The action dimension of Artin groups

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References

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This is joint work with Giang Le and Jingyin Huang based on earlier work (Avramidi, et. al.)

Let π be a discrete, torsion-free group.

The *geometric dimension* of π, denoted $\operatorname{gd}\pi$, is the minimum dimension of a CW model for $B\pi$.

Its *action dimension*, denoted $\operatorname{actdim}\pi$, is the minimum dimension of a manifold model for $B\pi$, ie, a thickening of $B\pi$ to a mfld with bdry. In other words, $\operatorname{actdim}\pi$ is the smallest n s.t. π acts properly on a contractible n-mfld.

Example

If $\pi = \pi_1$(closed aspherical mfld M), then

$$\operatorname{actdim}\pi = \dim M = \operatorname{cd}\pi = \operatorname{gd}\pi.$$

For general reasons, $\quad \operatorname{actdim}\pi \leq 2\operatorname{gd}\pi.$

Examples

- If $\pi = F_2$, then $\operatorname{gd}\pi = 1, \quad \operatorname{actdim}\pi = 2$.
- If $\pi = \underbrace{F_2 \times \cdots \times F_2}_{k}$, then $\operatorname{gd}\pi = k, \quad \operatorname{actdim}\pi = 2k$.
- If π is a lattice in a semisimple Lie gp G, then $\operatorname{actdim}\pi = \dim G/K$ (Bestvina-Feighn); if π is nonuniform, then $\operatorname{gd}\pi < \operatorname{actdim}\pi$.
- If π is a (torsion-free) subgp of finite index in the mapping class gp, $Mod(g)$, then $\operatorname{actdim}\pi = \dim(\text{Teichmüller space})$.
- If π is B_{n+1} (the braid group on $n+1$ strands), then $\operatorname{gd} B_{n+1} = n, \quad \operatorname{actdim} B_{n+1} = 2n-1$.

Methods for computing actdim π

Lower bound

Method of Bestvina-Kapovich-Kleiner:
1) Find a finite complex K s.t. "$K \subset \partial\pi$".
2) If the van Kampen obstruction for embedding K in S^m is not 0, then actdim $\pi \geq m + 2$.

Upper bound

God-given or else find a direct construction of manifold with boundary M s.t. $M \sim B\pi$.

"$K \subset \partial\pi$" means that $(\text{Cone } K)^0$ coarsely embeds in π, i.e., there is a proper, expanding, Lipschitz map, $(\text{Cone } K)^0 \to \pi$.

The van Kampen obstruction is a class $vk^m(K) \in H^m(\mathcal{C}(K); \mathbf{Z}/2)$, where $\mathcal{C}(K)$ is the configuration space $(K \times K - \Delta)/\mathbf{Z}/2$. K is an *m-obstructor* if $vk^m(K) \neq 0$.

If "$K \subset \partial\pi$", then $vk^m(K)$ is an obstruction for $\pi \curvearrowright$ contractible M^{m+1}. The *obstructor dimension* of π, denoted obdim π, is $= m + 2$, where m is the largest integer for which $\exists$ an m-obstructor K with "$K \subset \partial\pi$"

Coxeter groups

- (W, S) a Coxeter system. (m_{st}) associated Coxeter matrix.
- For $T \leq S$, $W_T = \langle T \rangle$ is a *special subgp*.
- T is a *spherical subset* if W_T is finite. The *nerve* L of (W, S) is the simplicial complex defined by $\mathcal{S}(L) = \{\text{simplices of } L\} = \{\text{spherical subsets of } S\}$

Artin groups

Given (W, S) the associated *Artin gp* has generating set $\{a_s\}_{s \in S}$ and relations: $\underbrace{a_s a_t \cdots}_{m_{st}} = \underbrace{a_t a_s \cdots}_{m_{st}}$.

For $T \leq S$, A_T is defined as before. A_T is *spherical* if W_T is spherical.

RACGs and RAAGs

A Coxeter gp or Artin gp is *right-angled* if all m_{st}, with $s \neq t$, are $= 2$ or $= \infty$.
If this is the case, then the nerve L is a flag cx.
A_L denotes the RAAG associated to a flag cx L.

Standard model for BA_L

$$BA_L = \bigcup_{\sigma \in \mathcal{S}(L)} T^\sigma,$$ where T^σ is a torus of dim $= \dim \sigma + 1$.

It follows that the RAAG A_L has geometric dimension $= \dim L + 1$ $(= \operatorname{cd} A_L)$.

Salvetti Complex

- Let T be a spherical subset of S, ie, (the vertex set of) a simplex of L. Let A_T be the associated spherical Artin gp. If the dim of simplex is d, then $d + 1 = \#T$.
- $\exists$ a CW cx X_T, with dim $X_T = d + 1$, called the "Salvetti cx". Deligne's Theorem $\implies$ $X_T \sim BA_T$. So, gd $A_T \leq d + 1$ (in fact, $=$).

A complex of groups

For a general Artin gp A, $\exists$ a poset of groups over the poset $\mathcal{S}(L)$, defined by $T \mapsto A_T$. Fact: $A = \operatorname{colim} A_T$. Put $X = \operatorname{colim} X_T$. ($X$ also is called "Salvetti cx").

$K(\pi, 1)$-Conjecture: $X \sim BA$.

(Proved when L is flag cx by Charney-D.)

Suppose $\dim L = d$.

Theorem (Avramidi-D.-Okun-Schreve)

Suppose A_L is a RAAG. *Then*
$H^d(L : \mathbf{Z}/2) \neq 0 \iff \text{actdim}\, A_L = 2(d+1)$.

Suppose $K(\pi, 1)$-Conjecture is true for A.

