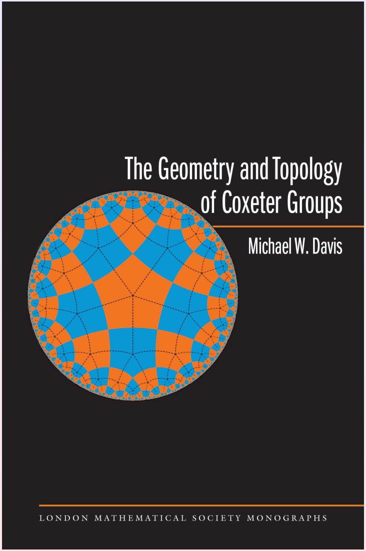
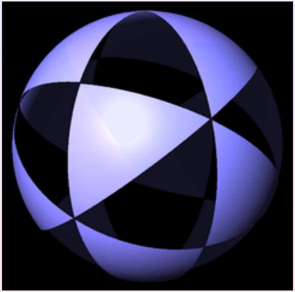
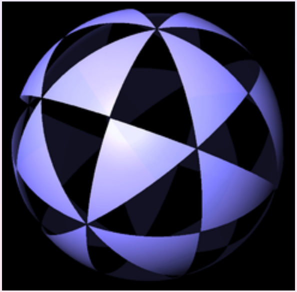
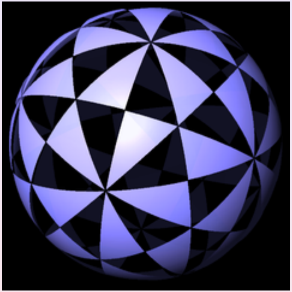
The Geometry and Topology of Coxeter Groups

Mike Davis

Invitations to Mathematics
OSU

September 18, 2013
http://www.math.osu.edu/∼davis.12/


The Geometry and Topology
of Coxeter Groups
Michael W. Davis
LONDON MATHEMATICAL SOCIETY MONOGRAPHS

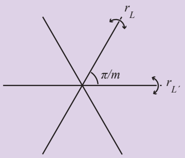
1. Geometric reflection groups
   - Some history
   - Properties

2. Abstract reflection groups
   - Coxeter systems
   - First realization: the Tits representation
   - Second realization: the cell complex Σ

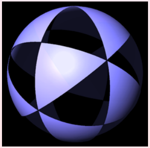
Dihedral groups

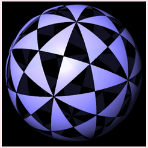
A *dihedral group* is any group which is generated by 2 involutions, call them s, t. Such a group is determined up to isomorphism by the order m of st (m is an integer ≥ 2 or ∞). Let $\mathbf{D}_m$ denote the dihedral group corresponding to m.

For $m \neq \infty$, $\mathbf{D}_m$ can be represented as the subgroup of $O(2)$ which is generated by reflections across lines L, L', making an angle of π/m.

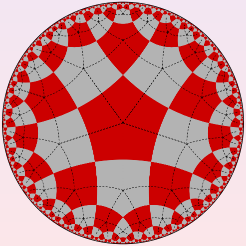


- In 1852 Möbius determined the finite subgroups of $O(3)$ generated by isometric reflections on the 2-sphere.
- The fundamental domain for such a group on the 2-sphere is a spherical triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, $\frac{\pi}{r}$, with p, q, r integers ≥ 2.
- Since the sum of the angles is $> \pi$, we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.
- For $p \geq q \geq r$, the only possibilities are: $(p, 2, 2)$ for any $p \geq 2$ and $(p, 3, 2)$ with $p = 3$, 4 or 5. The last three cases are the symmetry groups of the Platonic solids.





