Orbifolds 4

Mike Davis

CMS, Zhejiang University

July 3, 2008
http://www.math.ohio-state.edu/~mdavis/
1. Andreev’s Theorem
   • A dimension count

2. 3-dimensional orbifolds

3. Reflectofolds
A geometric reflection group on $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ is determined by its fundamental polytope. In the spherical case the fundamental polytope must be a simplex and in the Euclidean case it must be a product of simplices. There is nothing more to said in the spherical and Euclidean cases.

In the hyperbolic case we know what happens in dim 2: the fundamental polygon can be an $k$-gon for any $k \geq 3$ and almost any assignment of angles can be realized by a hyperbolic polygon (there are a few exceptions when $k = 3$ or 4). What happens in dim 3?

There is a beautiful theorem due to Andreev, which gives a complete answer.
Roughly, it says given a simple polytope $K$, for it to be the fundamental polytope of a hyperbolic reflection group,

- there is no restriction on its combinatorial type
- subject to the condition that the group at each vertex be finite, almost any assignment of dihedral angles to the edges of $K$ can be realized (provided a few simple inequalities hold).

In contrast to dim 2, the 3-dim hyperbolic polytope is uniquely determined, up to isometry, by its dihedral angles – the moduli space is a point.
Theorem (Thurston’s Conjecture, Perelman’s Theorem)

A closed 3-orbifold $Q^3$ with infinite $\pi_1^{orb}$ admits a hyperbolic structure iff it satisfies the following two conditions:

- $Q^3$ is developable.
- Every embedded 2-dim spherical suborbifold bounds a quotient of a 3-ball in $Q^3$ (\[\iff\] $Q^3$ is aspherical).
- There is no incompressible 2-dim Euclidean suborbifold in $Q^3$ (i.e., $Q^3$ is “atoroidal”).

(“Incompressible” means induces an injection on $\pi_1^{orb}( )$.)
Theorem (Andreev ~1967)

Suppose $K$ is (the combinatorial type of) a simple 3-dim polytope, different from a tetrahedron. $E$ is its edge set and $\theta : E \rightarrow (0, \pi/2]$ any function. Then $(K, \theta)$ can be realized as a convex polytope in $\mathbb{H}^3$ with dihedral angles as prescribed by $\theta$ if and only if the following conditions hold:

- At each vertex, the angles at the three edges $e_1, e_2, e_3$ which meet there satisfy $\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi$.
- If three faces intersect pairwise but do not have a common vertex, then the angles at the three edges of intersection satisfy $\theta(e_1) + \theta(e_2) + \theta(e_3) < \pi$.
- Four faces cannot intersect cyclically with all four angles $= \pi/2$ unless two of the opposite faces also intersect.
- If $K$ is a triangular prism the angles along base and top cannot all be $\pi/2$. 
Moreover, when \((K, \theta)\) is realizable, it is unique up to an isometry of \(\mathbb{H}^3\).

**Corollary**

Suppose \(K\) is (the combinatorial type of) a simple 3-polytope, different from a tetrahedron, that \(\{F_s\}_{s \in S}\) is its set of codim 1 faces and that \(e_{st}\) is the edge \(F_s \cap F_t\) (when \(F_s \cap F_t \neq \emptyset\)). Given an angle assignment \(\theta : E \to (0, \pi/2]\), with \(\theta(e_{st}) = \pi / m(s, t)\) and \(m(s, t)\) an integer \(\geq 2\), then \((K, \theta)\) is a hyperbolic orbifold iff the \(\theta(e_{st})\) satisfy Andreev’s Conditions. Moreover, the geometric reflection gp \(W\) is unique up to conjugation in \(\text{Isom}(\mathbb{H}^3)\).
Examples

- $K$ is a dodecahedron with all dihedral angles $= \pi/2$.
- $K$ is a cube with disjoint edges in different directions labeled by integers $> 2$ and all other edges labeled 2.

Exercise

Make up your own examples.
Dual form of Andreev’s Theorem

Let \( L \) be the triangulation of \( S^2 \) dual to \( \partial K \).

