Orbifolds 4

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A geometric reflection group on $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ is determined by its fundamental polytope. In the spherical case the fundamental polytope must be a simplex and in the Euclidean case it must be a a product of simplices. There is nothing more to said in the spherical and Euclidean cases.

In the hyperbolic case we know what happens in dim 2: the fundamental polygon can be an k-gon for any $k \geq 3$ and almost any assignment of angles can be realized by a hyperbolic polygon (there are a few exceptions when $k = 3$ or 4). What happens in dim 3?

There is a beautiful theorem due to Andreev, which gives a complete answer.

Roughly, it says given a simple polytope K, for it to be the fund polytope of a hyperbolic reflection gp,

- there is no restriction on its combinatorial type
- subject to the condition that the gp at each vertex be finite, almost any assignment of dihedral angles to the edges of K can be realized (provided a few simple inequalities hold).

In contrast to dim 2, the 3-dim hyperbolic polytope is uniquely determined, up to isometry, by its dihedral angles – the moduli space is a point.

Theorem (Thurston's Conjecture, Perelman's Theorem)

A closed 3-orbfld Q^3 with infinite π_1^{orb} admits a hyperbolic structure iff it satisfies the following two conditions:

- *Q^3 is developable.*
- *Every embedded 2-dim spherical suborbfld bounds a quotient of a 3-ball in Q^3 ($\implies$ Q^3 is aspherical).*
- *There is no incompressible 2-dim Euclidean suborbfld in Q^3 (i.e., Q^3 is "atoroidal").*

("Incompressible" means induces an injection on $\pi_1^{orb}(\ )$.)

Theorem (Andreev ∼1967)

Suppose K is (the combinatorial type of) a simple 3-dim polytope, different from a tetrahedron. E is its edge set and $\theta : E \to (0, \pi/2]$ *any function. Then* (K, θ) *can be realized as a convex polytope in* $\mathbb{H}^3$ *with dihedral angles as prescribed by* θ *if and only if the following conditions hold:*

- *At each vertex, the angles at the three edges* e_1, e_2, e_3 *which meet there satisfy* $\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi$.
- *If three faces intersect pairwise but do not have a common vertex, then the angles at the three edges of intersection satisfy* $\theta(e_1) + \theta(e_2) + \theta(e_3) < \pi$.
- *Four faces cannot intersect cyclically with all four angles* $= \pi/2$ *unless two of the opposite faces also intersect.*
- *If K is a triangular prism the angles along base and top cannot all be* $\pi/2$.

Moreover, when (K, θ) is realizable, it is unique up to an isometry of $\mathbb{H}^3$.

Corollary

Suppose K is (the combinatorial type of) a simple 3-polytope, different from a tetrahedron, that $\{F_s\}_{s\in S}$ is its set of codim 1 faces and that e_{st} is the edge $F_s \cap F_t$ (when $F_s \cap F_t \neq \emptyset$). Given an angle assignment $\theta : E \to (0, \pi/2]$, with $\theta(e_{st}) = \pi/m(s,t)$ and $m(s,t)$ an integer ≥ 2, then (K, θ) is a hyperbolic orbfld iff the $\theta(e_{st})$ satisfy Andreev's Conditions. Moreover, the geometric reflection gp W is unique up to conjugation in $\mathrm{Isom}(\mathbb{H}^3)$.

Examples

- K is a dodecahedron with all dihedral angles $= \pi/2$.
- K is a cube with disjoint edges in different directions labeled by integers > 2 and all other edges labeled 2

Exercise

Make up your own examples.

Dual form of Andreev's Theorem

Let L be the triangulation of S^2 dual to ∂K.

$$\begin{aligned} \mathrm{Vert}(L) &\longleftrightarrow \mathrm{Face}(K) \\ \mathrm{Edge}(L) &\longleftrightarrow \mathrm{Edge}(K) \\ \{\text{2-simplices in } L\} &\longleftrightarrow \mathrm{Vert}(K) \end{aligned}$$

Input data

$\theta : \mathrm{Edge}(L) \to (0, \pi/2]$

The condition that K have a spherical link at each vertex: if e_1, e_2, e_3 are the edges of a triangle, then $\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi$.

