THE GEOMETRY AND TOPOLOGY

OF COXETER GROUPS

Southampton, July 5 - 9, 2004

Lecture 1:

Overview

Finite reflection groups on $\mathbb{R}^n$.

Example. Take two lines in $\mathbb{R}^2$ making an angle of π/m. The gp $\mathbf{D}_m$ generated by the orthogonal reflections across these lines is the *dihedral group* of order $2m$.

Example. The symmetric group S_{n+1} acts on $\mathbb{R}^n$ $(= \mathbb{R}^{n+1}/\{x_1 = \cdots = x_{n+1}\})$ as a group generated by reflections. (The reflections are the transpositions.)

Suppose W is a finite group generated by orthogonal reflections on $\mathbb{R}^n$. Let

$$\begin{aligned} R &:= \{\text{reflections}\} \\ \text{open chamber} &:= \text{component of } \mathbb{R}^n - \bigcup_{r \in R} H_r \\ K &:= \overline{\text{chamber}} \\ S &:= \{\text{reflections across codim 1 faces of } K\} \end{aligned}$$

Main features:

- K is a simplicial cone.
- W acts simply transitively on $\{$chambers$\}$.
- $W = \langle S \rangle$ (i.e., S generates W.)
- $\mathbb{R}^n/W = K$.

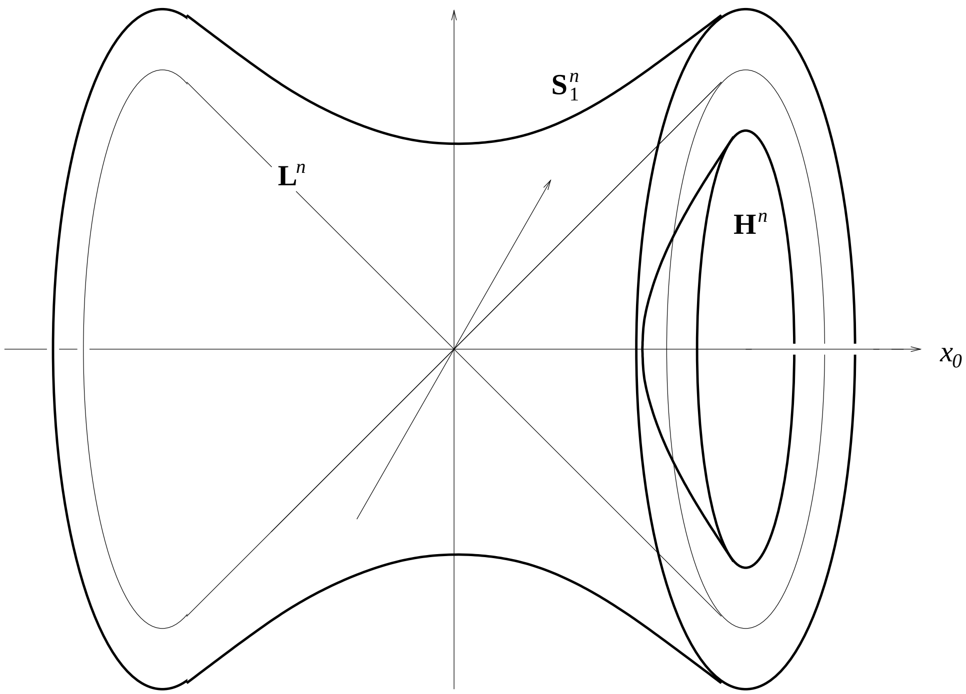
Finite reflection groups play a decisive role in

- the theories of Lie groups and algebraic groups,

- the classification of regular polytopes.

Reflections groups on $\mathbb{E}^n$. $W \subset \text{Isom}(\mathbb{E}^n)$ a discrete gp generated by reflections. We have a similar picture: a fundamental chamber K is convex and $W = \langle S \rangle$, where $S := \{\text{reflections across the codim 1 faces of } K\}$. $\mathbb{E}^n/W = K$. If, in addition, K is compact, then it is a polytope. If action does not split as a nontrivial product, then K is a simplex.

Reflections groups on $\mathbb{H}^n$. Similar remarks apply, except K need not be a product of simplices.



Abstract reflection groups. Is there a group theoretic characterization of relection groups?

First attempt: a group generated by involutions, more precisely, a pair (W, S) with $W = \langle S \rangle$, each element of S of order 2.

J. Tits proposed two different refinements of the above. The first hypothesis was that the Cayley graph of (W, S) had certain separation properties. (There are equivalent versions of this hypothesis which are purely combinatorial, e.g. the “Exchange Condition.”) The second hypothesis was that W admitted a presentation of a certain form. Incredibly these 2 hypotheses turn out to be equivalent. This is proved in the beginning of

N. Bourbaki, *Lie groups and Lie Algebras, Chapters 4–6*.

It will be the focus of Lecture 2. Some details of the hypotheses:

(1) Let $\Omega = \text{Cay}(W, S)$. Then $\forall s \in S$, the fixed set, Ω^s, separates Ω. (Recall that $\text{Cay}(W, S)$ is the graph with vertex set W and with v, w connected by an edge iff $v = ws$ for some $s \in S$.)

(2) For each pair $(s, t) \in S \times S$, let $m_{st} := \text{order}(st)$. (W, S) is a *Coxeter system* if it has a presentation of the form:

$$\langle S \mid (st)^{m_{st}} \rangle_{(s,t) \in S \times S}$$

(If $m_{st} = \infty$, the relation is omitted.)

The equivalence of these two definitions is not obvious. The meaning of (2) is that if we start with Cay(W, S) and fill in orbits of 2-cells corresponding to distinct pairs $\{s, t\}$ with $m_{st} \neq \infty$, then the resulting 2-dim cell cx is simply connected.

Definition. A *Coxeter matrix* on a set S is an $S \times S$ symmetric matrix (m_{st}), with entries in $\mathbb{N} \cup \{\infty\}$, with 1's on the diagonal and with off-diagonal entries ≥ 2.

Fact. Any Coxeter matrix is the matrix for a Coxeter system.

Representation of (W, S) by a geometric object.

Tits: $\exists$ a faithful representation $\rho : W \hookrightarrow GL(N, \mathbb{R})$ s.t.

- $\forall s \in S, \rho(s)$ is a (not necessarily orthogonal) linear reflection.

- W acts properly on the interior I of a convex cone in $\mathbb{R}^N$.

  (Basically, "proper" means finite stabilizers.)

- Hyperplanes corresponding to S bound a "chamber" $K \subset I$.

For many purposes this representation is completely satisfactory.

Corollary. W *is virtually torsion-free (i.e., it contains a torsion-free subgp of finite index).*

Proof. Selberg's Lemma asserts that any finitely generated subgp of $GL(N, \mathbb{R})$ has this property. □

Disadvantage: K is not compact.

The cell complex Σ. In Lecture 4, I will describe a cell cx Σ with a proper W-action.

Properties:

- $\exists$ a compact fundamental chamber K with $\Sigma/W \cong K$.
- $S = \{$"reflections across faces" of $K\}$.
- Σ is contractible.

- (The barycentric subdivision of) Σ is the "cocompact core" of the Tits' cone I.

- Suppose Σ^k denotes the k-skeleton ($:= \bigcup$ all cells of dim $\leq k$). Then $\Sigma^1 = \text{Cay}(W, S)$ and Σ^2 is the "Cayley 2-complex."

- More generally, $\exists$ an orbit of k-dimensional cells for each $T \subset S$ with $\text{Card}(T) = k$ and $\langle T \rangle$ finite.

Definition. For $T \subset S$, $W_T := \langle T \rangle$. It is called a *special subgroup* of W.

Fact. (W_T, T) is a Coxeter system.