Vert(\( L \)) ←→ Face(\( K \))
Edge(\( L \)) ←→ Edge(\( K \))
\{2-simplices in \( L \}\) ←→ Vert(\( K \))

Input data

\[ \theta : \text{Edge}(\( L \)) \rightarrow (0, \pi/2] \]

The condition that \( K \) have a spherical link at each vertex:
if \( e_1, e_2, e_3 \) are the edges of a triangle, then
\[ \theta(e_1) + \theta(e_2) + \theta(e_3) > \pi. \]
Theorem (Dual form of Andreev’s Thm)

Suppose $L$ is a triangulation of $S^2$ and $L \neq \partial \Delta^3$. Let $\theta : \text{Edge}(L) \rightarrow (0, \pi/2]$. Then dual polytope $K$ can be realized as convex polytope in $\mathbb{H}^3$ with prescribed dihedral angles $\iff$

- If $e_1, e_2, e_3$ are the edges of any triangle, then $\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi$.
- If $e_1, e_2, e_3$ are the edges of a 3-circuit $\neq \partial \Delta^2$, then $\theta(e_1) + \theta(e_2) + \theta(e_3) < \pi$.
- If $e_1, e_2, e_3, e_4$ are the edges of a 4-circuit $\neq \partial$bdry of union of 2 adjacent triangles, then all 4 $\theta(e_i)$ cannot $= \pi/2$.
- If $L$ is suspension of $\partial \Delta^2$, then all “vertical” edges cannot have $\theta(e_i) = \pi/2$. 
Given a convex 3-dim polytope $K$, Andreev’s Theorem asserts that a certain map $\theta$ from the space $C(K)$ of isometry classes convex polyhedra of the same combinatorial type as $K$ to a certain subset $A(K) \subset \mathbb{R}^E$ (where $E \equiv \text{Edge}(K)$ and where $A(K)$ is the convex subset defined by Andreev’s inequalities) is a homeomorphism. Let’s compute $\dim C(K)$.

For each $F \in \text{Face}(K)$, let $u_F \in S^{2,1}$ be the inward-pointing unit normal vector to $F$ (Here $S^{2,1} \equiv \{x \in \mathbb{R}^{2,1} \mid \langle x, x \rangle = 1\}$). The $(u_F)_{F \in \text{Face}(K)}$ determine $K$ (since $K$ is the intersection of the half-spaces determined by the $u_F$). The assumption that $K$ is simple means that the hyperbolic hyperplanes normal to the $u_F$ intersect in general position. So, a slight perturbation of the $u_F$ will not change the combinatorial type of $K$. That is to say, the set of $\mathcal{F}$-tuples $(u_F)$ which define a polytope combinatorially equivalent to $K$ is an open subset $Y$ of $(S^{2,1})^\text{Face}(K)$. 
Andreev's Theorem
3-dimensional orbifolds
Reflectofolds

A dimension count

\[ \dim C(K) \]

1. \( f = \#(\text{Face}(K)), \ e = \#(\text{Edge}(K)), \ v = \#(\text{vertex}(K)) \)
2. \( \text{Isom}(\mathbb{H}^3) = O(3, 1), \ \dim(O(3, 1)) = 6, \ \text{and} \ \dim S^{3,1} = 3 \)
3. Previous page \( \implies \dim C(K) = 3f - 6 \)

Since \( f - e + v = 2 \), \( 3f - 6 = 3e - 3v \). Since 3 edges meet at each vertex, \( 3v = 2e \).
\[ \therefore \ 3f - 6 = 3e - 3v = e. \]

So, \( \theta : C(K) \rightarrow A(K) \subset \mathbb{R}^E \) is a map between mflds (with bdry) of the same dimension.
Recall the list of 2-dim spherical orbifolds:

- $|Q^2| = D^2$: $(; )$, $(; m, m)$, $(; 2, 2, m)$, $(; 2, 3, 3)$, $(; 2, 3, 4)$, $(; 2, 3, 5)$, $(2; m)$, $(3; 2)$.
- $|Q^2| = S^2$: $(; )$, $(n, n)$, $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$.
- $|Q^2| = \mathbb{R}P^2$: $(; )$, $(n)$

The local models for 3-dim orbifolds are cones on any one of the above.

