Orbifolds 1

Mike Davis

CMS, Zhejiang University

June 30, 2008
http://www.math.ohio-state.edu/∼mdavis/

1. Transformation groups

2. Orbifolds
   - Definitions and terminology
   - Covering spaces and π_1^{orb}
   - 1- and 2-dimensional orbifolds

Definitions

An action of a topological gp G on a space X is a (continuous) map $G \times X \to X$, denoted $(g, x) \to gx$, st

- g(hx)=(gh)x,
- 1x=x.

(Write $G \curvearrowright X$.)

Given $g \in G$, define $\theta_g : X \to X$ by $x \to gx$. Since $\theta_g \circ \theta_{g^{-1}} = 1_X = \theta_{g^{-1}} \circ \theta_g$, the map θ_g is a homeomorphism and the map $\Theta : G \to \text{Homeo}(X)$ defined by $g \to \theta_g$ is a homomorphism of groups.

Given $x \in X$, $G_x := \{g \in G \mid gx = x\}$ is the *isotropy subgroup*. The action is *free* if $G_x = \{1\}$, $\forall x \in X$.

More Definitions

$G(x) := \{gx \in X \mid g \in G\}$ is the *orbit* of x. Note $G(x) \cong G/G_x$. The action is *transitive* if there is only one orbit. Given $x \in X$, the natural map $G/G_x \to G(x)$ defined by $gG_x \to gx$ is a continuous bijection.

The *orbit space* X/G is the set of orbits in X with the quotient topology (wrt the natural map $X \to X/G$).

A map $f : X \to Y$ of G-spaces is *equivariant* (or a *G-map*) if $f(gx) = gf(x)$

More definitions

Suppose $H \subset G$ is a subgp and Y is a H-space. Then H acts on $G \times Y$ via $h \cdot (g, x) = (gh^{-1}, hx)$. The orbit space is denoted $G \times_H Y$ and called the *twisted product*. The image of (g, x) in $G \times_H Y$ is denoted $[g, x]$. Note G acts on $G \times_H Y$.

A *slice* at a point $x \in X$ is a G_x-stable subset U_x st the map $G \times_{G_x} U_x \to X$ is an equivariant homeomorphism onto a neighborhood of $G(x)$. If U_x is homeomorphic to a disk, then $G \times_{G_x} U_x$ is an *equivariant tubular neighborhood* of $G(x)$.

Remark

A neighborhood of the orbit in X/G is homeomorphic to U_x/G_x.

The Differentiable Slice Theorem

Theorem

Suppose a compact Lie gp acts differentiably (= "smoothly") on a manifold M. Then every orbit has a G-invariant tubular neighborhood. More precisely, $\exists$ *a linear representation of* G_x *on a vector space S st that* $G \times_{G_x} S$ *is a tubular neighborhood of* $G(x)$*.*

Proof.

By integrating over the compact Lie group G we can find a G-invariant Riemannian metric. Then apply the usual proof using the exponential map. □

Proper actions of discrete groups

Γ a discrete gp, X a Hausdorff space and $\Gamma \curvearrowright X$.
The Γ-acton is *proper* if given any 2 points $x, y \in X$, $\exists$ open nbhds U of x and V of y st $\gamma U \cap V \neq \emptyset$ for only finitely many γ.

Exercise

Show that a Γ-action on X is proper iff

- X/Γ is Hausdorff,
- each isotropy subgp is finite
- each point $x \in X$ has a slice, ie, $\exists$ Γ_x-stable open nbhd U_x st $\gamma U_x \cap U_x = \emptyset, \quad \forall \gamma \in \Gamma - \Gamma_x$.

(This means that $\Gamma \times_{\Gamma_x} U_x$ is a nbhd of the orbit of x.)

Actions on manifolds

Suppose a discrete gp Γ acts properly on an n-dim mfld M^n.

A slice U_x at $x \in M^n$ is *linear* if $\exists$ a linear Γ_x-action on $\mathbf{R}^n$ st U_x is Γ_x-equivariantly homeomorphic to a Γ_x-stable nbhd of the origin in $\mathbf{R}^n$. The action is *locally linear* if every point has a linear slice.

Proposition

If $\Gamma \curvearrowright M^n$ properly and differentiablly, then action is locally linear.

Proof.

