Groups up to quasi-isometry

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Topological methods in group theory

group theory $\subset$ topology

via the fundamental group.

- $\forall$ group Γ, $\exists$ a topological space X with $\pi_1(X) = \Gamma$.
- Γ acts on the universal cover $\widetilde{X}$ with $\widetilde{X}/\Gamma = X$.
- X is not unique. We can choose it to be 2-dimensional, or so that $\widetilde{X}$ is contractible. If Γ is finitely presented, then we can choose X to be compact.
- Properties of Γ are reflected in properties of X or of $\widetilde{X}$

Geometric group theory

The field of geometric group theory has grown enormously in the last twenty five years - largely because of work of Gromov. In this field we are concerned with actions of groups on metric spaces via isometries (i.e., the gp action preserves distances).

The word metric

- Γ: a finitely generated group
- S: a finite set of generators (closed under taking inverses)
- Given $g \in \Gamma$, its *length*, $l(g)$ is the minimum integer k s.t. $g = s_1 \cdots s_k$, with $s_i \in S$.
- Define $d : \Gamma \times \Gamma \to \mathbb{N} \subset \mathbb{R}$ by $d(g, h) := l(g^{-1}h)$. (d is the *word metric*.)

Of course, d depends on the choice of generating set S.

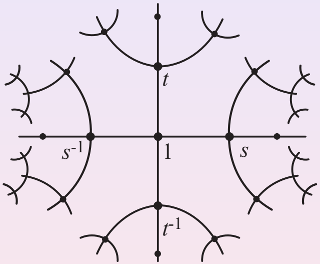
Another description of d

Definitions (The Cayley graph of (Γ, S))

$\text{Cay}(\Gamma, S)$ is graph with vertex set Γ and
Vertices g,h are connected by an edge $\iff h = gs$ or gs^{-1}, for some $s \in S$.

There is a natural metric d on $\text{Cay}(\Gamma, S)$.
Declare each edge to have length 1 and define $d(x, y)$ to be the length of the shortest path from x to y.
The restriction of this metric to the vertex set is the original word metric on Γ.

Cayley graph of the free group, F_2



Cay(F_2, S) is a regular 4-valent tree.

Definition

A (not necessarily continuous) map $f : X \to Y$ between metric spaces is a (L, A)-*quasi-isometry* if $\exists$ constants L, A so that $\forall x_1, x_2 \in X$ and $y \in Y$

$$\frac{1}{L}d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + A$$
$$\text{and} \quad d(y, f(X)) \leq A.$$

Notation

$X \sim_{\text{qi}} Y$ means X and Y are q.i.

Examples

- If S and S' are two (finite) sets of generators for Γ, then the resulting word metrics are q.i.
- If $H \subset \Gamma$ is a subgp of finite index, then $H \sim_{\text{qi}} \Gamma$. (A gp Γ *virtually* has some property if a finite index subgp H has it.)
- $\Gamma \sim_{\text{qi}} \text{Cay}(\Gamma, S)$.
- $\mathbb{Z} \sim_{\text{qi}} \mathbb{R}$.
- $\mathbb{Z}^n \sim_{\text{qi}} \mathbb{R}^n$.

Question

When are two groups q.i?

Definitions

A *geodesic* in a metric space X is a map $f : [a, b] \to X$ s.t. $\forall s, t \in [a, b]$,

$$d(f(s), f(t)) = |s - t|.$$

X is a *geodesic space* if any 2 points can be connected by a geodesic, or, equivalently, if the distance between any 2 points is the length of the shortest path connecting them.

X is a *proper* metric space if every closed ball is compact.

An isometric action of a gp Γ on a metric space X is *discrete* if $\forall x \in X$ and $R \in [0, \infty)$,

$$\{g \in \Gamma \mid d(gx, x) < R\}$$

is finite. The action is *cocompact* if X/Γ is compact.

Lemma (The Fundamental Lemma of Geometric Group Theory)

Suppose X, Y are proper geodesic metric spaces with Γ-actions which are discrete, cocompact and via isometries. Then $X \sim_{\mathrm{qi}} Y$. In particular, any such X is q.i to $\mathrm{Cay}(\Gamma, S)$.

