Aspherical manifolds that cannot be triangulated

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Although Kirby - Siebenmann showed that, in dimensions $\geq 5$, there exist manifolds which do not admit a PL structure, the possibility remained that all manifolds could be triangulated. In dimension 4, Freedman’s $E_8$ manifold (and others) are not PL; moreover, they cannot be triangulated. In 1991 Januszkiewicz and I applied Gromov’s hyperbolization technique to the $E_8$-manifold to show the existence of non-triangulable aspherical 4-manifolds. In dimensions $\geq 5$, the existence of non-triangulable manifolds depended on the nonexistence of homology 3-spheres with certain properties. In 2013 this question about homology spheres was resolved by Manolescu. So, there exist non-triangulable $M^n$ for $n \geq 5$. We use two versions of hyperbolization to show that, for $n \geq 6$, these can be chosen aspherical.
Why aspherical?

At one point the only examples of closed aspherical manifolds came from differential geometry or Lie groups; hence, were smooth manifolds. Gromov’s hyperbolization showed that you could convert simplicial complexes (hence, triangulated manifolds) into aspherical ones. But to get non PL manifolds you need a trick.
1 Introduction

2 Dimension 4
   - Rokhlin’s Theorem and the $\mu$-invariant
   - Freedman’s $E_8$-manifold
   - Hyperbolization

3 Dimensions $> 4$
   - Kirby - Siebenmann
   - Galewski - Stern + Manolescu
   - Relative hyperbolization

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Polyhedral homology mflds

Definitions

A simplicial cx $L^n$ is a polyhedral homology $n$-mfld (a PHM for short) if for each $k$-simplex $\sigma$, $Lk(\sigma, L)$ has the same homology as $S^{n-k-1}$. $L^n$ is a PL mfld if $\forall \sigma \in L$, $Lk(\sigma, L)$ is PL homeomorphic to $S^{n-\dim \sigma -1}$.

If the 4-dim PL Poincaré Conjecture is true, then we can drop “PL” and shorten “PL homeomorphic” to “homeomorphic” in the above.

Meaning of the Double Suspension Theorem

In dims $\geq 5$ top mflds can have triangulations (as PHMs) which are not PL.
Fact

If $B$ is an even, nondegenerate, symmetric bilinear form over $\mathbb{Z}$, then its signature, $\sigma(B)$, is divisible by 8.

Theorem (Rokhlin 1952)

If $M^4$ is a closed PL 4-mfld, with $w_1 = 0$ and $w_2 = 0$, then

$$\sigma(M^4) \equiv 0 \mod 16.$$
Fact

If $H^3$ is a homology 3-sphere, then $H^3 = \partial W^4$, where $W^4$ is a PL mfld with even intersection form.

The $\mu$-invariant

Define

$$\mu(H^3) = \frac{\sigma(W^4)}{8} \in \mathbb{Z}/2.$$ 

This defines a homomorphism $\mu : \Theta^H_3 \to \mathbb{Z}/2$, where $\Theta^H_3$ is the group of homology cobordism classes of homology 3-spheres.
**Introduction**

**Dimension 4**

**Freedman’s \(E_8\)-manifold**

**Hyperbolization**

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\[ Q(E_8) := \text{the } E_8 \text{ plumbing.} \]

It is a smooth 4-manifold with bdry.

\[ \partial Q(E_8) = H^3, \text{ Poincaré’s homology 3-sphere. } \sigma(Q(E_8)) = 8. \text{ Let } X^4 := Q(E_8) \cup c(H^3). \text{ It is a PHM of signature 8.} \]

**Theorem (Freedman 1982)**

\[ H^3 = \partial C^4, \text{ where } C^4 \text{ is a top contractible mfld. Put } M^4 = Q(E_8) \cup C^4, \text{ the “}E_8\text{-manifold”}. \]

By Rokhlin’s Thm, \(M^4\) does not have a PL structure.