Theorem (Dual form of Andreev's Thm)

Suppose L is a triangulation of S^2 and $L \neq \partial\Delta^3$.
$\theta : \text{Edge}(L) \to (0, \pi/2]$. *Then dual polytope K can be realized as convex polytope in $\mathbb{H}^3$ with prescribed dihedral angles $\iff$*

- *If e_1, e_2, e_3 are the edges of any triangle, then $\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi$.*
- *If e_1, e_2, e_3 are the edges of a 3-circuit $\neq \partial\Delta^2$, then $\theta(e_1) + \theta(e_2) + \theta(e_3) < \pi$.*
- *If e_1, e_2, e_3, e_4 are the edges of a 4-circuit $\neq$ to bdry of union of 2 adjacent triangles, then all 4 $\theta(e_i)$ cannot $= \pi/2$.*
- *If L is suspension of $\partial\Delta^2$, then all "vertical" edges cannot have $\theta(e_i) = \pi/2$.*

Given a convex 3-dim polytope K, Andreev's Theorem asserts that a certain map θ from the space $C(K)$ of isometry classes convex polyhedra of the same combinatorial type as K to a certain subset $A(K) \subset \mathbf{R}^E$ (where $E := \text{Edge}(K)$ and where $A(K)$ is the convex subset defined by Andreev's inequalities) is a homeomorphism. Let's compute $\dim C(K)$.

For each $F \in \text{Face}(K)$, let $u_F \in \mathbb{S}^{2,1}$ be the inward-pointing unit normal vector to F (Here $\mathbb{S}^{2,1} := \{x \in \mathbf{R}^{2,1} \mid \langle x, x \rangle = 1\}$). The $(u_F)_{F \in \text{Face}(K)}$ determine K (since K is the intersection of the half-spaces determined by the u_F). The assumption that K is simple means that the hyperbolic hyperplanes normal to the u_F intersect in general position. So, a slight perturbation of the u_F will not change the combinatorial type of K. That is to say, the set of $\mathcal{F}$-tuples (u_F) which define a polytope combinatorially equivalent to K is an open subset Y of $(\mathbb{S}^{2,1})^{\text{Face}(K)}$.

dim $C(K)$

- $f = \#(\text{Face}(K))$, $e = \#(\text{Edge}(K)$, $v = \#(vertex(K)$
- $\text{Isom}(\mathbb{H}^3) = O(3,1)$, $\dim(O(3,1) = 6$, and $\dim \mathbb{S}^{3,1} = 3$
- Previous page $\implies$ $\dim C(K) = 3f - 6$

Since $f - e + v = 2$, $3f - 6 = 3e - 3v$. Since 3 edges meet at each vertex, $3v = 2e$.
$\therefore\ 3f - 6 = 3e - 3v = e$.

So, $\theta : C(K) \to A(K) \subset \mathbf{R}^E$ is a map between mflds (with bdry) of the same dimension.

Recall the list of 2-dim spherical orbifolds:

- $|Q^2| = D^2$: $(\ ; \ )$, $(\ ; m, m)$, $(\ ; 2, 2, m)$, $(\ ; 2, 3, 3)$, $(\ ; 2, 3, 4)$, $(\ ; 2, 3, 5)$, $(2; m)$, $(3; 2)$.
- $|Q^2| = S^2$: $(\ )$, (n, n), $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$.
- $|Q^2| = \mathbf{R}P^2$: $(\ )$, (n)

The local models for 3-dim orbflds are cones on any one of the above.

For example, if $|Q^2| = S^2$ with (n, n), then the 3-dim model is an interval in D^3. For example, (quotients of) n-fold branched covers of knots or links in S^3 (or any other 3-mfld) have this form.