$T \subset S$ is a *spherical* subset if W_T is finite. Let

$$\mathcal{S} := \{T \subset S \mid T \text{ is spherical}\}.$$

In other words, Σ has one orbit of cells for each element of $\mathcal{S}$.

Therefore, the poset of cells in Σ is the poset of spherical cosets:

$$W\mathcal{S} := \coprod_{T \in \mathcal{S}} W/W_T.$$

Note: Σ is not the same as the "Coxeter complex." When W comes from an (irreducible) Euclidean reflection gp, they are the same; moreover, both $= \mathbb{E}^n$, tessellated by simplices.

Metric properties of Σ. In the late 1940 s Aleksandrov and others defined what it means for a singular metric space (i.e.,

$\neq$ Riem mfld) to be nonpositively curved – or in today's terminology, to be "locally CAT(0)." Gromov (circa 1986) recalled this definition and gave many constructions of CAT(0) polyhedra, i.e., of spaces with piecewise Euclidean metrics which are (globally) CAT(0). Perhaps, the nicest such examples are the complexes Σ which we are discussing.

Theorem. (Gromov, Moussong). *For any Coxeter system, the natural PE metric on the associated cell cx* Σ *is* CAT(0)*.*

Remark. A corollary is that Σ is contractible (as is any CAT(0) space). I will give a different argument for this in Lecture 5.

Remark. I don't plan to say much more about CAT(0) spaces.

Coxeter groups as a source of examples. Any 2 vertices of Σ have isomorphic nbhds (because W acts transitively on Vert(Σ)). Such a nbhd is $\cong$ to Cone(L), where L is a certain finite simplicial cx called the "link" of the vertex. For example, if L is homeomorphic to S^{n-1}, then Σ is an n-mfld. $L = L(W,S)$ can be defined in terms of the spherical subsets of S. In fact, $\text{Vert}(L) := S$ and a subset T of S spans a simplex σ_T of L iff $T \in \mathcal{S}$.

The reason Coxeter groups provide such a potent source of examples is that the simplicial cx L is essentially arbitrary. (Its topology is arbitrary!)

Definition. Suppose X is a space which is homotopy equivalent to a CW complex. X is *aspherical* if its universal cover is contractible. This is equivalent to the condition that $\pi_i(X) = 0$, $\forall i > 1$, (since $\pi_i(X) = \pi_i(\widetilde{X})$ for $i > 1$).

If $\Gamma \subset W$ is a torsion-free subgp of finite index, then

- Γ acts freely on Σ (since stabilizers are finite subgps of W).

- Σ/Γ is an aspherical cx (the universal cover Σ is contractible).

- Σ/Γ is a finite cx (since Σ/W is compact).

(W, S) is *right-angled* if each $m_{st} = 2$ or ∞. When this is the case, Σ turns out to be a cubical cx.

A more concrete version of Σ in the right-angled case. Let L be an arbitrary finite simplicial cx. Put $S := \operatorname{Vert} L$ and

$\mathcal{S}(L) := \{T \subseteq S \mid T$ is the vertex set of a simplex$\}$. $\square^S := [-1,1]^S$.

Let $X_L \subseteq \square^S$ be the union of all faces which are parallel to $\square^T$ for some $T \in \mathcal{S}(L)$.

The group $(\mathbb{Z}/2)^S$ acts as a reflection group on $\square^S$. X_L is $(\mathbb{Z}/2)^S$-stable. $\operatorname{Vert} X_L = \operatorname{Vert} \square^S = \{\pm 1\}^S$. The link of each vertex of X_L is $\cong L$. Let $p : \widetilde{X}_L \to X_L$ be the universal cover. Let W be the group of all lifts of elements of $(\mathbb{Z}/2)^S$ to $\widetilde{X}_L$. Then W

acts as a reflection group on $\widetilde{X}_L$. Identify an element of S with the lift of the corresponding reflection in $(\mathbb{Z}/2)^S$ which fixes the appropriate face of a lifted chamber. One checks that (W, S) is a Coxeter system where $m_{st} \neq \infty \iff \{s, t\}$ span an edge of L. Moreover, $\Gamma := \pi_1(X_L)$ is a torsion-free subgp of W (it is the commutator subgp). If L satisfies the condition of being a "flag cx" (to be defined in Lecture 4) then $\widetilde{X}_L = \Sigma$. (This condition is always satisfied if L is a barycentric subdivision). So, if this is the case, $X_L = \Sigma/\Gamma$ is aspherical. So, we have a machine for a converting simplicial complex L (with possibly interesting topology) into a finite aspherical cx X_L and gp W acting nicely on its universal cover. In Lecture 6 I will use this construction to give examples of

- a torsion-free group having different cohomological dimension over $\mathbb{Z}$ than over $\mathbb{Q}$,

- a closed aspherical mfld with universal cover $\neq \mathbb{R}^n$,

Calculation of the (co)homology of Σ. This is the main theme of these lectures. We have 4 different types of cohomology in mind:

- ordinary: $H^*(\Sigma)$

- compact supports: $H^*_c(\Sigma)$

- L^2 (reduced): $L^2\mathcal{H}^*(\Sigma)$

- weighted L^2 : $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$, depending on a positive real multi-parameter $\mathbf{q}$.

The last item is fairly new work. Let's discuss each of the 4 types in turn.

Ordinary (co)homology. The result is that Σ is acyclic. In other words, $H_*(\Sigma)$ and $H^*(\Sigma)$ are concentrated in dim 0. The reason for proving this is that showing Σ is contractible is equivalent to showing that it is both acyclic and simply connected.

Cohomology with compact supports. There is the following explicit formula:

$$H_c^*(\Sigma) \cong \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes \mathbb{Z}(W^T),$$

where K is a fundamental chamber, K^{S-T} is the union of mirrors

(= "codim 1 faces") of K which are indexed by elements of $S-T$, where $W^T \subset W$, W is the disjoint union of the W^T and where $\mathbb{Z}(W^T)$ means the free abelian gp on W^T. One application of this formula is that we can characterize when W is a virtual Poincaré duality group.

L^2-**(co)homology.** Here we are concerned with ordinary real-valued cellular cochains but with *square summable coefficients*.

The resulting (reduced) cohomology groups are usually infinite dimensional Hilbert spaces. Using the group action, it is possible to associate a nonnegative real number to such a Hilbert space. This number is called an "L^2- Betti number." Atiyah's Formula is that the alternating sum of the L^2- Betti numbers is $= \chi^{orb}(\Sigma/W)$. Not much else is known about these L^2- Betti numbers; however, there is the following special case of a well-known conjecture of Dodziuk and Singer.

Conjecture. *If* Σ *is an* n*-mfld, then the* L^2*- Betti numbers,* $L^2b^k(\Sigma)$*, vanish* $\forall k \neq \frac{n}{2}$.

Because of Atiyah's Formula the Singer Conjecture implies another well-known conjecture.

Conjecture. (The Euler Characteristic Conjecture). *Suppose* M^{2n} *is a closed aspherical mfld. Then* $(-1)^n\chi(M^{2n}) \geq 0$.

Weighted L^2**-cohomology.** This is a further refinement. Let

$I :=$ {conjugacy classes in S} and let $\mathbf{t}$ be an I-tuple of indeterminates. One can define $W(\mathbf{t})$, a power series in $\mathbf{t}$, called the "growth series." (It generalizes $W(t) := \sum t^{l(w)}$.) It is a rational function. Given $\mathbf{q} \in (0, \infty)^I$, there is an algebra $\mathbb{R}_{\mathbf{q}}W$, called the "Hecke algebra." It is a deformation of the group algebra. It is possible to define an inner product on finitely supported cochains which differs from the standard one by a weight function involving the multiparameter $\mathbf{q}$ and word length. As before one has square summable cochains, the resulting cohomology groups, $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$, and their associated von Neumann dimensions (w.r.t. the Hecke algebra) the "weighted L^2- Betti numbers, $b^k_{\mathbf{q}}(\Sigma)$. The groups $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ interpolate between ordinary cohomology and cohomology with compact supports in a very precise sense. Let $\mathcal{R}$ be the region of convergence of $W(\mathbf{t})$. Then

- If $\mathbf{q} \in \mathcal{R}$, then $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ is concentrated in dim 0.

