Aspherical manifolds that cannot be triangulated

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August 16, 2013

Although Kirby - Siebenmann showed that, in dimensions ≥ 5, $\exists$ mflds which do not admit a PL structures, the possibility remained that all mflds could be triangulated. In dim 4, Freedman's E_8 mfld (and others) are not PL; moreover, they cannot be triangulated. In 1991 Januszkiewicz and I applied Gromov's hyperbolization technique to the E_8-mfld to show the existence of nontriangulable aspherical 4-mflds. In dims ≥ 5 the existence of nontriangulable mflds depended on the nonexistence of homology 3-spheres with certain properties. In 2013 this question about homology spheres was resolved by Manolescu. So, $\exists$ nontriangulable M^n for $n \geq 5$. We use two versions of hyperbolization to show that, for $n \geq 6$, these can be chosen aspherical.

Why aspherical?

At one point the only examples of closed aspherical mflds came from differential geometry or Lie groups; hence, were smooth mflds. Gromov's hyperbolization showed that you could convert simplicial complexes (hence, triangulated mflds) into aspherical ones. But to get non PL mflds you need a trick.

1. Introduction

2. Dimension 4
   - Rokhlin's Theorem and the μ-invariant
   - Freedman's E_8-manifold
   - Hyperbolization

3. Dimensions > 4
   - Kirby - Siebenmann
   - Galewski - Stern $+$ Manolescu
   - Relative hyperbolization

Polyhedral homology mflds

Definitions

A simplicial cx L^n is a *polyhedral homology n-mfld* (a PHM for short) if for each k-simplex σ, $\mathrm{Lk}(\sigma, L)$ has the same homology as S^{n-k-1}. L^n is a *PL mfld* if $\forall \sigma \in L$, $\mathrm{Lk}(\sigma, L)$ is PL homeomorphic to $S^{n-\dim\sigma-1}$.

If the 4-dim PL Poincaré Conjecture is true, then we can drop "PL" and shorten "PL homeomorphic" to "homeomorphic" in the above.

Meaning of the Double Suspension Theorem

In dims ≥ 5 top mflds can have triangulations (as PHMs) which are **not** PL.

Fact

If B is an even, nondegenerate, symmetric bilinear form over $\mathbb{Z}$, then its signature, $\sigma(B)$, is divisible by 8.

Theorem (Rokhlin 1952)

If M^4 is a closed PL 4-mfld, with $w_1 = 0$ and $w_2 = 0$, then

$$\sigma(M^4) \equiv 0 \mod 16.$$

Fact

If H^3 is a homology 3-sphere, then $H^3 = \partial W^4$, where W^4 is a PL mfld with even intersection form.

The μ-invariant

Define

$$\mu(H^3) = \frac{\sigma(W^4)}{8} \in \mathbb{Z}/2.$$

This defines a homomorphism $\mu : \Theta_3^H \to \mathbb{Z}/2$, where Θ_3^H is the group of homology cobordism classes of homology 3-spheres.

$$Q(E_8) := \text{the } E_8 \text{ plumbing.}$$

It is a smooth 4-manifold with bdry.
$\partial Q(E_8) = H^3$, Poincaré's homology 3-sphere. $\sigma(Q(E_8) = 8$. Let $X^4 := Q(E_8) \cup c(H^3)$. It is a PHM of signature 8.

Theorem (Freedman 1982)

$H^3 = \partial C^4$, where C^4 is a top contractible mfld. Put $M^4 = Q(E_8) \cup C^4$, the "E_8-manifold".

By Rokhlin's Thm, M^4 does not have a PL structure.

Fact

Any triangulation of a 4-mfld is automatically PL. (Pf: By the Poincaré Conj, the link of any vertex is PL homeomorphic toS^3.) So, Freedman's M^4 is not triangulable.

Hyperbolization (Gromov)

A *hyperbolization procedure* is a functor $\mathfrak{h}$ from {simplicial complexes} to {locally CAT(0) spaces} together with a map $f : \mathfrak{h}(K) \to K$ with the following properties:

- $\mathfrak{h}$ preseves local structure: $\forall \sigma \in K$, $\mathrm{Lk}(\mathfrak{h}(\sigma)) \cong \mathrm{Lk}(\sigma)$ (Lk means "link".) In particular, if K is a mfld (or a PHM), then so is $\mathfrak{h}(K)$.
- f^* is a split injection on cohomology.
- When K is a mfld, f pulls back stable tangent bundle to stable tangent bundle. So, f^* pulls back characteristic classes of K to those of $\mathfrak{h}(K)$.

(D - Januszkiewicz)

- Apply $\mathfrak{h}$ to the E_8 homology mfld X^4.
- Resolve it to $N^4 = (\mathfrak{h}(X^4) - \text{nbhd of cone pt}) \cup C^4$.
- Then N^4 is aspherical and not triangulable.

Theorem (DJ 1991)

$\exists$ closed aspherical 4-mflds that cannot be triangulated. For $n \geq 5$, $\exists$ closed aspherical n-mflds which are not homotopy equivalent to PL mflds.

Proof of 2nd sentence.

$N^4 \times T^k$ is not PL. $\square$

Remark

By Double Suspension Thm, for $k > 0$, $N^4 \times T^k \cong X^4 \times T^k$ (where X^4 is the PHM). So, $N^4 \times T^k$ can be triangulated.

