L a flag cx $\implies$ $f^*(Y)$ is locally CAT(0). Moreover, $f^*(\widetilde{X})$ is homotopy equivalent to $f^*(Y)$; hence, univ cover of $f^*(\widetilde{X})$ is contractible. □

Remark

If X is a mfld with corners and L is a triangulation of sphere (or generalized homology sphere) then $f^*(X)$ is a mfld. So, these methods can be used to construct aspherical mflds.

Reference

Right angularity, flag complexes, asphericity.
arXiv:1002.2564.