- If $\mathbf{q} \in \mathcal{R}^{-1}$, then there is a formula for $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ similar to the formula for $H^*_c(\Sigma)$.

Relationship with buildings. One of the principal reasons for studying weighted L^2-cohomology groups is that they compute the ordinary L^2-cohomology groups of Tits buildings. Buildings come in different flavors or "types," where the *type* is a Coxeter system (W, S). The correct definition of the geometric realization of a bldg should be such that each "apartment" is $\cong \Sigma$. The difference between a bldg and Σ is that in a bldg more than two chambers ($=$ copies of K) can meet along a given mirror.

It turns out that the weighted L^2- Betti numbers of Σ compute the L^2- Betti numbers of the bldg. More precisely, if the bldg admits a chamber transitive automorphism group G and it has thickness $\mathbf{q}$, then the L^2- Betti numbers of the bldg (w.r.t. G) are given by the $b^k_{\mathbf{q}}$. So, for bldgs only integral values of the multiparameter $\mathbf{q}$ matter!

Lecture 2:

The combinatorial theory of Coxeter groups

The Cayley graph and the word metric

Suppose G is a gp and $G = \langle S \rangle$ (S is a set of generators for G).

Suppose $1 \notin S$.

Definition. The *Cayley graph*, Cay(G, S), is the graph with vertex set G s.t. a two element subset of G is an *edge* iff it has the form $\{g, gs\}$ for some $g \in G$, $s \in S$. Label the edge $\{g, gs\}$ by s.

An *edge path* γ in the graph $\Omega := \text{Cay}(G, S)$ is a sequence of vertices $\gamma = (g_0, g_1, \ldots, g_k)$ s.t. two successive vertices are connected by an edge. Associated to γ, we get a word $\mathbf{s}$ in $S \cup S^{-1}$ defined by

$$\mathbf{s} = ((s_1)^{\varepsilon_1}, \ldots, (s_k)^{\varepsilon_k}),$$

where s_i is the label on the edge between g_{i-1} and g_i and where $\varepsilon_i \in \{\pm 1\}$ is defined to be $+1$ if the edge is directed from g_{i-1} to

g_i (i.e., if $g_i = g_{i-1}s_i$) and to be -1 if it is oppositely directed. Given such a word s define an element $g(\mathrm{s}) \in G$ by

$$g(\mathrm{s}) = (s_1)^{\varepsilon_1} \ldots (s_k)^{\varepsilon_k}$$

and call it the *value* of the word s. Clearly, $g_k = g_0 g(\mathrm{s})$. In other words, there is a one-to-one correspondence between edge paths from g_0 to g_k and words s such that $g_k = g_0 g(\mathrm{s})$. Since S generates G, Ω is connected.

Example. Suppose S is a set and $G = F_S$, the free group on S. The Cayley graph Cay(F_S, S) is a tree. (Each vertex has valence 2$|S|$.)

Word length. We define a metric on $\Omega =$ Cay(G, S). Each edge has length 1. The length of any path in Ω is then defined in the obvious manner. Define a "path metric" $d : \Omega \times \Omega \to [0, \infty)$, by letting $d(x, y)$ be the length of the shortest path from x to y.

Restricting the metric to the vertex set of Ω, we get the *word metric* $d : G \times G \to \mathbb{Z}$, i.e., $d(h, g)$ is the smallest integer k such that $g = hg(\mathbf{s})$ for some $\mathbf{s}$, a word of length k in $S \cup S^{-1}$. The distance from g to the identity element 1 is called its *word length* and is denoted $l(g)$.

Dihedral groups

Definition. A *dihedral group* is a group which is generated by two distinct elements of order 2. If $\{s, t\} = \{\text{generators}\}$, put $m_{st} :=$ order of st.

Exercise. Two dihedral groups are isomorphic iff they have the same m_{st} $(= m)$. Both are isomorphic to

$$D_m := \langle s, t | s^2, t^2, (st)^m \rangle.$$

Moreover, $|D_m| = 2m$.

Example. L a line in $\mathbb{R}^2$. $r_L =$ orthogonal reflection across L.

$\theta =$ angle between L and L'. $r_L \circ r_{L'} =$ rotation through 2θ. So, if $\theta = \pi/m$, $m \in \mathbb{N}$, then $r_L \circ r_{L'}$ is rotation through $2\pi/m$ and consequently, is of order m. So, the subgroup of $O(2)$ generated by r_L and $r_{L'}$ is $\cong D_m$.

Example. (*The infinite dihedral group D_∞*). This group is generated by two isometric affine transformations of the real line. Let r and r' denote the reflections about the points 0 and 1, respectively (that is, $r(t) = -t$ and $r'(t) = 2 - t$). Then $r' \circ r$ is translation by 2 (and hence, is of infinite order).

Coxeter systems

Definition. A pair (W, S), with W a gp and $S \subset W$, is a *pre-Coxeter system* if $W = \langle S \rangle$ and each element of S is an involution.

m_{st} := order of st.

Definition. (Tits). A pre-Coxeter system (W, S) is a *Coxeter system* if

$$\langle S \mid (st)^{m_{st}} \rangle_{(s,t) \in S \times S}$$

is a presentation for W.

Definition. A pre-Coxeter system (W, S) is a *reflection system* if the fixed point set of each $s \in S$ separates Cay(W, S).

Theorem. *Coxeter system* $\iff$ *reflection system.*

Definition. Suppose (W, S) is a pre-Coxeter system. An element $r \in W$ is a *reflection* if it is a conjugate of an element of S. Let

$$R := \{\text{reflections}\}.$$

Suppose $\mathbf{s} := (s_1, \ldots, s_k)$ is a word in S. Put $w(\mathbf{s}) := s_1 \cdots s_k$.

We say $\mathbf{s}$ is a *reduced expression* for $w(\mathbf{s})$ if $k = l(w(\mathbf{s}))$.

Conditions (D), (E) and (F)

(D) If $\mathrm{s} = (s_1, \ldots, s_k)$ is a word in S for $w := w(\mathrm{s})$ and $k > l(w)$, then there are indices $i < j$ so that the subword

$$\mathrm{s}' = (s_1, \ldots, \widehat{s_i}, \ldots, \widehat{s_j}, \ldots s_k)$$

is another expression for w.

(E) Given a reduced expression $\mathbf{s} = (s_1, \ldots, s_k)$ for an element $w \in W$ and an element $s \in S$, either $l(sw) = k + 1$ or else there is an index i such that

$$w = ss_1 \cdots \widehat{s_i} \cdots s_k.$$

(F) Suppose $w \in W$ and $s, t \in S$ are s.t. $l(sw) = l(w) + 1$ and $l(wt) = l(w) + 1$. Then either $l(swt) = l(w) + 2$ or $swt = w$.

Exercise. Conditions (D), (E) and (F) are equivalent.

Lemma. *Reflection system* $\iff$ *(D)*

Suppose $\mathbf{s} = (s_1, \ldots, s_k)$ is a word in S. For $0 \le i \le k$, set

$$w_i := s_1 \cdots s_i$$

$$r_i := w_{i-1} s_i w_{i-1}^{-1}$$

$$\Phi(\mathbf{s}) := (r_1, \ldots, r_k).$$

Then $r_i \ldots r_1 = w_i = s_1 \cdots s_i$ and $(w_0, \ldots, w_k)$ is an edge path.