Since Γ_x is finite, we can find a Γ_x-invariant Riemannian metric on M. The exponential map, $exp : T_xM \to M$ is Γ_x-equivariant and takes a small disk about the origin homeomorphically onto a neighborhood U_x of x. If the disk is small enough, U_x is a slice. □

Definition

An *orbifold chart* on a space X is a 4-tuple $(\widetilde{U}, G, U, \pi)$ where

- U is open subset of X
- $\widetilde{U}$ is open in $\mathbf{R}^n$ and G is finite gp of homeomorphisms of $\widetilde{U}$
- $\pi : \widetilde{U} \to U$ is a map which can be factored as $\pi = \overline{\pi} \circ p$ where $p : \widetilde{U} \to \widetilde{U}/G$ is the orbit map and $\overline{\pi} : \widetilde{U}/G \to U$ is a homeo.

The chart is *linear* if $G \curvearrowright \mathbf{R}^n$ linearly.

For $i = 1, 2$, suppose $(\widetilde{U}_i, G_i, U_i, \pi_i)$ are orbifold charts on X. The charts are *compatible* if given points $\tilde{u}_i \in \widetilde{U}_i$ with $\pi_1(\tilde{u}_1) = \pi_2(\tilde{u}_2)$, $\exists$ homeo h from nbhd of $\tilde{u}_1$ in $\widetilde{U}_1$ onto nbhd of $\tilde{u}_2$ in $\widetilde{U}_2$ st $\pi_1 = \pi_2 \circ h$ on the nbhd.

Definition

An *orbifold atlas* on X is a compatible collection $\{(\widetilde{U}_i, G_i, U_i, \pi_i)\}_{i \in I}$ of orbifold charts which cover X.

Definition

An orbifold Q consists of an *underlying space* $|Q|$ together with a maximal atlas of charts.

A *smooth* orbifold means the groups act via diffeomorphisms and the charts are compatible via diffeomorphisms. A *locally linear* orbifold means all charts are equivalent to linear ones.

From now on, all orbifolds will be locally linear

Exercise

Suppose $\Gamma \curvearrowright M^n$ properly. By choosing slices we can cover M/Γ by orbifold charts. Show this gives the underlying space M/Γ the structure of orbifold which we denote by $M/\!\!/\Gamma$.

The local group

There is more info in an orbifold than just its underlying space. For example, if $q \in |Q|$ and $x = \pi^{-1}(q)$ is a point in the inverse image of q in some local chart, then the isotropy subgp G_x is independent of the chart up to isomorphism of gps. With this ambiguity, we call it the *local group at q* and denote it G_q.

A manifold is an orbifold in which each local gp is trivial.

Strata

In transformation gps, if $G \curvearrowright X$ and $H \subset G$, then

$$X_{(H)} := \{x \in X \mid G_x \text{ is conjugate to } H\}$$

is the set of points of *orbit type* G/H. The image of $X_{(H)}$ in X/G is a *stratum* of X/G. This image can be described as follows. First, take the fixed set X^H $(:= \{x \in X \mid hx = x, \forall h \in H\})$. Remove points x with $G_x \supsetneq H$ to get $X^H_{(H)}$. Then divide by the free action of $N(H)/H$ to get $X^*_{(H)}$, the *stratum of type* (H) in X/G. In an orbifold, Q, a *stratum of type* (H) is the subspace of $|Q|$ consisting of all points with local gp iso to H.

Proposition

If Q is a locally linear orbifold, then each stratum is a mfld.

Proof.

Suppose a finite gp $G \curvearrowright \mathbf{R}^n$ linearly and $H \subset G$. Then $(\mathbf{R}^n)^H$ is a linear subspace; hence, $(\mathbf{R}^n)^H_{(H)}$ is a mfld. Dividing by the free action of $N(H)/H$, we see that $(\mathbf{R}^n)^*_{(H)}$ is a mfld. □

The origin of the word "orbifold"

The true story

Near the beginning of his graduate course in 1976 Bill Thurston wanted to introduce a word to replace Satake's "V-manifold". His first choice was "manifolded". This turned out not to work for talking - the word could not be distinguished from "manifold". His next idea was "foldimani". People didn't like this. So, Bill said we would have an election after people made various suggestions for a new name for this concept. Chuck Giffen suggested "origam", Dennis Sullivan "spatial dollop" and Bill Browder "orbifold". There were many other suggestions. The election had several rounds with the names having the lowest number of votes being eliminated.

Finally, there were only 4 names left, origam, orbifold, foldimani and one other (maybe "V-manifold"). After the next round of voting "orbifold" and the other name were to be eliminated. At this point I spoke up and said something like "Wait you can't eliminate orbifold because the other two names are ridiculous. " So, "orbifold" was left on the list. After my impassioned speech, it won easily in the next round of voting.