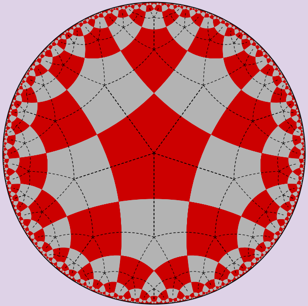
Later work by Riemann and Schwarz showed there were discrete groups of isometries of $\mathbb{E}^2$ or $\mathbb{H}^2$ generated by reflections across the edges of triangles with angles integral submultiples of π. Poincaré and Klein: similarly for polygons in $\mathbb{H}^2$.

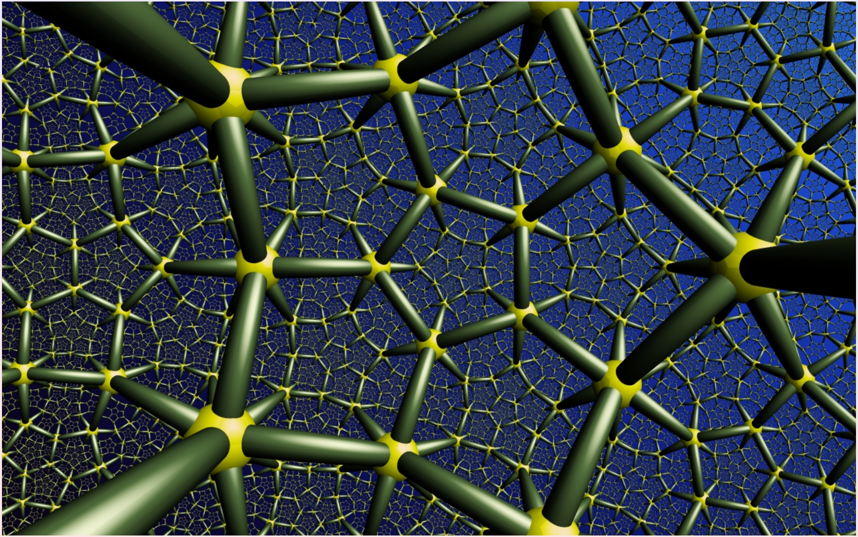


In 2^{nd} half of the 19^{th} century work began on finite reflection groups on $\mathbb{S}^n$, $n > 2$, generalizing Möbius' results for $n = 2$. It developed along two lines.

- Around 1850, Schläfli classified regular polytopes in $\mathbb{R}^{n+1}$, $n > 2$. The symmetry group of such a polytope was a finite group generated by reflections and as in Möbius' case, the projection of a fundamental domain to $\mathbb{S}^n$ was a spherical simplex with dihedral angles integral submultiples of π.
- Around 1890, Killing and E. Cartan classified complex semisimple Lie algebras in terms of their root systems. In 1925, Weyl showed the symmetry group of such a root system was a finite reflection group.
- These two lines were united by Coxeter in the 1930's. He classified discrete groups reflection groups on $\mathbb{S}^n$ or $\mathbb{E}^n$.

Let K be a fundamental polytope for a geometric reflection group. For $\mathbb{S}^n$, K is a simplex ($=$ generalization of a triangle). For $\mathbb{E}^n$, K is a product of simplices. For $\mathbb{H}^n$ there are other possibilities, eg, a right-angled pentagon in $\mathbb{H}^2$ or a right-angled dodecahedron in $\mathbb{H}^3$.





- Conversely, given a convex polytope K in $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ so that all dihedral angles have form $\pi/$integer, there is a discrete group W generated by isometric reflections across the codimension 1 faces of K.
- Let S be the set of reflections across the codim 1 faces of K. For $s, t \in S$, let $m(s,t)$ be the order of st. Then S generates W. The faces corresponding to s and t intersect in a codim 2 face iff $m(s,t) \neq \infty$, and for $s \neq t$, the dihedral angle along that face is $\pi/m(s,t)$. ($m(s,t)$ is an $S \times S$ symmetric matrix called the *Coxeter matrix*.) Moreover,
- $$\langle S \mid (st)^{m(s,t)}, \ \text{ where } (s,t) \in S \times S \rangle$$