Let $n(r, \mathbf{s}) :=$ the number of occurrences of r in $\Phi(\mathbf{s})$.

The implication: **Coxeter system** $\Longrightarrow$ **reflection system**.

If (W, S) is a reflection system, then the set of "half-spaces" can be identified with $R \times \{\pm 1\}$. (For each $r \in R$, its fixed set divides the Cayley graph into two components. The "positive" one corresponding to $+1$ is the component containing the identity element.) Given that (W, S) is a Coxeter system, the trick is to define an action on its putative set of half-spaces.

Lemma. *Suppose (W, S) is a Coxeter system.*

(i) *For any word* $\mathbf{s}$ *with* $w = w(\mathbf{s})$ *and any element* $r \in R$, *the number* $(-1)^{n(r,\mathbf{s})}$ *depends only on the endpoint* w. *We denote this number by* $\eta(r, w) \in \{\pm 1\}$.

(ii) *There is a homomorphism,* $w \to \phi_w$ *from* W *to the group of permutations of the set* $R \times \{\pm 1\}$, *where the permutation*

ϕ_w *is defined by the formula*

$$\phi_w(r,\varepsilon) = (wrw^{-1}, \eta(r,w^{-1})\varepsilon).$$

Sketch of Proof. Note (ii) $\Longrightarrow$ (i). So, consider (ii). For each $s \in S$, define $\phi_s \in \text{Perm}(R \times \{\pm 1\})$ by

$$\phi_s(r,\varepsilon) = (srs, \varepsilon(-1)^{\delta(s,r)} \ ,$$

where $\delta(s,r)$ is the Kronecker delta. If $\mathrm{s} = (s_1, \ldots, s_k)$, put

$$v := w(\mathrm{s})^{-1} = s_k \cdots s_1 \quad \text{and}$$

$$\phi_{\mathrm{s}} := \phi_{s_k} \circ \cdots \circ \phi_{s_1}.$$

Claims:

- $\phi_{\mathrm{s}}(r, \varepsilon) = (vrv^{-1}, \varepsilon(-1)^{n(\mathrm{s},r)}).$

- $\mathrm{s} \to \phi_{\mathrm{s}}$ descends to a homomorphism $W \to \mathrm{Perm}(R \times \{\pm 1\})$.

$\square$

Corollary. *Coxeter system $\implies$ reflection system.*

Idea of proof. The previous lemma $\implies$ (W, S) is a reflection system. Let $\Omega = \text{Cay}(W, S)$. Must show that for each $r \in R$, Ω^r separates Ω. Indeed, a vertex w belongs to the same component of $\Omega - \Omega^r$ iff $\eta(r, w) = +1$. $\square$

The word problem: Given a word $\mathbf{s}$ in S, is there an algorithm for determining if its value $w(\mathbf{s})$ is $= 1 \in W$?

We give Tits' solution to the word problem for reflection systems. Suppose (W, S) is a pre-Coxeter system and $M = (m_{st})$. Suppose further that (W, S) satisfies (E) (i.e., it is a reflection system).

Definition. An *elementary M-operation* on a word in S is one of the following two types of operations:

(I) Delete a subword of the form (s, s)

(II) Replace an alternating subword of the form $(s, t, \dots)$ of length m_{st} by the alternating word $(t, s, \dots)$, of length m_{st}.

A word is *M-reduced* if it cannot be shortened by a sequence of elementary M-operations.

Theorem. (Tits). *Suppose* (W,S) *satisfies (E).*

(i) *A word* $\mathbf{s}$ *is a reduced expression iff it is M-reduced.*

(ii) *Two reduced expresions* $\mathbf{s}$ *and* $\mathbf{t}$ *represent the same element of* W *iff one can be transformed into the other by a sequence of elementary M-operations of type (II).*

The proof is by induction on $l(w)$.

Corollary. *Reflection system $\Longrightarrow$ Coxeter system.*

Proof. Suppose (W, S) is a pre-Coxeter system, (m_{st}) the associated Coxeter matrix. $(\widetilde{W}, \widetilde{S})$ is the Coxeter system given by the presentation associated to (m_{st}). $p : \widetilde{W} \to W$ the natural surjection. Show that p is injective. Let $\tilde{w} \in \mathrm{Ker}(p)$ and let $\tilde{\mathrm{s}} = (\tilde{s}_1, \ldots, \tilde{s}_k)$ be reduced expression for $\tilde{w}$. Then $\tilde{\mathrm{s}}$ is

M-reduced. Let $\mathbf{s} = (s_1, \ldots, s_k)$ be the corresponding word in S. Since (W, S) and $(\widetilde{W}, \widetilde{S})$ have the same Coxeter matrices, the notion of M-operations coincide and so, $\mathbf{s}$ is also M-reduced. But since $\mathbf{s}$ represents the identity element in W, it must be the empty word. Consequently, $\tilde{\mathbf{s}}$ is also the empty word and so $\tilde{w} = 1$. $\square$

Special subgroups

A simple consequence of the solution to the word problem: for any $w \in W$, the set of letters which can occur in a reduced expression of w does not change as we vary the choice of reduced expression.

Proposition. *For each $w \in W$, there is a subset $S(w) \subset S$ so that for any reduced expression $(s_1, \ldots, s_k)$ for w, $S(w) = \{s_1, \ldots, s_k\}$.*

For each $T \subset S$, let

$$W_T := \langle T \rangle$$

be the subgroup generated by T. W_T is called a *special subgroup* of W.

Corollary. *W_T consists of those $w \in W$ such that $S(w) \subset T$.*

Corollary. *(W_T, T) is a Coxeter system.*

Lecture 3:

Geometric reflection groups

The spherical case. $W \subset O(n+1)$, a finite group generated by reflections. W acts on $\mathbb{R}^{n+1}$ and on $\mathbb{S}^n$. Assume the action is *essential*, i.e.,

$$(\mathbb{R}^{n+1})^W = \{0\}.$$

$\{H_r\} = \{\text{hyperplanes of reflection}\}$. Let P be the closure of a component of $\mathbb{S}^n - \bigcup H_r$. P is a spherical polytope. Let $\{P_i\} = \{\text{codim 1 faces of } P\}$, s_i the reflection across P_i and $S = \{s_i\}$.

Some facts:

- $P = \Delta$, a spherical n-simplex.

- (W, S) is a Coxeter system.

- Δ is a strict fundamental domain.

- If $m_{ij} = \text{order}(s_i s_j)$, then $\measuredangle(\Delta_i, \Delta_j) = \pi/m_{ij}$.

Corollary. *Suppose* $W \subset O(n+1)$ *is finite reflection group on* $\mathbb{R}^{n+1}$. *Then a fundamental domain (= "chamber") is a simplicial cone.*

The Euclidean case. $W \subset \text{Isom}(\mathbb{E}^n)$ a discrete subgroup generated by reflections. Assume the W-action on $\mathbb{E}^n$ is cocompact.

- If W is irreducible, then $P = \Delta$, a Euclidean n-simplex.
- In general, P is a product of simplices.
- As before, (W, S) is a Coxeter system and P is a fundamental domain.

In the spherical case, why is P a simplex?