Example: Borromean rings

A flat orbfld

Consider the 3 families of lines in $\mathbb{E}^3$ of the form $(t, n, m + \frac{1}{2})$, $(m + \frac{1}{2}, t, n)$ and $(n, m + \frac{1}{2}, t)$, where $t \in \mathbf{R}$ and $n, m \in \mathbb{Z}$. Let Γ be the subgp of $\mathrm{Isom}(\mathbb{E}^3)$ generated by rotation by π about each of these lines. A fundamental domain is the unit cube. The orbfld $\mathbb{E}^3/\!\!/\Gamma$ is obtained by "folding up" the cube to get the 3-sphere. The image of the lines ($=$ the singular set) are 3 circles in S^3 each labeled by 2 (meaning C_2, the cyclic gp of order 2). These 3 circles form the Borromean rings.

Definition

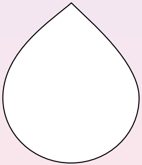
An n-dim orbfld Q is a *reflectofold* if is locally modeled on finite linear reflection groups $\curvearrowright \mathbf{R}^n$.

If $W \curvearrowright \mathbf{R}^n$ as a finite reflection gp, then $\mathbf{R}^n/W$ is a simplicial cone, ie, up to linear isomorphism it looks like $[0, \infty)^n$. It follows that the underlying space of a reflectofold Q is a mfld with corners. Conversely, to give a mfld with corners the structure of a reflectofold, essentially all we need to do is label its codim 2 strata by integers ≥ 2 in such a way that the strata of higher codim correspond to *finite* Coxeter gps.

It follows from the description of $\pi_1^{orb}(Q)$ in Lecture 1 that $\pi_1^{orb}(Q)$ is generated by reflections $\iff \pi(|Q|) = 1$. (Here "reflection" means an involution with codim 1 fixed set.)

Henceforth, let's assume this (that $|Q|$ is simply connected) unless we say otherwise.

If Q is developable, then any codim 2 stratum is contained in the closures of 2 distinct codim 1 strata. Otherwise we would have a nondevelopable suborbifold pictured to the right.



Similarly, developable $\implies$ if intersection of 2 codim 1 strata contains 2 distinct codim 1 strata, then they are labeled by the same integer.

Aspherical orbifolds

Definition

An orbfld is *aspherical* if its universal cover is a contractible mfld.

Question

Is it true that a contractible orbfld is automatically a mfld?

I think so, but I have never seen it written down.

Remark

A 2 dim orbfld Q^2 is aspherical $\iff \chi^{orb}(Q^2) \leq 0$.

My favorite conjecture

Conjecture (Hopf, Chern, Thurston)

Suppose Q^{2n} is a closed aspherical orbifold. Then $(-1)^n \chi^{orb}(Q^{2n}) \geq 0$.

The set up

Q a reflectofold. Denote the underlying space by K (instead of $|Q|$). Let S index the set of mirrors ($= \{$codim1 strata$\}$). K_s the closed mirror corresponding to s. $m(s,t)$ the label on the codim 2 strata of $K_s \cap K_t$. $m(s,t) = \infty$ if $K_s \cap K_t = \emptyset$. (W,S) the Coxeter system defined by the presentation (ie, $W = \pi_1^{orb}(Q)$). For each $T \subset S$, W_T is the subgp generated by T. Let $\mathcal{S} := \{T \subset S \mid \mathrm{Card}(W_T) < \infty\}$. Put

$$K_T = \bigcap_{s \in T} K_s.$$

Since Q is an orbfld, $K_T \neq \emptyset \implies W_T \in \mathcal{S}$.

Theorem

The reflectofold Q is aspherical $\iff$

- $K_T \neq \emptyset \iff T \in \mathcal{S}$ *(ie, W_T is finite).*
- *For each $T \in \mathcal{S}$, K_T is acyclic (ie, $\overline{H}_*(K_T) = 0$).*

(Note: $\emptyset \in \mathcal{S}$ and $K_\emptyset = K$. Since K is simply connected and acyclic it is contractible.)