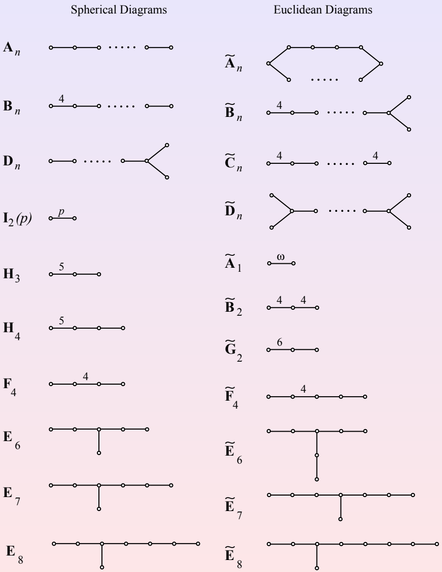
  is a presentation for W.

Coxeter diagrams

Associated to (W, S), there is a labeled graph Γ called its "Coxeter diagram."

$$\mathrm{Vert}(\Gamma) := S.$$

Connect distinct elements s, t by an edge iff $m(s, t) \neq 2$. Label the edge by $m(s, t)$ if this is > 3 or $= \infty$ and leave it unlabeled if it is $= 3$. (W, S) is *irreducible* if Γ is connected. (The components of Γ give the irreducible factors of W.)
The next slide shows Coxeter's classification of irreducible spherical and cocompact Euclidean reflection groups.


Spherical Diagrams
Euclidean Diagrams
$\mathbf{A}_n$
$\mathbf{B}_n$
4
$\mathbf{D}_n$
$\mathbf{I}_2(p)$
p
$\mathbf{H}_3$
5
$\mathbf{H}_4$
5
$\mathbf{F}_4$
4
$\mathbf{E}_6$
$\mathbf{E}_7$
$\mathbf{E}_8$
$\widetilde{\mathbf{A}}_n$
$\widetilde{\mathbf{B}}_n$
4
$\widetilde{\mathbf{C}}_n$
4
4
$\widetilde{\mathbf{D}}_n$
$\widetilde{\mathbf{A}}_1$
ω
$\widetilde{\mathbf{B}}_2$
4
4
$\widetilde{\mathbf{G}}_2$
6
$\widetilde{\mathbf{F}}_4$
4
$\widetilde{\mathbf{E}}_6$
$\widetilde{\mathbf{E}}_7$
$\widetilde{\mathbf{E}}_8$

Question

Given a group W and a set S of involutions which generates it, when should (W, S) be called an “abstract reflection group”?

Two answers

- Let $\mathrm{Cay}(W, S)$ be the Cayley graph (ie, its vertex set is W and $\{w, v\}$ spans an edge iff $v = ws$ for some $s \in S$)). **First answer**: for each $s \in S$, the fixed set of s separates $\mathrm{Cay}(W, S)$.
- **Second answer**: W has a presentation of the form:

$$\langle S \mid (st)^{m(s,t)}, \text{ where } (s,t) \in S \times S \rangle.$$

These two answers are equivalent!

Explanations of the terms

Cayley graphs

Given a group G and a set of generators S, let $\mathrm{Cay}(G, S)$ be the graph with vertex set G which has a (directed) edge from g to gs, $\forall g \in G$ and $\forall s \in S$. The group G acts on $\mathrm{Cay}(G, S)$ (written $G \curvearrowright \mathrm{Cay}(G, S)$), the action is simply transitive on the vertex set and the edges starting at a given vertex can be labelled by the elements of S or S^{-1}.

Presentations

Suppose S is a set of letters and $\mathcal{R}$ is a set of words in S. Let F_S be the free group on S and let N be the smallest normal subgroup containing $\mathcal{R}$. Then put $G := F_S/N$ and write $G = \langle S \mid \mathcal{R} \rangle$. It is a *presentation* for G.