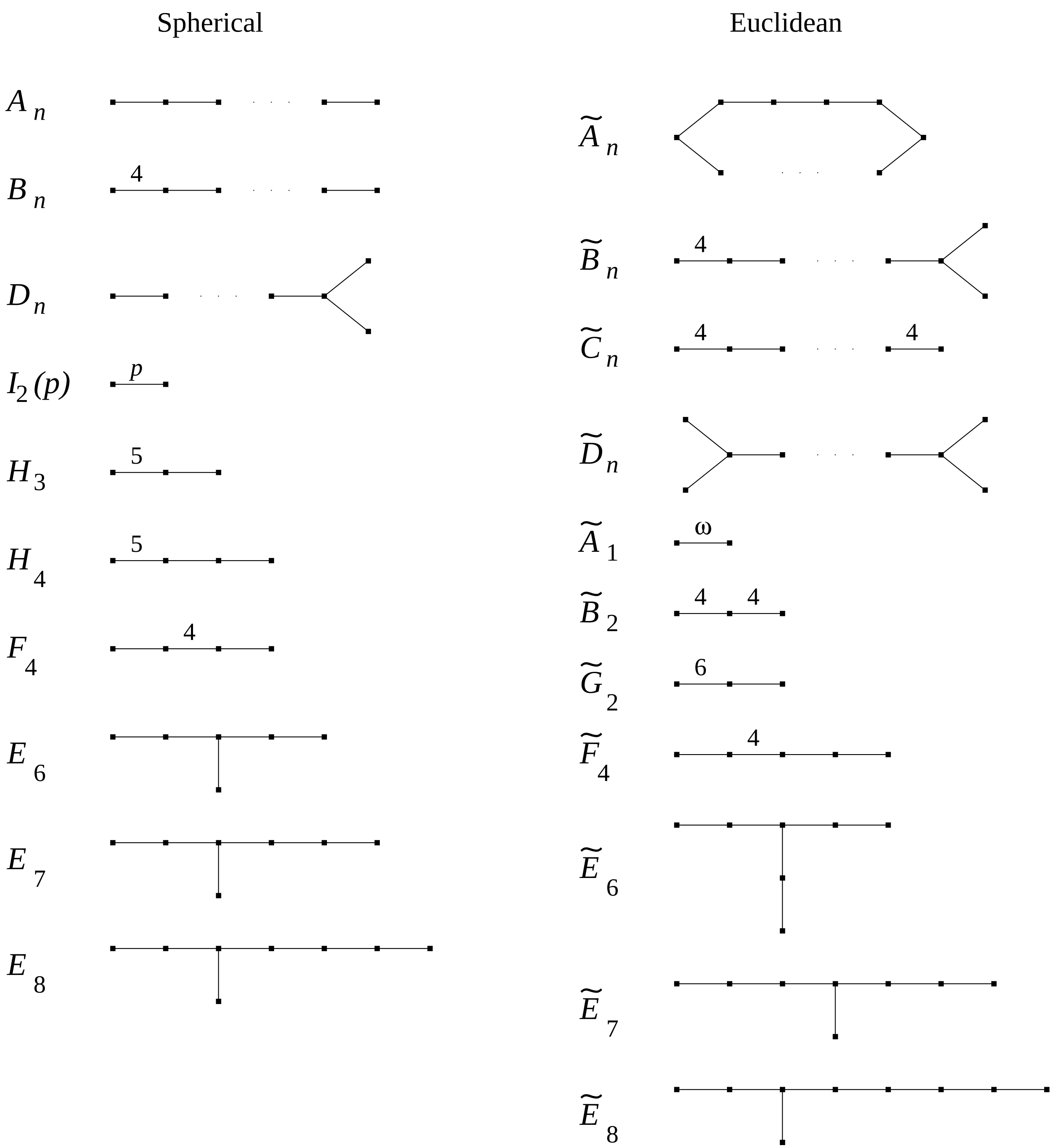
Lemma. *Suppose $P \subset \mathbb{S}^n$ is spherical polytope with all dihedral angles $\leq \pi/2$. Then P is a simplex.*

Proof. Linear algebra. □

Similarly in the Euclidean case.

Definition. Suppose that $M = (m_{ij})$ is a Coxeter matrix on a set I. We associate to M a graph Γ $(= \Gamma_M)$ called its *Coxeter graph*. The vertex set of Γ is I. Distinct vertices i and j are connected by an edge if and only if $m_{ij} \geq 3$. The edge $\{i, j\}$ is labeled by m_{ij} if $m_{ij} \geq 4$. (If $m_{ij} = 3$, the edge is left unlabeled.) The graph Γ together with the labeling of its edges is called the *Coxeter diagram* asssociated to M.

Theorem. (Coxeter). *The Coxeter diagrams of the irreducible spherical and Euclidean Coxeter groups are given in the tables.*


Spherical
Euclidean
A_n
B_n
4
D_n
$I_2(p)$
p
H_3
5
H_4
5
F_4
4
E_6
E_7
E_8
$\tilde{A}_n$
$\tilde{B}_n$
4
$\tilde{C}_n$
4
4
$\tilde{D}_n$
$\tilde{A}_1$
ω
$\tilde{B}_2$
4
4
$\tilde{G}_2$
6
$\tilde{F}_4$
4
$\tilde{E}_6$
$\tilde{E}_7$
$\tilde{E}_8$

The basic construction. A *mirror structure* on a space X consists of an index set S and a family of closed subspaces $(X_s)_{s \in S}$. The subspaces X_s are the *mirrors* of X. For each $x \in X$, let

$$S(x) = \{s \in S \mid x \in X_s\}.$$

For each nonempty subset $T \subset S$, let X_T (resp. X^T) denote the intersection (resp. union) of the mirrors indexed by T, that is,

$$X_T = \bigcap_{t \in T} X_t, \qquad \text{and}$$
$$X^T = \bigcup_{t \in T} X_t.$$

Also, for $T = \emptyset$, put $X_\emptyset = X$ and $X^\emptyset = \emptyset$. We shall sometimes call a subspace X_T a *coface* of X.

Suppose (W, S) is a pre-Coxeter system. Define an equivalence relation $\sim$ on $W \times X$ by $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w \in W_{S(x)}$. Let $\mathcal{U}(W, X)$ denote the quotient space:

$$\mathcal{U}(W, X) = (W \times X)/\sim .$$

Let $[w, x]$ denote the equivalence class of (w, x).

W acts on $\mathcal{U}$ via $w \cdot [w', x] := [ww', x]$.

Exercise. (i) The map $x \to [1, x]$ is an embedding of X as a closed subspace of $\mathcal{U}$.