**Fact**

Any triangulation of a 4-mfld is automatically PL. (Pf: By the Poincaré Conj, the link of any vertex is PL homeomorphic to \(S^3\).) So, Freedman’s \(M^4\) is not triangulable.
Hyperbolization (Gromov)

A hyperbolization procedure is a functor $\mathcal{H}$ from \{simplicial complexes\} to \{locally CAT(0) spaces\} together with a map $f: \mathcal{H}(K) \to K$ with the following properties:

- $\mathcal{H}$ preserves local structure: $\forall \sigma \in K$, $\text{Lk}(\mathcal{H}(\sigma)) \cong \text{Lk}(\sigma)$ (Lk means “link”.) In particular, if $K$ is a mfld (or a PHM), then so is $\mathcal{H}(K)$.
- $f^*$ is a split injection on cohomology.
- When $K$ is a mfld, $f$ pulls back stable tangent bundle to stable tangent bundle. So, $f^*$ pulls back characteristic classes of $K$ to those of $\mathcal{H}(K)$.
Rokhlin’s Theorem and the $\mu$-invariant

Freedman’s $E_8$-manifold

Hyperbolization

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**Introduction**

**Dimension 4**

**Dimensions $>4$**

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### (D - Januszkiewicz)

- Apply $\iota$ to the $E_8$ homology mfld $X^4$.
- Resolve it to $N^4 = (\iota(X^4) - \text{nbhd of cone pt}) \cup C^4$.
- Then $N^4$ is aspherical and not triangulable.

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**Theorem (DJ 1991)**

\[ \exists \text{ closed aspherical 4-mflds that cannot be triangulated. For } n \geq 5, \exists \text{ closed aspherical } n\text{-mflds which are not homotopy equivalent to PL mflds}. \]

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**Proof of 2nd sentence.**

$N^4 \times T^k$ is not PL.

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Remark

By Double Suspension Thm, for $k > 0$, $N^4 \times T^k \cong X^4 \times T^k$ (where $X^4$ is the PHM). So, $N^4 \times T^k$ can be triangulated.
Theorem (Kirby - Siebenmann 1969)

A top n-mfld, \( n \geq 5 \), admits a PL structure \( \iff \) an obstruction \( \Delta \in H^4(M^n; \mathbb{Z}/2) \) vanishes.

In other words, \( TOP/PL \) is the Eilenberg-MacLane space \( K(\mathbb{Z}/2, 3) \).
Polyhedral Mfld Characterization Theorem

Theorem (Edwards 1978 + Perelman)

A PHM (of dim > 2) is a top mfld $\iff$ the link of each vertex is simply connected.

Example

The double suspension of a homology sphere, with $\pi_1 \neq 1$.

Such triangulations are not PL.

In the early seventies such considerations led Siebenmann to ask if all mflds could actually be triangulated (before the Double Suspension Thm or Freedman’s $E_8$ 4-mfld were known).

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I will describe highlights of a theory worked out in the 1970s by several people, Siebenmann, Matumoto, most notably Galewski - Stern (with important contributions by others, eg, Cohen, Sullivan, Martin, Maunder).

Suppose $X$ is a PHM. Let $\lambda \in H^4(X; \Theta^H_3)$ be the cohomology class which associates to the “dual cell” of a codim 4 simplex $\sigma$, the class of $\text{Lk}(\sigma)$ in $\Theta^H_3$ (where $\Theta^H_3$ is the group of homology cobordism classes of homology 3-spheres). $\lambda$ is the obstruction to finding an “acyclic resolution” of $X$ by a PL manifold.
Consider the coefficient sequence:

\[ 0 \to \text{Ker } \mu \to \Theta_3^H \xrightarrow{\mu} \mathbb{Z}/2 \to 0. \]

**Fact 1**

When \( X \) is a top mfld, \( \mu_* \) takes \( \lambda \in H^4(X; \Theta_3^H) \) to the Kirby-Siebenmann obstruction \( \Delta \in H^4(X; \mathbb{Z}/2) \).