Theorem (Kirby - Siebenmann 1969)

A top n-mfld, $n \geq 5$, admits a PL structure $\iff$ an obstruction $\Delta \in H^4(M^n; \mathbb{Z}/2)$ vanishes.

In other words, TOP/PL is the Eilenberg-MacLane space $K(\mathbb{Z}/2, 3)$.

Polyhedral Mfld Characterization Theorem

Theorem (Edwards 1978 + Perelman)

A PHM (of dim > 2) is a top mfld $\iff$ the link of each vertex is simply connected.

Example

The double suspension of a homology sphere, with $\pi_1 \neq 1$.

Such triangulations are not PL.

In the early seventies such considerations led Siebenmann to ask if all mflds could actually be triangulated (before the Double Suspension Thm or Freedman's E_8 4-mfld were known).

I will describe highlights of a theory worked out in the 1970 s by several people, Siebenmann, Matumoto, most notably Galewski - Stern (with important contributions by others, eg, Cohen, Sullivan, Martin, Maunder).

Suppose X is a PHM. Let $\lambda \in H^4(X; \Theta_3^H)$ be the cohomology class which associates to the "dual cell" of a codim 4 simplex σ, the class of $\mathrm{Lk}(\sigma)$ in Θ_3^H (where Θ_3^H is the group of homology cobordism classes of homology 3-spheres).
λ is the obstruction to finding an "acyclic resolution" of X by a PL manifold.

Consider the coefficient sequence:

$$0 \to \operatorname{Ker}\mu \longrightarrow \Theta_3^H \xrightarrow{\mu} \mathbb{Z}/2 \to 0.$$

Fact 1

When X is a top mfld, μ_* takes $\lambda \in H^4(X; \Theta_3^H)$ to the Kirby-Siebenmann obstruction $\Delta \in H^4(X; \mathbb{Z}/2)$.

Fact 2

If M is a top mfld, then the obstruction to triangulation is $\beta(\Delta) \in H^5(M; \operatorname{Ker}\mu)$, where β is the Bockstein associated to the above coefficient sequence.

Theorem (Galewski-Stern $\sim$ 1980)

In dim $n > 4$, $\exists$ nontriangulable M^n iff the sequence $0 \to \mathrm{Ker}\,\mu \to \Theta_3^H \to \mathbb{Z}/2 \to 0$ does not split, ie, iff $\nexists$ a homology 3-sphere H^3 with $\mu(H^3) \neq 0$ and with $H^3 \# H^3 = 0$ in Θ_3^H.

Theorem (Manolescu 2013)

The sequence does not split.

Galewski-Stern mflds

It suffices to consider the Bockstein associated to

$$0 \to \mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \to 0.$$

This Bockstein is Sq^1 (the first Steenrod square).

$\exists$ a (nonorientable) PHM P^5 with bdry st

- int P^5 is a top mfld (this uses Edwards' Thm).
- $\Delta(P^5) = \mu_*(\lambda(P^5)) \neq 0$ in $H^4(P^5; \mathbb{Z}/2)$ and $\Delta(\partial P^5) = 0$.
- $Sq^1(\Delta) \neq 0$ in $H^5(P^5, \partial P^5; \mathbb{Z}/2)$.
- $\exists$ a PHM bordism U from ∂P^5 to a PL mfld V^4, and V^4 is bdry of PL mfld W^5.
- $M^5 := P^5 \cup U \cup W^5$ is not triangulable.

Relative hyperbolization (D - Januszkiewicz - Weinberger, 2001)

Let $(M, \partial M)$ be a triangulated mfld with bdry. Put

$$\mathcal{H}(M, \partial M) := \mathfrak{h}(M \cup c(\partial M)) - (\text{nbhd of cone point})$$

Key properties

- $\mathcal{H}(M, \partial M)$ is mfld with bdry; its bdry is ∂M.
- $\pi_1(\partial M) \to \pi_1(\mathcal{H}(M, \partial M))$ is injective.
- $\mathcal{H}(M, \partial M)$ is aspherical iff ∂M is aspherical.

Corollary (DJW)

If an aspherical mfld bounds a triangulable mfld, then it bounds an aspherical mfld.

GS mflds in dimensions ≥ 6

Put $P^6 := P^5 \times S^1$. Since $\Delta(\partial P^6) = 0$, ∂P^6 admits a PL structure.

Put $M^6 = P^6 \cup U \cup W$, where U is the mapping cylinder of a (necessarily non-PL) homeomorphism from ∂P^6 to a PL mfld V^5 and W is a PL 6-mfld bounded by V^5.

Theorem (D-Fowler-Lafont)

In each dim $n \geq 6$, $\exists$ an aspherical mfld N^n that cannot be triangulated.

Proof.

Start with $\mathfrak{h}(P^6)$. Then $\mathfrak{h}(\partial P^6)$ is homeomorphic to a PL mfld V^5. Let U be the mapping cylinder of a homeomorphism $V^5 \to \mathfrak{h}(\partial P^6)$. V^5 is bdry of a PL 6-mfld W. Put

$$N^6 := \mathfrak{h}(P^6) \cup U \cup \mathcal{H}(W, V).$$

We check immediately that

- N^6 is aspherical.
- $\Delta(N^6) \neq 0$ and $Sq^1(\Delta(N^6)) \neq 0$.

So, N^6 cannot be triangulated. □

Thank you.