Examples of buildings constructed via covering spaces

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HAPPY BIRTHDAY STEVE!

Coxeter systems

Let L^1 be a simplicial graph with $\mathrm{Vert}(L^1) = S$. Label each edge $\{s, t\}$ of L^1 by an integer, $m(s, t)$, ≥ 2. This is the data for a presentation of a group W with generating set S and with relations:

$$s^2 = 1, \forall s \in S, \text{ and}$$
$$(st)^{m(s,t)} = 1, \forall \{s, t\} \in \mathrm{Edge}(L^1)$$

The pair (W, S) is a *Coxeter system*.

Cayley graph of (W, S)

- Start with $K^1 := \mathrm{Cone}(S)$.
- Paste together copies of K^1, one for each element of W. The result is the Cayley graph.
- If $W = \mathbf{D}_m$ (the dihedral group of order $2m$), then $\mathrm{Cay}(W, S)$ is the boundary of a $2m$-gon.
- Filling in the 2-cells we obtain the "Cayley 2-complex" (which is simply connected). Here $K^2 := \mathrm{Cone}(L^1)$. (We shall also call the Cayley 2-complex the "dual of the codimension 2-skeleton of ...")
- Can continue to obtain a contractible complex Σ with a proper cocompact W-action.

- (W, S) is a "thin" bldg (ie "thickness" $q = 1$). This means each "chamber" ($=$ vertex in $\mathrm{Cay}(W, S)$) is s-adjacent to exactly one other.
- Any bldg has an associated Coxeter system. However, for "thick" bldgs ($q > 1$), it is unclear which Coxeter gps can occur. For example thick bldgs of type $(\mathbf{D}_m, \{s, t\})$ exist only for $m = 2, 3, 4, 6, 8$ or ∞.
- Classical bldgs are associated to spherical or Euclidean reflection gps (ie, relatively few Coxeter gps occur for the classical types). We will use the classical examples to construct many more bldgs.

As a space a bldg should be thought of as follows: edges in the Cayley graph are replaced by cones over $q+1$ points, 2-cells are replaced by cones on “generalized m-gons” ($=$ bldgs of type $\mathbf{D}_m$). The fundamental blocks K^1, K^2 are the same as for Coxeter systems.

180 4 Buildings as Chamber Complexes

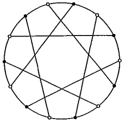


Fig. 4.1. The incidence graph of the Fano plane.

On the level of Coxeter systems our construction is to omit some edges of L^1, keeping the same vertices, ie, some $m(s,t)$'s are changed to ∞. This changes (W,S) to a new Coxeter system $(\widetilde{W},S)$. The Cayley 2-complex for $(\widetilde{W},S)$ is constructed by removing 2-cells from the 2-complex for (W,S) and taking the universal cover.

On the level of bldgs, the procedure is to omit the corresponding cones on generalized m-gons from the dual codim 2-skeleton of a bldg of type (W,S) and then take the universal cover. The resulting space is the dual codim 2-skeleton of a new bldg of type $(\widetilde{W},S)$. (This is what must be proved).

1. Definitions
   - Chamber systems
   - Buildings
2. The construction
   - In detail
   - Previous examples
   - Automorphism groups
3. Products
   - Graph products
   - $\square$-products

What is a building?

To begin with, it is **not** a simplicial complex or even a topological space; rather, it is a “chamber system.”

Definition

A *chamber system* over a (finite) set S is a set $\mathcal{C}$ (of “chambers”), together with a family of equivalence relations indexed by S. (In other words, there is one equivalence relation for each $s \in S$.) Chambers, C and C' in $\mathcal{C}$ are *s-adjacent* if they are s-equivalent and not equal.

Example

Let G be a group, $B \subset G$ a subgp and $\{G_s\}_{s\in S}$ a family of subgps each $\supset B$. Put $\mathcal{C} := G/B$.

Call chambers gB and $g'B$, *s-equivalent* iff $gG_s = g'G_s$.

Special case

$G = W$, $B = \{1\}$, $G_s = \langle s\rangle$. Then w, w' are equivalent iff $w = w's$.