If either of the two answers holds, (W, S) is a *Coxeter system* and W a *Coxeter group*. The second answer is usually taken as the official definition:
W has a presentation of the form:

$$\langle S \mid (st)^{m(s,t)}, \text{ where } (s,t) \in S \times S \rangle.$$

where $m(s,t)$ is a *Coxeter matrix*.

Question

Does every Coxeter system have a geometric realization?

Answer

Yes. In fact, there are two different ways to do this:

- the Tits representation
- the cell complex Σ.

Both realizations use the following construction.

The basic construction

A *mirror structure* on a space X is a family of closed subspaces $\{X_s\}_{s\in S}$. For $x \in X$, put $S(x) = \{s \in S \mid x \in X_s\}$. Define

$$\mathcal{U}(W, X) := (W \times X)/\sim ,$$

where $\sim$ is the equivalence relation: $(w, x) \sim (w', x') \iff x = x'$ and $w^{-1}w' \in W_{S(x)}$ (the subgroup generated by $S(x)$). $\mathcal{U}(W, X)$ is formed by gluing together copies of X (the *chambers*). $W \curvearrowright \mathcal{U}(W, X)$. (Think of X as the fundamental polytope and the X_s as its codimension 1 faces.)

Properties a geometric realization should have

It should be an action of W on a space $\mathcal{U}$ so that

- W acts as a reflection group, i.e., $\mathcal{U} = \mathcal{U}(W, X)$.
- The stabilizer of each $x \in \mathcal{U}$ should be a finite group.
- $\mathcal{U}$ should be contractible.
- $\mathcal{U}/W$ $(= X)$ should be compact.

The Tits representation

Linear reflections

Two pieces of data determine a (not necessarily orthogonal) reflection on $\mathbb{R}^n$:

- linear form $\alpha \in (\mathbb{R}^n)^*$ (the fixed hyperplane is $\alpha^{-1}(0)$).
- a (-1)-eigenvector $h \in \mathbb{R}^n$ (normalized so that $\alpha(h) = 2$).

The formula for the reflection is then

$$v \mapsto v - \alpha(v)h.$$

Symmetric bilinear form

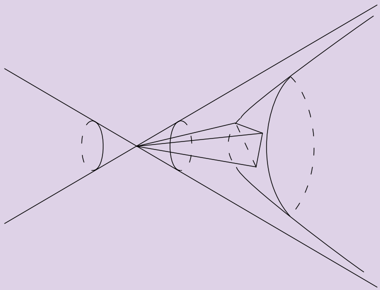
Let $(e_s)_{s \in S}$ be the standard basis for $(\mathbb{R}^S)^*$. Given a Coxeter matrix $m(s,t)$ define a symmetric bilinear form B on $(\mathbb{R}^S)^*$ by $B(e_s, e_t) = -2\cos(\pi/m(s,t))$.

For each $s \in S$, we have a linear reflection $r_s : v \mapsto v - B(e_s, v)e_s$. Tits showed this defines a linear action $W \curvearrowright (\mathbb{R}^S)^*$. We are interested in the dual representation $\rho : W \to GL(\mathbb{R}^S)$ defined by $s \mapsto \rho_s := (r_s)^*$.

Properties of Tits representation $W \to GL(\mathbb{R}^S)$

- The ρ_s are reflections across the faces of the standard simplicial cone $C \subset \mathbb{R}^S$.
- $\rho : W \hookrightarrow GL(\mathbb{R}^S)$, that is, ρ is injective.
- WC $(= \bigcup_{w \in W} wC)$ is a convex cone and if $\mathcal{I}$ denotes the interior of the cone, then
- $\mathcal{I} = \mathcal{U}(W, C^f)$, where C^f denotes the complement of the nonspherical faces of C (a face is *spherical* if its stabilizer is finite).
- So, W is a "discrete reflection group" on $\mathcal{I}$.

A hyperbolic triangle group



One consequence

W is virtually torsion-free. (This is true for any finitely generated linear group.)