Thurston's big improvement over Satake's earlier version was to show that the theory of covering spaces and fundamental groups worked for orbifolds. (When I was a grad student a few years before, this was "well-known" *not* to work.)

The local model for a covering projection between n-dimensional mflds is the identity map $id : U \to U$ on an open subset $U \subset \mathbf{R}^n$. Similarly, the local model for an *orbifold covering projection* is the natural map $\mathbf{R}^n/H \to \mathbf{R}^n/G$ where a finite gp G acts on $\mathbf{R}^n$ and $H \subset G$ is a subgp.

Proposition

If $\Gamma \curvearrowright M$ properly and $\Gamma' \subset \Gamma$ is a subgp, then $M/\!\!/\Gamma' \to M/\!\!/\Gamma$ is an orbifold covering projection.

Definition

An orbfld Q is *developable* if it is covered by a mfld. As we will see, this is equivalent to the condition that Q is the quotient of a discrete gp acting properly on a mfld. (In Thurston's terminology, Q is a "good" orbfld.)

Remark

Not every orbfld is developable (eg, later we will describe the "tear drop," the standard counterexample).

Definition

Q is *simply connected* if it is connected and does not admit a nontrivial orbfld covering, ie, if $p : Q' \to Q$ is a covering with $|Q'|$ connected, then p is a homeomorphism.

Fact

Any connected orbifold Q admits a simply connected orbfld covering $\pi : \widetilde{Q} \to Q$. This has the usual universal property, ie, if we pick a "generic" base point $q \in Q$ and $p : Q' \to Q$ is another covering with base points $q' \in Q'$ and $\tilde{q} \in \widetilde{Q}$ lying over q, then π factors through Q' via a covering proj $\widetilde{Q} \to Q'$ taking $\tilde{q}$ to q'. In particular, $\widetilde{Q} \to Q$ is a regular covering in the sense that its group of deck transformations is simply transitively on $\pi^{-1}(q)$.

Definition of the orbifold fundamental group

Definition

$\pi_1^{orb}(Q)$ is the group of deck transformations of the universal orbfld cover, $p : \widetilde{Q} \to Q$

Later I will give a definition in terms of generators and relations. A third defn: "homotopy classes" of loops $[0,1] \to Q$. (To do this we must first define what is meant by a map from a space to Q - it should be a continuous map to $|Q|$ together with a choice of a "local lift" (up to equivalence) for each orbfld chart for Q. A fourth defn: if $\mathcal{G}_Q$ is the groupoid associated to Q and $B\mathcal{G}_Q$ is its classifying space, then $\pi_1^{orb}(Q) := \pi_1(B\mathcal{G}_Q)$.

Developability and the local group

For each $x \in |Q|$, let G_x denote the local gp at x. (It a subgp of $GL(n, \mathbf{R})$, well-defined up to conjugation. We can identify G_x with the fundamental group of a nbhd of the form $\widetilde{U}_x/G_x$ where $\widetilde{U}_x$ is a ball in some linear representation. So, G_x is the "local fundamental group" at x. The inclusion of the nbhd induces a homomorphism $G_x \to \pi_1^{orb}(Q)$.

Proposition

Q is developable $\iff$ each local gp injects (ie, for each $x \in |Q|$, the map $G_x \to \pi_1^{orb}(Q)$ is injective).

Dimension 1

The only finite gp which acts linearly (and effectively) on $\mathbf{R}^1$ is the cyclic gp of order 2, C_2. It acts via the reflection $x \mapsto -x$. The orbit space $\mathbf{R}^1/C_2$ is identified with $[0, \infty)$.

It follows that every 1-dimensional orbfld Q is either a 1-mfld or a 1-mfld with boundary. If Q is compact and connected, then it is either a circle or an interval (say, $[0, 1]$).

The infinite dihedral gp, D_∞ is the gp generated by 2 distinct affine reflections on $\mathbf{R}^1$. $\mathbf{R}^1/D_\infty \cong [0, 1]$.

2-dimensional linear groups

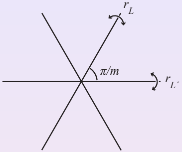
A finite gp $G \curvearrowright \mathbf{R}^n$ linearly. Then G is conjugate to subgp of $O(n)$. (Pf: By averaging we get an invariant inner product). Hence, G acts on the unit sphere $S^{n-1} \subset \mathbf{R}^n$.

Suppose $G \subset O(2)$. Then $S^1/\!\!/G = S^1$ or $S^1/\!\!/G = [0,1]$.