Example

The fundamental group of any closed Riemannian manifold M is q.i to its universal cover $\widetilde{M}$. As before, we see

- $S^1 = \mathbb{R}/\mathbb{Z}$, $\pi_1(S^1) = \mathbb{Z}$; so $\mathbb{Z} \sim_{\mathrm{qi}} \mathbb{R}$
- $T^n := \mathbb{R}^n/\mathbb{Z}^n$, $\pi_1(T^n) = \mathbb{Z}^n$; so $\mathbb{Z}^n \sim_{\mathrm{qi}} \mathbb{R}^n$.

Features of hyperbolic n-space, $\mathbb{H}^n$

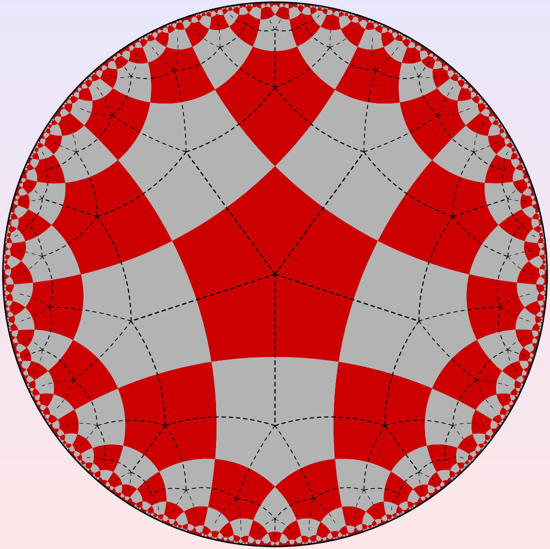
- The most important object in mathematics is the hyperbolic plane.
- $\mathbb{H}^n$ is the model for (n-dimensional) non-Euclidean geometry.
- $\mathbb{H}^n$ is a Riemannian manifold of constant curvature -1. (If we also require it to be complete & simply connected, then it is the unique such n-mfld.)
- It has a large isometry gp, which, in particular, is transitive on $\{(\text{pt}, \text{orthonormal frame})\}$, e.g., when $n = 2$, $\text{Isom}_+(\mathbb{H}^2) = SL(2, \mathbb{R})/\{\pm id\}$.

The Poincaré disk model

There are several models for $\mathbb{H}^n$. (For simplicity, let's say $n = 2$.) One is the *Poincaré disk model*. The points are points in the interior of the unit 2-disk, D^2. The metric is defined by

$$ds = \frac{2}{(1 - r^2)} dx,$$

where dx is the element of Euclidean arc length and r is distance from the origin. Geodesics (i.e., "lines") are circles $\perp$ bdry.



Non-Euclidean features of $\mathbb{H}^2$

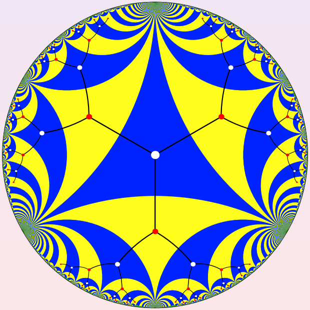
- (Anti-Parallel Postulate). Given a line and a point not on the line, there are an infinite number of (ultra)-parallel lines through the point.
- For any triangle, $\sum(\text{angles}) < \pi$.

Groups, Graphs and Trees

An Introduction to the Geometry of Infinite Groups

John Meier

Jon McCammond



Surface groups

Let S_g be a closed surface of genus $g > 1$. Then $\exists$ a discrete subgp $\Gamma \subset \mathrm{Isom}_+(\mathbb{H}^2)$ s.t $\quad \mathbb{H}^2/\Gamma \cong S_g$ (i.e., $\Gamma \cong \pi_1(S_g)$).
So, $\Gamma \sim_{\mathrm{qi}} \mathbb{H}^2$.

Similarly, if M^n is any closed Riemannian mfld of constant curvature -1, then its universal cover is isometric to $\mathbb{H}^n$.
Consequently, $\pi_1(M^n) \sim_{\mathrm{qi}} \mathbb{H}^n$.