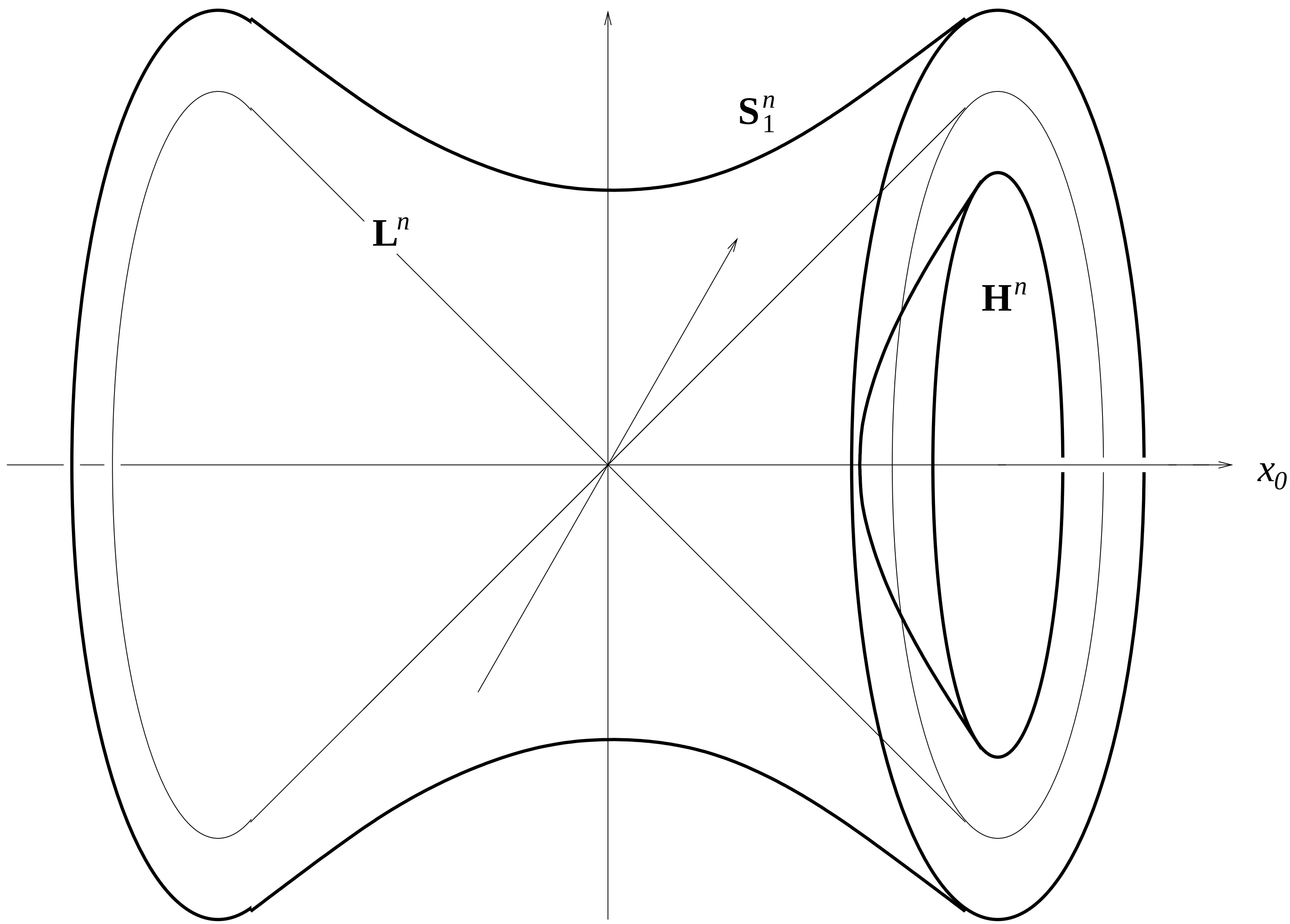
(ii) X is a strict fundamental domain, in the sense that it intersects each W-orbit in exactly one point.

(iii) The map $\mathcal{U} \to X$ defined by $[w, x] \to x$ induces a homeomorphism $\mathcal{U}/W \cong X$.

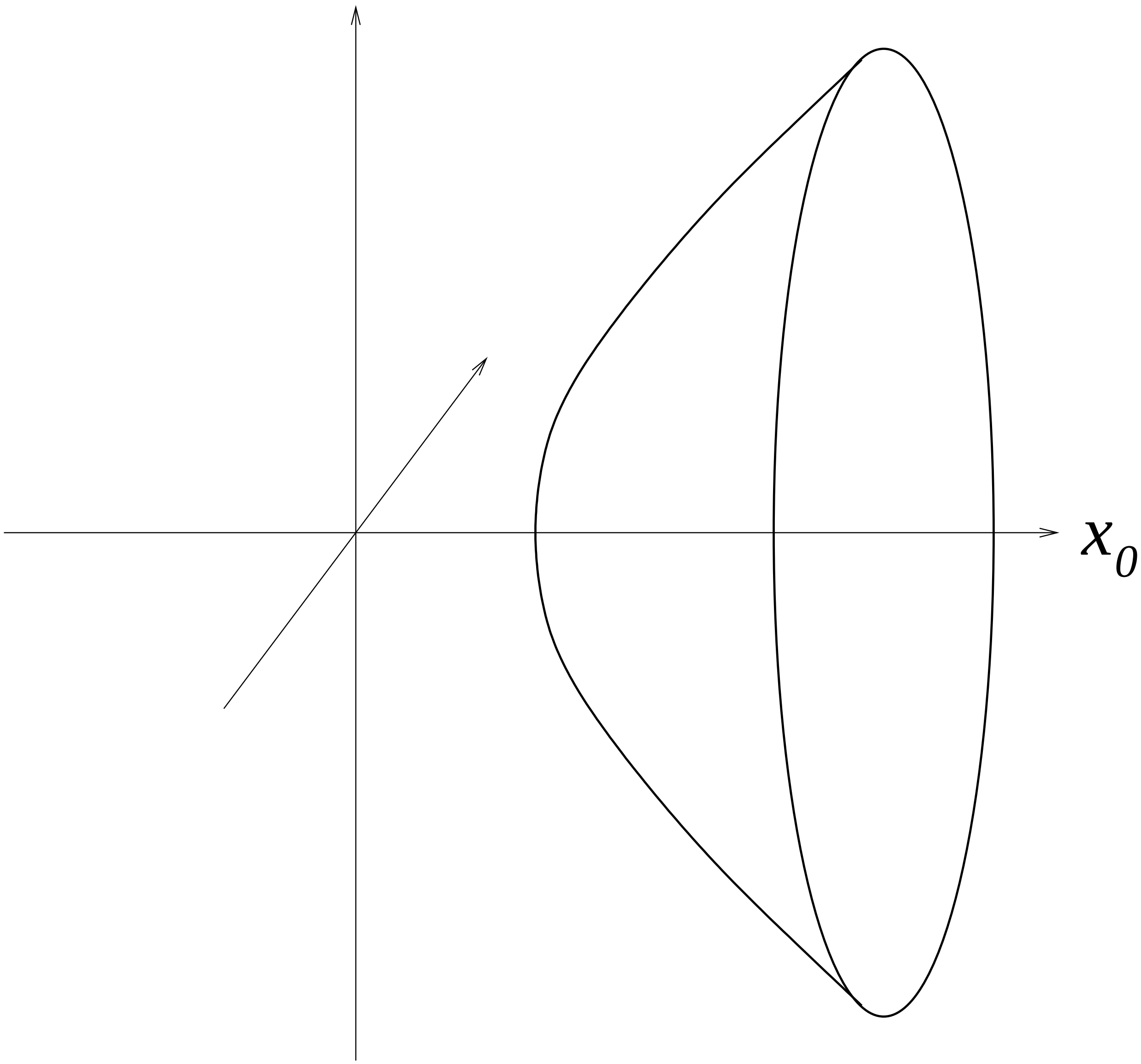
The image of $w \times X$ in $\mathcal{U}$ is denoted wX and called a *chamber* (for W on $\mathcal{U}$).

Spaces of constant curvature. $\mathbb{R}^{n,1}$ is $\mathbb{R}^{n+1}$ equipped with the indefinite quadratic form $-x_0^2 + x_1^2 + \cdots + x_n^2$. *Hyperbolic n-space* $\mathbb{H}^n$ is one sheet of the "sphere of radius i,"

$$-x_0^2 + x_1^2 + \cdots + x_n^2 = -1.$$



$\mathbf{S}_1^n$
$\mathbf{L}^n$
$\mathbf{H}^n$
x_0



x_0

$\mathbb{X}^n$ stands for $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$. $P^n \subset \mathbb{X}^n$ is a convex polytope.

$\{F_i\}_{i\in I} = \{$codim 1 faces$\}$.

For $i \in I$, let $\overline{s}_i$ be the isometric reflection of $\mathbb{X}^n$ across the hyperplane supported by F_i. $\overline{W} := \langle\{\overline{s}_i\}_{i\in I}\rangle \subset \text{Isom}(\mathbb{X}^n)$. When is $\overline{W}$ a discrete subgroup Isom($\mathbb{X}^n$)? When is P^n a fundamental domain for the $\overline{W}$-action on $\mathbb{X}^n$? A necessary condition: if the hyperplanes supported by F_i and F_j intersect, then the dihedral subgroup $\overline{W}_{ij}$ generated by s_i and s_j must be finite and the sector bounded by these hyperplanes and containing P^n must be a fundamental domain for the $\overline{W}_{ij}$-action. So, the dihedral angle between F_i and F_j must be an integral submultiple of π. This forces all the dihedral angles of P^n to be $\leq \pi/2$.

