1 Introduction

2 Dimension 4
   - Rokhlin’s Theorem and the $\mu$-invariant
   - Freedman’s $E_8$-manifold
   - Hyperbolization

3 Dimensions $> 4$
   - Kirby - Siebenmann
   - Galewski - Stern + Manolescu
   - Relative hyperbolization
Although Kirby - Siebenmann showed that $\exists$ mflds in dimensions $\geq 5$ which do not admit a PL structure, the possibility remained that all mflds could be triangulated. In dimension 4, Freedman’s $E_8$ mfld (and others) are not PL; moreover, they cannot be triangulated. In 1991 Januszkiewicz and I used Gromov’s hyperbolization technique to show the existence of nontriangulable aspherical 4-mflds. In dimensions $\geq 5$ the existence of nontriangulable mflds depended on the nonexistence of homology 3-spheres with certain properties. In 2013 this question was resolved by Manolescu. So, $\exists$ nontriangulable $M^n$ for $n \geq 5$. We use hyperbolization to show that, for $n \geq 6$, these can be chosen aspherical.
Polyhedral homology mflds

Definitions

A simplicial cx $L^n$ is a polyhedral homology $n$-mfld (a PHM for short) if for each $k$-simplex $\sigma$, $\text{Lk}(\sigma, L)$ has the same homology as $S^{n-k-1}$. $L^n$ is a PL mfld if $\forall \sigma \in L$, $\text{Lk}(\sigma, L)$ is PL homeomorphic to $S^{n-\dim \sigma - 1}$.

If the 4-dimensional PL Poincaré Conjecture is true, then we can shorten “PL homeomorphic” to “homeomorphic” in the above.

Meaning of the Double Suspension Theorem

In dimensions $\geq 5$ top mflds can have non PL triangulations (as PHM s).
Fact

If $B$ is an even, nondegenerate, symmetric bilinear form over $\mathbb{Z}$, then its signature, $\sigma(B)$, is divisible by 8.

Theorem (Rokhlin 1952)

If $M^4$ is a closed PL 4-mfld, with $w_1 = 0 = w_2$, then

$$\sigma(M^4) \equiv 0 \mod 16.$$
Fact
If $H^3$ is a homology 3-sphere, then $H^3 = \partial W^4$, a PL manifold with even intersection form.

The $\mu$-invariant
Define
\[ \mu(H^3) = \frac{\sigma(W^4)}{8} \in \mathbb{Z}/2. \]

This defines a homomorphism $\mu : \Theta^H_3 \to \mathbb{Z}/2$, where $\Theta^H_3$ is the group of homology cobordism classes of homology 3-spheres.
Let $Q(E_8) = \text{the } E_8 \text{ plumbing.}$  
It is a smooth 4-manifold with bdry.  
$\partial Q(E_8) = H^3$, Poincaré’s homology 3-sphere.  
$\sigma(Q(E_8)) = 8$.  
Let $X^4 := Q(E_8) \cup c(H^3)$.  
It is a PHM of signature 8.

**Theorem (Freedman 1982)**  
$H^3 = \partial C^4$, where $C^4$ is a topological contractible mfld.  
Put $M^4 = Q(E_8) \cup C^4$, the “$E_8$ manifold”.

By Rokhlin’s Thm, $M^4$ does not have a PL structure.  
In fact, any triangulation is automatically PL.  
(Pf: By the Poincaré Conjecture, the link of any vertex is PL homeomorphic $S^3$.  
So, any triangulation of a 4-mfld is PL.)
Hyperbolization (Gromov)

A hyperbolization procedure is a functor $h$ from \{simplicial complexes\} to \{locally CAT(0) spaces\} together with a map $f : h(K) \to K$ with the following properties:

- $h$ preserves local structure: $\forall \sigma \in K$, $\text{Lk}(h(\sigma)) \cong \text{Lk}(\sigma)$ (Lk means “link”). In particular, if $K$ is a mfld, then so is $h(K)$.
- $f^*$ is a split injection on cohomology.
- When $K$ is a mfld, $f$ pulls back stable tangent bundle to stable tangent bundle. So, $f^*$ pulls back characteristic classes of $K$ to those of $h(K)$. 

Mike Davis (joint with Jim Fowler and Jean Lafont) Aspherical manifolds that cannot be triangulated
Rokhlin’s Theorem and the $\mu$-invariant
Freedman’s $E_8$-manifold
Hyperbolization

(D - Januszkiewicz)

- Apply $\mathfrak{h}$ to the $E_8$ homology mfld $X^4$.
- Put $N^4 = (\mathfrak{h}(X^4) - \text{nbhd of cone pt}) \cup C^4$
- Then $N^4$ is aspherical and not triangulable

Theorem (DJ 1991)

$\exists$ closed aspherical 4-mflds that cannot be triangulated. For $n \geq 5$, $\exists$ closed aspherical $n$-mflds which are not homotopy equivalent to PL mflds.

Proof.