Advantages

$\mathcal{I}$ is contractible (since it is convex) and W acts properly (ie, with finite stabilizers) on it.

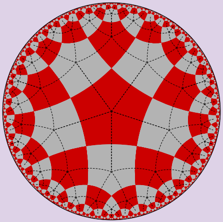
Disadvantage

$\mathcal{I}/W$ is not compact (since C^f is not compact).

Remark

By dividing by scalar matrices, we get a representation $W \to PGL(\mathbb{R}^S)$. So, $W \curvearrowright P\mathcal{I}$, the image of $\mathcal{I}$ in projective space. When W is infinite and irreducible, this is a proper convex subset of $\mathbb{R}P^n$, $n+1 = \#S$.

Vinberg showed one can get linear representations across the faces of more general polyhedral cones. As before, $W \curvearrowright \mathcal{I}$, where $\mathcal{I}$ is a convex cone; $P\mathcal{I}$ is a open convex subset of $\mathbb{R}P^n$. The fundamental chamber is a convex polytope with some faces deleted. Sometimes it can be a compact polytope, for example, a pentagon.



Question

For (W, S) to have a a reflection representation into $PGL(n+1, \mathbb{R})$ with fundamental chamber a compact convex polytope P^n there is a necessary condition: the simplicial complex L given by the spherical subsets of S must be dual to ∂P^n for some polytope P^n. Is this sufficient? (Probably not.)

Question

Are there irreducible, non-affine examples of such $W \subset PGL(n+1, \mathbb{R})$ and $P^n \subset \mathbb{R}P^n$ for n arbitrarily large?

The cell complex Σ

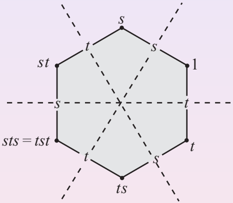
The second answer is to construct of contractible cell complex Σ on which W acts properly and cocompactly as a group generated by reflections. Its advantage is that Σ/W will be compact.

There are two dual constructions of Σ.

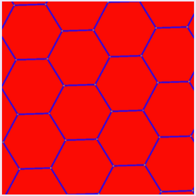
- Build the correct fundamental chamber K with mirrors K_s, then apply the basic construction, $\mathcal{U}(W, K)$.
- "Fill in" the Cayley graph of (W, S).

Filling in the Cayley graph

The Cayley graph of a finite dihedral group



Cayley graph of an infinite Coxeter group



Let $W_{\{s,t\}}$ be the dihedral subgroup $\langle s, t\rangle$. Whenever $m(s,t) < \infty$ each coset of $W_{\{s,t\}}$ spans a polygon in $\mathrm{Cay}(W, S)$. If we fill in these polygons, we get a simply connected 2-dimensional complex, which is the 2-skeleton of Σ.

If we want to obtain a contractible space then we have to fill in higher dimensional polytopes (”cells”).

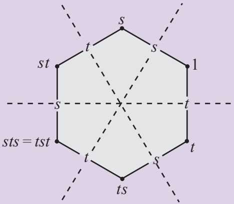
Definition

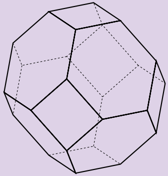
A subset $T \subset S$ is *spherical* if the subgroup W_T, which is generated by T, is finite. Let $\mathcal{S}$ denote the poset of spherical subsets of S.

Corresponding to a spherical subset T with $\#T = k$, there is a k-dimensional convex polytope called a *Coxeter zonotope*. When $k = 2$ it is the polygon associated to the dihedral group.

Coxeter zonotopes

Suppose W_T is finite reflection group on $\mathbb{R}^T$. Choose a point x in the interior of fundamental simplicial cone and let P_T be convex hull of $W_T x$.





The 1-skeleton of P_T is Cay(W_T, T).

When $W_T = (\mathbb{Z}/2)^n$, then P_T is an n-cube.