Theorem (D., Huang, Le)

For general A:

- $H^d(L; \mathbf{Z}/2) \neq 0 \implies \text{actdim}\, A = 2(d+1)$
- $H^d(L; \mathbf{Z}) = 0 \implies \text{actdim}\, A \leq 2d+1$ *(If $d = 2$, we also must assume that $\pi_1(L) \cong H_1(L; \mathbf{Z})$).*

Octahedralization

Given a simplicial cx L with $\mathrm{Vert}(L) = S$, there is another simplicial cx OL with vertex set $S \times \{\pm 1\}$, called its "octahedralization." If $\Delta(S)$ and $O(S)$ denote, respectively, the simplex and octahedron on S , then OL is defined by:

$$\begin{array}{ccc} OL & \longrightarrow & O(S) \\ \downarrow & & \downarrow p \\ L & \longrightarrow & \Delta(S) \end{array}$$

ie, $OL = p^{-1}(L) \leq O(S)$.

In the case of a RAAG, where $BA_L = \bigcup_{\sigma \in \mathcal{S}(L)} T^\sigma$, the lift of T^σ to universal cover (through a given basepoint) is a flat subspace which gives an octahedron in sphere at ∞. Thus, $OL \subset \partial A_L$ is determined by this union of standard flat subspaces through a given basepoint.

Lemma (ADOS)

If $H^d(L; \mathbf{Z}/2) \neq 0$, then OL is a $2d$-obstructor, ie, $vk^{2d}(OL) \neq 0$.

A spherical Artin gp A_T is *irreducible* if the Coxeter diagram **D** of (W_T, T) is connected.

The center of a spherical Artin group

The center of an irreducible spherical Artin gp is infinite cyclic, generated either by an element Δ or Δ^2 where, $\Delta \in A_{\mathbf{D}}$ projects to the element of longest length in $W_{\mathbf{D}}$.

Definition of standard abelian subgps in spherical Artin gp

By induction on $\#\,\mathrm{Vert}\,\mathbf{D}$, define a *standard maximal abelian subgp* Λ of $A_{\mathbf{D}}$. If $A_{\mathbf{D}}$ has more than one irreducible factor, say, $A_{\mathbf{D}} = A_{\mathbf{D}_1} \times \cdots \times A_{\mathbf{D}_k}$, then $\Lambda = \Lambda_1 \times \cdots \times \Lambda_k$, where Λ_i is max abelian in $A_{\mathbf{D}_i}$. If **D** is irreducible and $\mathbf{D}'$ is any subdiagram of corank 1, then one of the $\Lambda < A_{\mathbf{D}}$ has the form $\Lambda_{\mathbf{D}'} \times Z(A_{\mathbf{D}})$.

A standard maximal abelian subgroup Λ in $A_{\mathbf{D}}$ is free abelian of rank $d+1$, where $d+1 = \#\,\mathrm{Vert}(\mathbf{D})$ $((=\mathrm{rk}(W_{\mathbf{D}}))$. A basis for Λ consists of central elements of various irreducible subdiagrams.

Corollary (Assuming the $K(\pi, 1)$-Conjecture for A)

$\mathrm{cd}\, A = \dim L + 1 = \mathrm{gd}\, A.$

The standard abelian subgroups of an Artin group are indexed by the simplices of a subdivision $L_{\oslash}$ of L. The vertices of $L_{\oslash}$ correspond to (barycenters of) irreducible spherical simplices. The cx $L_{\oslash}$ is defined inductively by taking joins of irreducible factors or by coning to a barycenter.

Lemma (Huang-D.)

Suppose Λ_α and Λ_β are abelian subgroups corresponding to simplices α and β in $L_\oslash$. Then $\Lambda_\alpha \cap \Lambda_\beta = \Lambda_{\alpha\cap\beta}$.

It follows that $L_\oslash$ gives the pattern of intersection of standard abelian subgps. These abelian subgps should give a pattern spheres in a possible bdry of A, ie, a copy of $OL_\oslash$ in the possible bdry.

Proposition (Huang-D)

"$OL_\oslash \subset \partial A$".

Theorem (Huang-D.)

Let $d = \dim L$. *If* $H^d(L; \mathbf{Z}/2) \neq 0$, *then* $\operatorname{actdim} A = 2d + 2$ *(the maximum possible).*

Proof.

$L_\oslash$ is homeomorphic to L. By [ADOS], $vk^{2d}(OL_\oslash) \neq 0$, ie, $OL_\oslash$ is a $2d$-obstructor. Hence, $\operatorname{actdim} A \geq \operatorname{obdim} A \geq 2d + 2$. □

Theorem (Le)

Suppose $\dim L = d$. *Then* $H^d(L; \mathbf{Z}) = 0 \implies BA \sim M^{2d+1}$. *(If* $d = 2$*, we also must assume that* $\pi_1(L) \cong H_1(L; \mathbf{Z})$*.)*

Ideas in proof

- Hypothesis implies that L embeds in a contractible simplicial cx L' of the same dimension. Use simplices of L' to get another poset of groups with the same aspherical realization as $\mathcal{S}(L)$.
- Since L' is contractible, the posets of gps over $\mathcal{S}(L')$ and $\mathcal{S}(L')_{>\emptyset}$ have homotopy equivalent aspherical realizations.

Ideas in proof, continued

- If T is a spherical simplex in L, then BA_T is homotopy equivalent to a mfld with bdry M_T of dimension $2k + 1$, where $k = \dim T$. (This is proved by using the complement of the hyperplane arrangement.) In particular, if T has highest dimension d, then $\dim M_T = 2d + 1$.
- Use the complex of gps over $\mathcal{S}(L')_{>\emptyset}$ and "thickened dual cones" to glue together the M_T s to obtain a $(2d + 1)$-mfld with bdry M homotopy equivalent to BA.