Remarks about “word hyperbolic” groups

- The geometry of $\mathbb{H}^n$ imposes many conditions on discrete cocompact subgps of Isom($\mathbb{H}^n$).
- Similar properties hold for fundamental groups of negatively curved manifolds.
- Rips and independently, Cooper defined the notion of a “word hyperbolic group” in terms of the metric on the Cayley graph. This notion was popularized by Gromov. It is supposed to be a strictly group theoretic definition of negative curvature “in the large.”
- Word hyperbolicity is a quasi-isometry invariant.

Answers

Theorem

- $\mathbb{R}^n \sim_{\mathrm{qi}} \mathbb{R}^m \iff n = m.$
- $\mathbb{H}^n \sim_{\mathrm{qi}} \mathbb{H}^m \iff n = m.$
- $\mathbb{H}^n \nsim_{\mathrm{qi}} \mathbb{R}^n$ *(for $n \neq 1$).*

Theorem (Gromov, Pansu)

Any gp q.i with $\mathbb{R}^n$ is virtually $\mathbb{Z}^n$.

Theorem (Stallings)

If a gp is q.i with F_2, then it acts properly on some locally finite tree.

Theorem (Sullivan, Gromov, Tukia, . . .)

If Γ is finitely generated gp q.i to $\mathbb{H}^n$, then $\exists$ a discrete, cocompact, isometric Γ-action on $\mathbb{H}^n$.

Theorem (Kleiner-Leeb, . . .)

Suppose G is a semisimple Lie gp of noncompact type and $\widetilde{X}$ is the corresponding symmetric space. Then $\Gamma \sim_{\mathrm{qi}} \widetilde{X} \iff \widetilde{X}$ admits a proper , cocompact isometric Γ-action (hence, Γ is an extension of a cocompact lattice in G by a finite gp).

Symmetric spaces

Let G be a semisimple Lie gp w/o compact factors and $K \subset G$ a maximal compact subgp. $G/K := \widetilde{X}$ is the associated *symmetric space*. $\widetilde{X}$ is homeomorphic to $\mathbb{R}^n$. It admits a metric s.t $\quad \mathrm{Isom}(\widetilde{X}) = G \quad$ (essentially).
A discrete subgp $\Gamma \subset G$ is a *cocompact lattice* (or *a uniform lattice*) if $\widetilde{X}/\Gamma$ is compact.

Examples

- $G = SL(n, \mathbb{R})$, $K = SO(n)$.
- $G = O_+(n, 1)$, $K = O(n)$, $G/K = \mathbb{H}^n$.

To show that two gps are not q.i we need to develop properties which are invariant under quasi-isometries. One such property is the rate of (volume) growth in a group.

Given a finitely generated gp Γ, put

$$n_\Gamma(r) := \#\{g \in \Gamma \mid l(g) \leq r\}.$$

Γ has *polynomial growth* if $n_\Gamma(r) \leq Cr^n$ for some constant C and $n \in \mathbb{Z}$.

Γ has *exponential growth* if $e^{Cr} \leq n_\Gamma(r)$ for some positive constant C. (It's automatic that $n_\Gamma(r) < e^{Dr}$, for some D.)

These notions are obviously q.i invariants.

Exponential growth

One of the first results in this area was the following result of Milnor.

Theorem (Milnor, circa 1968)

The fundamental group of a compact, negatively curved manifold has exponential growth.

Current interest in GGT started with the following theorem of Gromov. It is still one of the best results in the field.

Polynomial growth

Theorem (Gromov, circa 1980)

Γ *has polynomial growth* $\iff$ *it is virtually nilpotent.*

Example (of a nilpotent gp)

$$\begin{pmatrix} 1 & m & n \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix} \qquad m, n, p \in \mathbb{Z}$$

Gromov's Theorem means that any gp of polynomial growth has a subgp of finite index which is a subgp of the gp of upper triangular integral matrices with 1's on the diagonal.

Other q.i invariant properties

- We have $\partial\mathbb{H}^n = S^{n-1}$. Any q.i $\mathbb{H}^n \to \mathbb{H}^n$ induces a homeo of the bdry.
  Similarly, for any word hyperbolic gp. (In particular, such a gp has a well-defined bdry.)
- # ends of Cay(Γ, S).
- Other topological invariants "at ∞," in the case when Γ acts properly, cocompactly on a contractible space, $\widetilde{X}$, e.g. $H^*_c(\widetilde{X})$.

Direction for current and future research

Pick some class of gps and then classify the gps in this class up to quasi-isometry.