Geometric realization of a poset

Associated to any poset $\mathcal{P}$ there is a simplicial complex $|\mathcal{P}|$ called its *geometric realization*.

Filling in Cay(W, S)

Let $W\mathcal{S}$ denote the disjoint union of all spherical cosets (partially ordered by inclusion):

$$W\mathcal{S} := \coprod_{T \in \mathcal{S}} W/W_T \quad \text{and} \quad \Sigma := |W\mathcal{S}|.$$

Filling in Cay(W, S)

There is a cell structure on Σ with $\{\text{cells}\} = W\mathcal{S}$.

This follows from fact that poset of cells in P_T is $\cong W_T\mathcal{S}_{\leq T}$. The cells of Σ are defined as follows: the geometric realization of subposet of cosets $\leq wW_T$ is $\cong$ barycentric subdivision of P_T.

Properties of this cell structure on Σ

- W acts cellularly on Σ.
- Σ has one W-orbit of cells for each spherical subset $T \in \mathcal{S}$ and $\dim(\text{cell}) = \text{Card}(T)$.
- The 0-skeleton of Σ is W
- The 1-skeleton of Σ is $\text{Cay}(W, S)$.
- The 2-skeleton of Σ is the Cayley 2 complex of the presentation.
- If W is right-angled (i.e., each $m(s,t)$ is $1, 2$ or ∞), then each Coxeter zonotope is a cube.
- Moussong: the induced piecewise Euclidean metric on Σ is $\mathrm{CAT}(0)$ (meaning that it is nonpositively curved).

More properties

- Σ is contractible. (This follows from the fact it is $\mathrm{CAT}(0)$).
- The W-action is proper (by construction each isotropy subgroup is conjugate to some spherical W_T).
- Σ/W is compact.
- If W is finite, then Σ is a Coxeter zonotope.

Typical application of $\mathrm{CAT}(0)$-ness

$\exists$ nonpositively curved (polyhedral) metric on a manifold that is not homotopy equivalent to a nonpositivley curved Riemannian manifold.

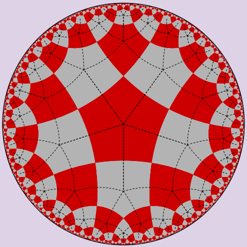
The dual construction of Σ

- Recall $\mathcal{S}$ is the poset of spherical subsets of S. The *fundamental chamber* K is defined by $K := |\mathcal{S}|$. (K is the cone on the barycentric subdivision of a simplicial complex L.)
- Mirror structure: $K_s := |\mathcal{S}_{\geq \{s\}}|$.
- $\Sigma := \mathcal{U}(W, K)$.
- So, K is homeomorphic to Σ / W.

The construction of Σ is very useful for constructing examples. The basic reason is that the chamber K is the cone over a fairly arbitrary simplicial complex (for example, L can be any barycentric subdivision). This means we can construct Σ with whatever local topology we want. (So K can be very far from a polytope.)

Relationship with geometric reflection groups

If W is a geometric reflection group on $\mathbb{X}^n = \mathbb{E}^n$ or $\mathbb{H}^n$, then K can be identified with the fundamental polytope, Σ with $\mathbb{X}^n$ and the cell structure is dual to the tessellation of Σ by translates of K.



Relationship with Tits representation

- Suppose W is infinite. Then K is subcomplex of $b\Delta$, the barycentric subdivision of the simplex $\Delta \subset C$.
- Consider the vertices which are barycenters of spherical faces. They span a subcomplex of $b\Delta$. This subcomplex is K. It is a subset of Δ^f.
- So, $\Sigma = \mathcal{U}(W, K) \subset \mathcal{U}(W, \Delta^f) \subset \mathcal{U}(W, C^f) = \mathcal{I}$.
- Σ is the "cocompact core" of $\mathcal{I}$.

Book

M.W. Davis, *The Geometry and Topology of Coxeter Groups*, Princeton Univ. Press, 2008.