Associating a space to $\mathcal{C}$

Choose

- a space X (as a "model chamber")
- a family of subspaces $\{X_s\}_{s\in S}$ indexed by S (called a "mirror structure").

Let $\mathcal{U}(\mathcal{C}, X)$ be the result of pasting together copies of X, one for each element of $\mathcal{C}$, so that copies corresponding to s-adjacent elements of $\mathcal{C}$ are pasted together along X_s.

The classical choice for X is a simplex Δ of dimension $|S| - 1$, with $\{\Delta_s\}_{s \in S} = \{\text{codim 1 faces}\}$.

Once we bring a Coxeter system (W, S) into the picture, another choice is $X = K$, the geometric realization of the poset of "spherical subsets" of S

Definition (Galleries)

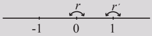
A *gallery* in a chamber system $\mathcal{C}$ is a sequence $\mathbf{C} = (C_0, C_1, \dots, C_n)$ of adjacent chambers. Its *type* is the word $\mathbf{s} = (s_1, \dots, s_n)$ in S defined by: C_{i-1} is s_i-adjacent to C_i.

Definition

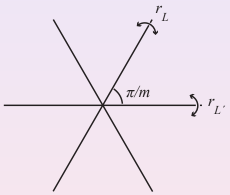
Suppose (W, S) is a Coxeter system. A *building of type* (W, S) is a pair $(\mathcal{C}, \delta)$ consisting of a chamber system $\mathcal{C}$ over S and a function $\delta : \mathcal{C} \times \mathcal{C} \to W$ (called *Weyl distance*) so that for any reduced expression $w = s_1 \dots s_n$ and chambers $C, D \in \mathcal{C}$, $\exists$ a gallery of type $\mathbf{s} = (s_1, \dots, s_n)$ from C to D iff $\delta(C, D) = w$.

Example (Trees)

Suppose $S = \{s, t\}$ and $W = D_\infty$ (the infinite dihedral group).



Let $\mathcal{T}$ be a tree w/o terminal vertices. Any tree is bipartite – color the vertices with $\{s, t\}$. Then $\mathcal{C} := \mathrm{Edge}(\mathcal{T})$ is a building of type (D_∞, S). The Weyl-distance between $C, D \in \mathcal{C}$ is defined as follows. Take the edge path w/o backtracking from C to D. Its type gives a word in $\{s, t\}$; hence, an element $w \in W$. Set $\delta(C, D) = w$.



Example (Projective planes over finite fields)

$S = \{s, t\}$ and $W = D_3$ (dihedral gp of order 6). Let Ω be the incidence graph of a projective plane over a finite field. $\mathcal{C} := \text{Edge}(\Omega)$.

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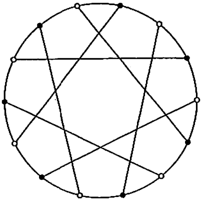

Fig. 4.1. The incidence graph of the Fano plane.

Recall L^1 is a simplicial graph with $\mathrm{Vert}(L^1) = S$. Each edge $\{s, t\}$ is labeled by an integer, $m(s, t)$, which is ≥ 2.

The 2-skeleton of the standard realization

Given (W, S), define $K^1 := \mathrm{Cone}(S)$ and $K^2 := \mathrm{Cone}(L^1)$. Also, K^2_s denotes the closed star of s in the barycentric subdiv of L^1. Then $\mathcal{U}(W, K^1)$ is the Cayley graph of (W, S) and $\mathcal{U}(W, K^2)$ is the standard 2-complex associated to its presentation.

The construction in a nutshell

Consider again the incidence graph of a projective plane. It is a bldg of type $(D_3, \{s, t\})$. Its standard realization is the cone over the graph on the right. In a nutshell, remove the cells containing the cone point (getting the graph) and take the universal cover. The result is a trivalent tree, ie, a bldg of type $(D_\infty, \{s, t\})$.

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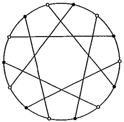

Fig. 4.1. The incidence graph of the Fano plane.