- In the first case, $S^1 \to S^1/\!\!/G = S^1$ is an n-fold cover, where $n = |G|$, and G is the cyclic gp C_n acting by rotations.
- In the second case, the composition, $\mathbf{R}^1 \to S^1 \to S^1/\!\!/G = [0,1]$, is the univ cover with gp of deck transformations D_∞. It follows that $G = D_m$ (the dihedral gp of order $2m$) or $G = C_2$ $(= D_1)$ acting by reflection across a line.

Theorem of Leonardo da Vinci

Any finite subgp of $O(2)$ is conjugate to either C_n or D_m.



Question

What does $\mathbf{R}^2 /\!/ G$ look like?

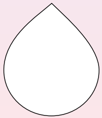
- $\mathbf{R}^2$ $(G = \{1\})$
- cone $(G = C_n)$
- half-space $(G = D_1)$
- sector $(G = D_m)$

In half-space case, codim 1 stratum is a *mirror*. In the sector case, codim 2 stratum is a *corner reflector*

2-dimensional orbifolds

Here is the picture: the underlying space of a 2-dim orbfld Q is a 2-mfld, possibly with bdry. Certain points in the interior of the $|Q|$ are "cone points" labeled by an integer n_i specifying that the local gp is C_{n_i}. The codim 1 strata are the mirrors; their closures cover $\partial|Q|$. The closures of two mirrors intersect in a corner reflectors (where local gp is D_{m_i}).

The picture to the right is possible. It is not developable.



General orbifolds

- $G \subset O(n)$, $D^n \subset \mathbf{R}^n$ the unit disk, $G \curvearrowright D^n$
- $D^n = \text{Cone}(S^{n-1})$, so $D^n /\!\!/ G = \text{Cone}(S^{n-1} /\!\!/ G)$
  $\therefore$ a point in a general orbfld has conical nbhd of this form.

Example

Suppose $G = C_2$ acting via antipodal map, $x \mapsto -x$. Then $D^n /\!\!/ C_2 = \text{Cone}(\mathbf{R}P^{n-1})$

Suppose Q is an n-dim orbfld, $Q_{(2)}$ is the complement of the strata of codim > 2. The description of $Q_{(2)}$ is similar to a 2-dim orbfld. $|Q_{(2)}|$ is an n-mfld with bdry; the bdry is a union of (closures of) mirrors; the codim 2 strata in the interior are codim 2 submflds.

Examples of orbifold coverings

Suppose $X \to |Q|$ is an ordinary covering of topological spaces. Pullback strata of Q to strata in X obtaining an orbfld Q'. (Specific example: Q is $\mathbf{R}P^2$ with one cone point labeled n. $S^2 \to \mathbf{R}P^2$ is the double cover. The single cone point pulls back to two cone points in S^2 labeled n.

Double $|Q|$ along its bdry to get a 2-fold orbfld covering $Q' \to Q$ w/o codim 1 strata. (Eg, if Q is a triangle, Q' is a 2-sphere with 3 cone points.)

The n-fold branched cover of Q along a codim 2 stratum labeled by the cyclic gp of order n.

Generators and relations for $\pi_1^{orb}(Q)$

Remark

$\pi_1^{orb}(Q) = \pi_1^{orb}(Q_{(2)})$. (Pf: general position)

Let $\hat{Q}$ denote the complement in $|Q|$ of the strata of codim ≥ 2 (retain the mirrors on $\partial|Q|$). Choose a base point x_0 in interior $\hat{Q}$. We are going to construct $\pi_1^{orb}(Q)$ from $\pi_1(\hat{Q}, x_0)$ by adding generators and relations.

- For each component T of a codim 2 stratum in interior of $|Q|$, choose a loop α_T starting at x_0 which makes a small loop around T. Let $n(T)$ be the order of the cyclic gp labeling T.

Generators and relations, continued

- Suppose P is a codim 2 stratum contained in $M \cap N$ (st P is a corner reflector). Let $m(P)$ be the label on P (st the dihedral gp at P has order $2m(P)$).

- For each mirror M and homotopy class of paths γ_M from x_0 to M introduce a new generator $\beta_{(M,\gamma_M)}$.

Relations

$$[\alpha_T]^{n(T)} = 1, \quad [\beta_{(M,\gamma_M)}]^2 = 1, \text{ and } \quad ([\beta_{(M,\gamma_M)}][\beta_{(N,\alpha_N)}])^{m(P)} = 1,$$

where P is a component of $\overline{M} \cap \overline{N}$ and γ_M and α_N are homotopic as paths from x_0 to P.