$N^4 \times T^k$ is not PL.
Remark

By Double Suspension Thm, for $k > 0$, $N^4 \times T^k \cong X^4 \times T^k$. So, $N^4 \times T^k$ can be triangulated.
Theorem (Kirby - Siebenmann 1969)

A topological $n$-mfd, $n \geq 5$ admits a PL structure $\iff$ an obstruction $\Delta \in H^4(M^n; \mathbb{Z}/2)$ vanishes.
Polyhedral Mfld Characterization Theorem

Theorem (Edwards 1978)
A polyhedral homology mfld $L$ (of dim $\geq 2$) is a topological mfld $\iff$ the link of each vertex is simply connected.

Example
The double suspension of a homology sphere, with $\pi_1 \neq 1$.

Such triangulations are not PL.

In the early seventies such considerations led Siebenmann to ask if all mflds could actually be triangulated (before the Double Suspension Thm or Freedman’s $E_8$ 4-mfld were known).
I will describe highlights of a theory worked out in the 1970s by several people, Siebenmann, Matumoto, most notably Galewski - Stern (with important contributions by others, eg, Cohen, Sullivan, Martin, Maunder).

Suppose $X$ is a PHM. Let $\lambda \in H^4(X; \Theta^H_3)$ be the cohomology class which associates to the “dual cell” of a codimension 4 simplex $\sigma$, the class of $\text{Lk}(\sigma)$ in $\Theta^H_3$ (where $\Theta^H_3$ is the group of homology cobordism classes of homology 3-spheres). $\lambda$ is the obstruction to finding an “acyclic resolution” of $X$ by a PL manifold.
Consider the coefficient sequence:

\[ 0 \to \text{Ker} \, \mu \to \Theta^H_3 \to \mathbb{Z}/2 \to 0. \]

**Fact 1**

When \( X \) is a top mfld, \( \mu_* \) takes \( \lambda \in H^4(X; \Theta^H_3) \) to the Kirby-Siebenmann obstruction \( \Delta \in H^4(X; \mathbb{Z}/2) \).

**Fact 2**

If \( M \) is a top mfld, then the obstruction to triangulation is \( \beta(\Delta) \in H^5(M; \text{Ker} \, \mu) \), where \( \beta \) is the Bockstein associated to the above coefficient sequence.
Theorem (Galewski-Stern \sim 1980)

In dim \( n > 4 \), \( \exists \) nontriangulable \( M^n \) iff the sequence
\[
0 \to \text{Ker } \mu \to \Theta^H_3 \to \mathbb{Z}/2 \to 0
\]
does not split, ie, iff \( \nexists \) a homology 3-sphere \( H^3 \) with \( \mu(H^3) \neq 0 \) and \( H^3 \# H^3 = 0 \) in \( \Theta^H_3 \)

Theorem (Manolescu 2013)

The sequence does not split.
It suffices to consider the Bockstein associated to

\[ 0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0. \]

This Bockstein is \( Sq^1 \).

∃ a (nonorientable) PHM \( P^5 \) with bdry st

- \( \text{int } P^5 \) is a top mfld (Edwards’ Thm).
- \( \Delta(P^5) \neq 0 \) in \( H^4(P^5; \mathbb{Z}/2) \).
- \( Sq^1(\Delta) \neq 0 \).
- ∃ bordism \( U \) from \( \partial P^5 \) to a PL mfld \( V^4 \) and \( V^4 \) is bdry of PL mfld \( W^5 \).
- \( M^5 := P^5 \cup U \cup W^5 \) is not triangulable.
Relative hyperbolization (D - Januszkiewicz - Weinberger, 2001)

Let \((M, \partial M)\) be a triangulated-mfld with bdry. Put

\[
\mathcal{H}(M, \partial M) := h(M \cup c(\partial M)) - (\text{nbhd of cone point})
\]

Key properties

- \(\mathcal{H}(M, \partial M)\) is mfld with bdry; its bdry is \(\partial M\).
- \(\pi_1(\partial M) \rightarrow \pi_1(\mathcal{H}(M, \partial M))\) is injective.
- \(\mathcal{H}(M, \partial M)\) is aspherical iff \(\partial M\) is aspherical.

Corollary (DJW)

If an aspherical-mfld bounds a triangulable-mfld, then it bounds an aspherical-mfld.
GS mflds in dimensions $\geq 6$

Put $P^6 := P^5 \times S^1$. Since $\Delta(\partial P^6) = 0$, $\partial P^6$ admits a PL structure.

Put $M^6 = P^6 \cup U \cup W$, where $U$ is the mapping cylinder of a (necessarily non-PL) homeomorphism of $\partial P^6$ and $W$ is a PL 6-mfld bounded by $\partial P^6$ with its PL structure.
Theorem (DFL)

*In each dimension* \( n \geq 6 \), \( \exists \) an aspherical manifold \( N^n \) that cannot be triangulated.
Proof.

Start with $\mathfrak{h}(P^6)$. Then $\mathfrak{h}(\partial P^6)$ is homeomorphic to a PL mfld $V^5$. Let $U$ be the mapping cylinder of a homeomorphism $V^5 \to \mathfrak{h}(\partial P^6)$. $V^5$ is bdry of a PL 6-mfld $W$. Put

$$N^6 := \mathfrak{h}(P^6) \cup U \cup \mathcal{H}(W, V).$$

We check immediately that

- $N^6$ is aspherical.
- $\Delta(N^6) \neq 0$ and $Sq^1(\Delta(N^6)) \neq 0$.

So, $N^6$ cannot be triangulated.