- $\overline{\mathcal{C}}$ a bldg of type $(\overline{W}, S)$. $\overline{K}^2$ $(= K^2(\overline{W}, S)) =$ (2-skeleton) of standard chamber for $(\overline{W}, S)$.
- Omit some edges of $\overline{L}^1$ $(= L^1(\overline{W}, S))$, ie, change some $m(s, t)$'s to ∞.
- $(W, S) =$ new Coxeter system. $K^2 =$ new chamber. (K^2 is obtained by "removing 2-cells" from $\overline{K}^2$.)
- Form $\mathcal{U}(\overline{\mathcal{C}}, K^2)$. Let $\widetilde{\mathcal{U}} \to \mathcal{U}(\overline{\mathcal{C}}, K^2)$ be the universal cover.
- Then $\mathcal{C} := \{$chambers in $\widetilde{\mathcal{U}}\}$ is a bldg of type (W, S). (This is the Main Theorem.)

What needs to be proved.

Given $C, D \in \mathcal{C}$, choose a minimal gallery from C to D. Let $\mathbf{s} = (s_1, \dots, s_n)$ be its type. Put $w = s_1 \cdots s_n$ and $\delta(C, D) = w$. Must show $\delta : \mathcal{C} \times \mathcal{C} \to W$ is well-defined and that it is a Weyl distance. □

Example (of the construction)

Suppose $\overline{W}$ is finite, ie, suppose $\overline{\mathcal{C}}$ is a spherical bldg. Then $\overline{L}^1$ is the complete graph on S. Omit edges of $\overline{L}^1$ at random, obtaining a new Coxeter system (W, S). The universal cover construction gives a new bldg $\mathcal{C}$ of type (W, S).

Previous state of examples

Spherical: $\mathcal{C}$ comes from an algebraic gp over a finite field.

Affine: $\mathcal{C}$ comes from an algebraic gp over a local field.

Hyperbolic: Some isolated examples.

RAB's: (W, S) is right-angled. $\exists$ regular bldgs with each s-equivalence class having $q_s + 1$ elements, q_s an arbitrary integer ≥ 1.

Kac-Moody: $\exists$ bldgs associated to "Kac-Moody gps." (Here $m(s,t) \in \{2, 3, 4, 6, \infty\}$, $q_s =$ order of finite field.)

An *automorphism* of $\mathcal{C}$ is a self-bijection which preserves s-equivalence classes.

- Suppose $\overline{G} \subset \mathrm{Aut}(\overline{\mathcal{C}})$. Then $\overline{G}$ acts on $\mathcal{U}(\overline{\mathcal{C}}, K^2)$. Let G be the group of all lifts of elements of $\overline{G}$ to $\widetilde{\mathcal{U}}$. So, $G \subset \mathrm{Aut}(\mathcal{C})$.
- $\mathrm{Aut}(\mathcal{C})$ is a topological gp. (A small nbhd of *id* consists of all auto's which fix a large combinatorial ball.)
- Suppose, from now on, that $\mathcal{C}$ has finite thickness (ie, each s-equivalence class is finite). Then $\mathrm{Aut}(\mathcal{C})$ is locally compact; hence, it has a Haar measure.
- $G \subset \mathrm{Aut}(\mathcal{C})$ is a *lattice* if it is discrete and has finite covolume. It is *uniform* if $\mathcal{C}/G$ is a finite set.

Proposition

If $\mathrm{Aut}(\overline{\mathcal{C}})$ *admits a uniform lattice, then so does* $\mathrm{Aut}(\mathcal{C})$*. In particular, if* $\overline{\mathcal{C}}$ *is spherical, then* $\mathrm{Aut}(\mathcal{C})$ *admits a torsion-free uniform lattice.*

Proof.