**Fact 2**

If \( M \) is a top mfld, then the obstruction to triangulation is \( \beta(\Delta) \in H^5(M; \text{Ker } \mu) \), where \( \beta \) is the Bockstein associated to the above coefficient sequence.
Introduction

Dimension 4

Kirby - Siebenmann
Galewski - Stern + Manolescu
Relative hyperbolization

Theorem (Galewski-Stern ~ 1980)

In \( \text{dim } n > 4 \), \( \exists \) nontriangulable \( M^n \) iff the sequence
\[
0 \rightarrow \text{Ker } \mu \rightarrow \Theta^H_3 \rightarrow \mathbb{Z}/2 \rightarrow 0 \text{ does not split, i.e., iff } \nexists \text{ a homology } 3\text{-sphere } H^3 \text{ with } \mu(H^3) \neq 0 \text{ and with } H^3 \# H^3 = 0 \text{ in } \Theta^H_3.
\]

Theorem (Manolescu 2013)

The sequence does not split.
Galewski-Stern mflds

It suffices to consider the Bockstein associated to

\[ 0 \to \mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0. \]

This Bockstein is \( Sq^1 \) (the first Steenrod square).

∃ a (nonorientable) PHM \( P^5 \) with bdry st

- \( \text{int } P^5 \) is a top mfdld (this uses Edwards’ Thm).
- \( \Delta(P^5) = \mu_*(\lambda(P^5)) \neq 0 \) in \( H^4(P^5; \mathbb{Z}/2) \) and \( \Delta(\partial P^5) = 0. \)
- \( Sq^1(\Delta) \neq 0 \) in \( H^5(P^5, \partial P^5; \mathbb{Z}/2) \).
- ∃ a PHM bordism \( U \) from \( \partial P^5 \) to a PL mfdld \( V^4 \), and \( V^4 \) is bdry of PL mfdld \( W^5 \).
- \( M^5 := P^5 \cup U \cup W^5 \) is not triangulable.
Relative hyperbolization (D - Januszkiewicz - Weinberger, 2001)

Let \((M, \partial M)\) be a triangulated mfld with bdry. Put

\[\mathcal{H}(M, \partial M) := h(M \cup c(\partial M)) - (\text{nbhd of cone point})\]

Key properties
- \(\mathcal{H}(M, \partial M)\) is mfld with bdry; its bdry is \(\partial M\).
- \(\pi_1(\partial M) \rightarrow \pi_1(\mathcal{H}(M, \partial M))\) is injective.
- \(\mathcal{H}(M, \partial M)\) is aspherical iff \(\partial M\) is aspherical.

Corollary (DJW)

*If an aspherical mfld bounds a triangulable mfld, then it bounds an aspherical mfld.*
Put $P^6 := P^5 \times S^1$. Since $\Delta(\partial P^6) = 0$, $\partial P^6$ admits a PL structure.

Put $M^6 = P^6 \cup U \cup W$, where $U$ is the mapping cylinder of a (necessarily non-PL) homeomorphism from $\partial P^6$ to a PL mfld $V^5$ and $W$ is a PL 6-mfld bounded by $V^5$. 
Theorem (D-Fowler-Lafont)

In each dim $n \geq 6$, $\exists$ an aspherical mfld $N^n$ that cannot be triangulated.
Proof.

Start with $\mathfrak{h}(P^6)$. Then $\mathfrak{h}(\partial P^6)$ is homeomorphic to a PL mfld $V^5$. Let $U$ be the mapping cylinder of a homeomorphism $V^5 \to \mathfrak{h}(\partial P^6)$. $V^5$ is bdry of a PL 6-mfld $W$. Put

$$N^6 := \mathfrak{h}(P^6) \cup U \cup \mathcal{H}(W, V).$$

We check immediately that

- $N^6$ is aspherical.
- $\Delta(N^6) \neq 0$ and $Sq^1(\Delta(N^6)) \neq 0$.

So, $N^6$ cannot be triangulated.
Thank you.