$P^n \subset \mathbb{X}^n$ is *simple* if exactly n codim 1 faces meet at each vertex (i.e., ∂P is dual to a simplicial complex).

Lemma. *If all dihedral angles of P^n are $\leq \pi/2$, then P^n is simple.*

Proof. Follows from spherical case. □

So, suppose P^n is simple and that whenever $F_i \cap F_j \neq \emptyset$, the dihedral angle along F_{ij} is of the form π/m_{ij}, for some integer $m_{ij} \geq 2$. If $F_i \cap F_j = \emptyset$, set $m_{ij} = \infty$.

(m_{ij}) is a Coxeter matrix. Let (W, S) be the corresponding Coxeter system, with generating set $S = \{s_i\}_{i \in I}$. Since order$(\overline{s}_i \overline{s}_j) = m_{ij}$, $s_i \to \overline{s}_i$ extends to a homomorphism $\phi : W \to \overline{W}$. There is a *tautological mirror structure* on P^n: the mirror corresponding to i is F_i. The inclusion $\iota : P^n \to \mathbb{X}^n$ induces a ϕ-equivariant map $\tilde{\iota} : \mathcal{U}(W, P^n) \to \mathbb{X}^n$, given by $[w, x] \to \phi(w)x$.

Theorem. *$\tilde{\iota}$ is a homeomorphism.*

Sketch of proof. Induction on n. Spherical case in dimension $n - 1 \implies \tilde{\iota}$ is a local homeomorphism. Complete metric on $\mathcal{U}(W, P^n) \implies \tilde{\iota}$ is a covering map. Since $\mathbb{X}^n$ ($\neq \mathbb{S}^1$) is simply connected, we are done. $\square$

Corollary. *$\phi : W \to \overline{W}$ is an isomorphism (hence, $\overline{W}$ is a Coxeter group).*

Corollary. *P^n is a strict fundamental domain for $\overline{W}$ on $\mathbb{X}^n$.*

Polygon groups. The exterior angles in a Euclidean polygon sum to 2π. So, if P^2 is an m-gon in $\mathbb{E}^2$ with interior angles

$\alpha_1, \ldots, \alpha_m$, then $\sum(\pi - \alpha_i) = 2\pi$ or equivalently;

$$\sum_{i=1}^{m} \alpha_i = (m-2)\pi \ .$$

Suppose

$$\mathbb{X}^2_\varepsilon = \begin{cases} \mathbb{S}^2, & \text{if } \varepsilon = 1; \\ \mathbb{E}^2, & \text{if } \varepsilon = 0; \\ \mathbb{H}^2, & \text{if } \varepsilon = 1. \end{cases}$$

If $P^2 \subset \mathbb{X}^2_\varepsilon$ is an m-gon, then the Gauss–Bonnet Theorem asserts

$$\varepsilon \, \text{Area}(P^2) + \sum(\pi - \alpha_i) = 2\pi.$$

Hence, $\sum \alpha_i$ is $>$, $=$ or $<$ $(m-2)\pi$, as $\mathbb{X}^2_\varepsilon$ is, respectively, $\mathbb{S}^2$, $\mathbb{E}^2$

or $\mathbb{H}^2$.

Consider convex m-gon $P^2 \subset \mathbb{X}^n_\varepsilon$ with vertices $v_1, \ldots, v_m$ where the angle α_i at v_i is of the form $\alpha_i = \pi/m_i$, for an integer $m_i \geq 2$.

Example. (*Spherical triangle groups*). Suppose $\varepsilon = 1$. Since $\alpha_i \leq \pi/2$, the condition $\sum \alpha_i > (m-2)\pi$ forces $m < 4$, i.e., P^2 is a triangle. What are the possibilities for α_i? The inequality

$\pi/m_1 + \pi/m_2 + \pi/m_3 > \pi$ can be rewritten as

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} > 1.$$

Exercise. Supposing $m_1 \leq m_2 \leq m_3$, show that the only triples (m_1, m_2, m_3) of integers ≥ 2 satisfying the above are: $(2,3,3)$, $(2,3,4)$, $(2,3,5)$ and $(2,2,n)$ for any integer $n \geq 2$.

Each such triple corresponds to a spherical reflection group. The first 3 triples are the symmetry groups of the Platonic solids:

$(2,3,3)$ gives the symmetry group of the tetrahedron, $(2,3,4)$ the symmetry group of the cube (or octahedron) and $(2,3,5)$ the symmetry group of the dodecahedron (or icosahedron).

Example. (2-*dimensional Euclidean groups*). $\sum \alpha_i = (m-2)\pi$ forces $m \leq 4$; moreover, if $m = 4$, there is only one possibility for the m_i, namely, $m_1 = m_2 = m_3 = m_4 = 2$. Thus, in this case, P^2 is a rectangle and we get a standard rectangular tiling of $\mathbb{E}^2$. The corresponding Coxeter group is $\mathbf{D}_\infty \times \mathbf{D}_\infty$. If $m = 3$, the relevant equation is

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 1.$$

As before, it is easy to see that there are only three such triples:

$(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$.

Example. (*Hyperbolic polygon groups*). Given any assignment of angles of the form π/m_i to the vertices of a combinatorial

m-gon such that

$$\sum_{i=1}^{m} \frac{1}{m_i} < m - 2 \ ,$$

we can find a convex realization of it in $\mathbb{H}^2$. This yields a corresponding reflection group on $\mathbb{H}^2$. If $m > 3$, there is a continuous family (moduli) of hyperbolic polygons with the same angles and hence, a moduli space of representations of the Coxeter group.

The conclusion to be drawn from these examples is that any assignment of angles of the form π/m_i to the vertices of an m-gon can be realized by a convex polygon in a 2-dimensional space $\mathbb{X}^2$ of constant curvature; moreover, apart from a few exceptional cases, the space is $\mathbb{H}^2$.

Simplicial Coxeter groups. Suppose the fundamental polytope is a simplex $\Delta^n \subset \mathbb{X}^n$ and that dihedral angle between Δ_i and Δ_j is π/m_{ij}. The *cosine matrix* (associated to the Coxeter matrix (m_{ij}) is the matrix (c_{ij}) defined by

$$c_{ij} := -\cos(\pi/m_{ij})$$

(where $m_{ii} := 1$ so that $c_{ii} = -\cos(\pi) = 1$). Let u_i be the outward pointing unit normal vector to Δ_i. Then

$$u_i \cdot u_j = c_{ij}.$$

The matrix $(u_i \cdot u_j)$ is the *Gram matrix* of Δ.

Proposition. *Suppose* $\Delta^n \subset \mathbb{X}^n_\varepsilon$.

- *If* $\mathbb{X}^n = \mathbb{S}^n$, *then* (c_{ij}) *is positive definite.*

- *If* $\mathbb{X}^n = \mathbb{E}^n$, *then* (c_{ij}) *is positive semidefinite of rank* n.

- *If* $\mathbb{X}^n = \mathbb{H}^n$, *then* (c_{ij}) *is indefinite of type* $(n, 1)$.

In the 3 respective cases, let us say (c_{ij}) is type $\varepsilon = 1, 0$ or -1.

In 1950 Lannér classified those Coxeter groups W which can act as proper reflection groups on a simply connected space with fundamental chamber a simplex. In other words, he determined which W act properly on $\mathcal{U}(W, \Delta^n)$. We call such a (W, S), $S = \{s_i\}_{i \in I}$, a *simplicial Coxeter group*. It turns out that any such W is a geometric reflection group generated by reflections across the faces of an n-simplex in either $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$.