Suppose $\overline{G} \subset \mathrm{Aut}(\overline{\mathcal{C}})$ and $G = \{\text{lifts of } \overline{G}\}$.
$\overline{G}$ discrete $\implies$ G discrete. Since $\mathcal{C}/G = \overline{\mathcal{C}}/\overline{G}$,
$\overline{G}$ uniform $\implies$ G uniform.
For the last sentence, we can take $\overline{G} = \{1\}$. □

Direct products

- Suppose $\mathcal{C}_0, \ldots, \mathcal{C}_p$ are bldgs, with $\mathcal{C}_i$ of type (W_i, S_i).
- Then $\mathcal{C}_0 \times \cdots \times \mathcal{C}_p$ is a bldg of type $(\overline{W}, S)$, where $\overline{W} := W_0 \times \cdots \times W_p$ and $S := S_0 \coprod \cdots \coprod S_p$.

For $i \neq j$ and $s_i \in S_i$, $s_j \in S_j$, we have $m(s_i, s_j) = 2$. Change some of these to ∞ to get a new Coxeter system (W, S). Then apply our construction to the direct product $\overline{\mathcal{C}} := \mathcal{C}_0 \times \cdots \times \mathcal{C}_p$ to get a new bldg, $\mathcal{C}$, called a *partial product* of the $\mathcal{C}_i$.

Graph products

- Suppose J is a simplicial graph and $\{G_i\}_{i \in \mathrm{Vert}(J)}$ is a family of groups. Their *graph product*, $\prod_J G_i$, is the quotient of the free product of the G_i by the relations: $[g_i, g_j] = 1$, whenever $\{i, j\} \in \mathrm{Edge}(J)$.
- Suppose, as before, that $\overline{\mathcal{C}} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_p$, where $i \in \mathrm{Vert}(J)$. Whenever $\{i, j\} \notin \mathrm{Edge}(J)$, change all $m(s_i, s_j)$ from 2 to ∞ (and make no other changes).
- Then $W = \prod_J W_i$. If $\mathcal{C}$ denotes the set of chambers in $\widetilde{\mathcal{U}}$, define $\prod_J \mathcal{C}_i$ to be $\mathcal{C}$. It is a bldg of type (W, S).

Free products

Consider the special case, where there are no edges in J. Then W is the free product of the W_i and $\prod_J \mathcal{C}_i$ is called the *free product* of the $\mathcal{C}_i$. The chamber K^2 for the free product is the 1-point union of the $K^2(W_i, S_i)$

RAB's

The graph product over J of cyclic gps of order 2 is the *right-angled Coxeter gp* ($=$ RACG) associated to J. A bldg $\mathcal{C}_i$ of type $(\mathbf{C}_2, \{s_i\})$ (ie, a bldg of rank 1) is just a finite set (with all elements s_i-equivalent). In this case, the graph product of the $\mathcal{C}_i$ is called a *RAB* of type (W, S).

For simplicity assume we only have two bldgs $\mathcal{C}_0$ and $\mathcal{C}_1$. Also, suppose (W_0, S_0) and (W_1, S_1) have the same rank. Choose a bijection $\theta : S_0 \to S_1$.

$\Box$-products

For $s_0 \in S_0$ and $s_1 \in S_1$, define

$$m(s_0, s_1) = \begin{cases} \infty, & \text{if } s_1 = \theta(s_0); \\ 2, & \text{otherwise.} \end{cases}$$

The new Coxeter gp is denoted by $W_0 \Box W_1$ and the new bldg by $\mathcal{C}_0 \Box \mathcal{C}_1$.

If (W_0, S_0) and (W_1, S_1) are spherical Coxeter gps of rank n, then K_i $(:= K(W_i, S_i))$ is the cone on a $(n-1)$-simplex and $K = K_0 \,\Box\, K_1$ is a combinatorial n-cube. (Hence, its 2-skeleton K^2 is the 2-skeleton of an n-cube.) The bldgs $\mathcal{C}_0$ and $\mathcal{C}_1$ correspond to the links of $\mathcal{U}(\mathcal{C}_0 \,\Box\, \mathcal{C}_1, K)$ at opposite vertices of K.

Books

- P. Abramenko and K. Brown, *Buildings*, Springer, 2008.
- M.W. Davis, *The Geometry and Topology of Coxeter Groups*, Princeton Univ. Press, 2007.
- M. Ronan, *Lectures on Buildings*, Perspectives in Mathematics, vol. 7, Academic Press, San Diego, 1989.