For example, if $|Q^2| = S^2$ with $(n, n)$, then the 3-dim model is an interval in $D^3$. For example, (quotients of) $n$-fold branched covers of knots or links in $S^3$ (or any other 3-mfld) have this form.
A flat orbifold

Consider the 3 families of lines in $\mathbb{E}^3$ of the form $(t, n, m + \frac{1}{2})$, $(m + \frac{1}{2}, t, n)$ and $(n, m + \frac{1}{2}, t)$, where $t \in \mathbb{R}$ and $n, m \in \mathbb{Z}$. Let $\Gamma$ be the subgp of $\text{Isom}(\mathbb{E}^3)$ generated by rotation by $\pi$ about each of these lines. A fundamental domain is the unit cube. The orbifold $\mathbb{E}^3//\Gamma$ is obtained by “folding up” the cube to get the 3-sphere. The image of the lines (= the singular set) are 3 circles in $S^3$ each labeled by 2 (meaning $C_2$, the cyclic gp of order 2). These 3 circles form the Borromean rings.
Definition

An $n$-dim orbfld $Q$ is a reflectofold if is locally modeled on finite linear reflection groups $\rtimes \mathbb{R}^n$.

If $W \rtimes \mathbb{R}^n$ as a finite reflection gp, then $\mathbb{R}^n/W$ is a simplicial cone, ie, up to linear isomorphism it looks like $[0, \infty)^n$. It follows that the underlying space of a reflectofold $Q$ is a mfld with corners. Conversely, to give a mfld with corners the structure of a reflectofold, essentially all we need to do is label its codim 2 strata by integers $\geq 2$ in such a way that the strata of higher codim correspond to finite Coxeter gps.
It follows from the description of $\pi_1^{\text{orb}}(Q)$ in Lecture 1 that

$\pi_1^{\text{orb}}(Q)$ is generated by reflections $\iff \pi(|Q|) = 1$. (Here “reflection” means an involution with codim 1 fixed set.)

Henceforth, let’s assume this (that $|Q|$ is simply connected) unless we say otherwise.

If $Q$ is developable, then any codim 2 stratum is contained in the closures of 2 distinct codim 1 strata. Otherwise we would have a nondevelopable suborbifold pictured to the right.

Similarly, developable $\implies$ if intersection of 2 codim 1 strata contains 2 distinct codim 1 strata, then they are labeled by the same integer.
Aspherical orbifolds

Definition
An orbifold is *aspherical* if its universal cover is a contractible manifold.

Question
*Is it true that a contractible orbifold is automatically a manifold?*

I think so, but I have never seen it written down.

Remark
A 2 dim orbifold $Q^2$ is aspherical $\iff \chi^{orb}(Q^2) \leq 0.$
Conjecture (Hopf, Chern, Thurston)

Suppose $Q^{2n}$ is a closed aspherical orbifold. Then
$(-1)^n \chi^{orb}(Q^{2n}) \geq 0$. 

Mike Davis
Orbifolds 4
The set up

$Q$ a reflectofold. Denote the underlying space by $K$ (instead of $|Q|$). Let $S$ index the set of mirrors ($= \{ \text{codim1 strata} \}$). $K_s$ the closed mirror corresponding to $s$. $m(s, t)$ the label on the codim 2 strata of $K_s \cap K_t$. $m(s, t) = \infty$ if $K_s \cap K_t = \emptyset$. $(W, S)$ the Coxeter system defined by the presentation (ie, $W = \pi_{1}^{\text{orb}}(Q)$). For each $T \subset S$, $W_T$ is the subgp generated by $T$. Let $S := \{ T \subset S \mid \text{Card}(W_T) < \infty \}$. Put

$$K_T = \bigcap_{s \in T} K_s.$$ 

Since $Q$ is an orbfld, $K_T \neq \emptyset \implies W_T \in S.$
The reflectofold $Q$ is aspherical $\iff$

- $K_T \neq \emptyset \iff T \in S$ (ie, $W_T$ is finite).
- For each $T \in S$, $K_T$ is acyclic (ie, $\overline{\mathcal{H}}_*(K_T) = 0$).

(Note: $\emptyset \in S$ and $K_{\emptyset} = K$. Since $K$ is simply connected and acyclic it is contractible.)