Suppose the Coxeter group W is defined by an $I \times I$ Coxeter matrix (m_{ij}), $I := \{0, \ldots, n\}$. Let (c_{ij}) be the associated cosine matrix. W acts properly on $\mathcal{U}(W, \Delta^n)$ if and only if the isotropy subgroup at each vertex is finite and this is the case if and only if each principal submatrix of (c_{ij}) is positive definite. So, the problem is reduced to finding all Coxeter diagrams with the property that every proper subdiagram is positive definite. We note that if a diagram has this property and if it not connected, then each of its components is positive definite. It follows that there are only three possibilities depending on the determinant of the cosine matrix $C = (c_{ij})$:

- If $\det C > 0$, then C is positive definite.

- If $\det C = 0$, then the diagram is connected and C is positive semidefinite of corank 1.

- If $\det C < 0$, then the diagram is connected and C is type $(n, 1)$

Proposition. (Lannér). *Any simplicial Coxeter group can be represented as a geometric reflection group with fundamental chamber an n-simplex in either $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$.*

Proposition. (Lannér). *Hyperbolic simplicial Coxeter groups occur only in dimensions ≤ 4. On $\mathbb{H}^2$ we have the hyperbolic tri-*

angle groups. There are 9 *tetrahedral groups on* $\mathbb{H}^3$ *and* 5 *more on* $\mathbb{H}^4$*.*

Remark. In the exercises of Bourbaki, a hyperbolic simplicial Coxeter group is called a Coxeter group of "compact hyperbolic type." If its fundamental simplex is not required to be compact but is still required to have finite volume (possibly with ideal vertices), then it is said to be of "hyperbolic type." Similar terminology is still widely used. It is very misleading. It gives the false impression that the fundamental chamber of a hyperbolic reflection group is always a simplex, completely ignoring the polygonal reflection groups well as the 3-dimensional examples discussed below.

Dimension 3. Let us review. A geometric reflection group on $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ is determined by its fundamental chamber. This chamber is a convex polytope with dihedral angles integral submultiples of π and any such polytope gives a reflection group. In the spherical case the fundamental polytope must be a simplex and in the Euclidean case it must be a a product of simplices. In the hyperbolic case all we know so far is that the polytope must be simple. In the converse direction if the fundamental chamber is a simplex (i.e., if the reflection group is simplicial), then we have a complete classification in all three cases. So, there is nothing more to said in the spherical and Euclidean cases. In the hyperbolic case we know what happens in dimension 2: the fundamental polygon can be an m-gon for any $m \geq 3$ and almost any assignment of angles can be realized by a hyperbolic polygon (there are a few restrictions when $m = 3$ or 4).

What happens in dimension $n \geq 3$?

Andreev's Theorem. There is a beautiful theorem, due to Andreev, which gives a complete answer in dimension 3. Roughly, it says that for a simple polytope P^3 to be the fundamental polytope of a hyperbolic reflection group, (a) there is no restriction on its combinatorial type and (b) subject to the condition that the isotropy group at each vertex be finite, almost any assignment of dihedral angles to the edges of P^3 can occur (provided a few simple inequalities hold). Moreover, in contrast to the picture in dimension 2, the 3-dimensional hyperbolic polytope is uniquely determined, up to isometry, by its dihedral angles – the moduli space is a point.

This situation reflects the nature of the relationship between geometry and topology in dimensions 2 and 3. A closed 2-manifold admits a hyperbolic structure if and only if its Euler characteristic is < 0 and there is a moduli space of such hyperbolic structures. In dimension 3 there is Thurston's Geometrization Conjecture (now Perelman's Theorem?). Roughly, it says that a closed 3-manifold M^3 admits a hyperbolic structure if and only if it satisfies certain simple topological conditions. Moreover, in contrast to the situation in dimension 2, the hyperbolic structure on M^3 is uniquely determined, up to isometry, by $\pi_1(M^3)$. (This is a consequence of the Mostow Rigidity Theorem, which is not true in dimension 2.)

Andreev's inequalities on the dihedral angles of P^3 precisely correspond to the topological restrictions on M^3 in Thurston's Conjecture. Thus, Andreev's Theorem is a special case of (an orbifoldal version of) Thurston's Conjecture.

Conjecture. (Thurston's Geometrization Conjecture). *A closed*

3-manifold M^3 admits a hyperbolic structure if and only if it satisfies the following two conditions:

(a) *Every embedded 2-sphere bounds a 3-ball in M^3.*

(b) *There is no incompressible torus in M^3 (i.e., M^3 is* atoroidal*).*

(An *incompressible torus* is an embedded $\mathbb{T}^2$ in M^3 such that the inclusion induces an injection $\pi_1(\mathbb{T}^2) \to \pi_1(M^3)$.)

Theorem. (Andreev). *Suppose that* P^3 *is the combinatorial type of a simple polytope, different from a tetrahedron. Let* E *be its edge set and* $\theta : E \to (0, \pi/2]$ *any function. Then* (P^3, θ) *can be realized as a convex polytope in* $\mathbb{H}^3$ *with dihedral angles as prescribed by* θ *if and only if the following conditions hold:*

(i) *At each vertex, the angles at the three edges* e_1, e_2, e_3 *which meet there satisfy* $\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi$.

(ii) *If three faces intersect pairwise but do not have a common vertex, then the angles at the three edges of intersection satisfy* $\theta(e_1) + \theta(e_2) + \theta(e_3) < \pi$.

(iii) *Four faces cannot intersect cyclically with all four angles* $= \pi/2$ *unless two of the opposite faces also intersect.*

(iv) *If* P^3 *is a triangular prism, then the angles along the base* *and top cannot all be* $\pi/2$.

Moreover, when the (P^3, θ) *is realizable, it is unique up to an isometry of* $\mathbb{H}^3$.

Corollary. *Suppose* P^3 *is the combinatorial type of a simple polytope, different from a tetrahedron and* $\{F_i\}_{i \in I}$ *its set of codimension one faces. Let* e_{ij} *be the edge* $F_i \cap F_j$ *(when* $F_i \cap F_j \neq \emptyset$*) and let* $E := \{e_{ij}\}$. *Given an angle assignment* $\theta : E \to (0, \pi/2]$, *with* $\theta(e_{ij}) = \pi/m_{ij}$ *where* m_{ij} *is an integer* ≥ 2*, then* (P^3, θ) *is the fundamental polytope of a hyperbolic reflection group* $W \subset \text{Isom}(\mathbb{H}^3)$ *if and only if the* $\theta(e_{ij})$ *satisfy Andreev's conditions. Moreover,* W *is unique up to conjugation in* $\text{Isom}(\mathbb{H}^3)$*.*

Lecture 4:

The cell complex Σ

We will define a cell complex Σ with the following:

Properties:

- W acts on Σ as a reflection group with finite isotropy subgroups and compact fundamental chamber.

- $\text{Vert}(\Sigma) = W$, the 1-skeleton, Σ^1, is $\text{Cay}(W, S)$ and Σ^2 is the Cayley 2-complex of the presentation.

- Σ is contractible.

- If (W, S) is spherical (i.e., if W is finite), then Σ is a convex polytope and $\partial\Sigma$ is the dual cell complex to the canonical triangulation of $\mathbb{S}^n$ into chambers.

- If (W, S) can be realized as a geometric reflection group on $\mathbb{E}^n$ or $\mathbb{H}^n$, then Σ is the dual cell complex to the subdivision into chambers.

- Σ has a canonical piecewise Euclidean (= "PE") cell structure which is $CAT(0)$ (= nonpositively curved).

Coxeter polytopes. For the next several slides, suppose W is finite. Consider the canonical representation of W on $\mathbb{R}^S$ ($\mathbb{R}^S \cong \mathbb{R}^n$, where $n = \text{Card}(S)$). Choose a point x in the interior of a fundamental chamber.

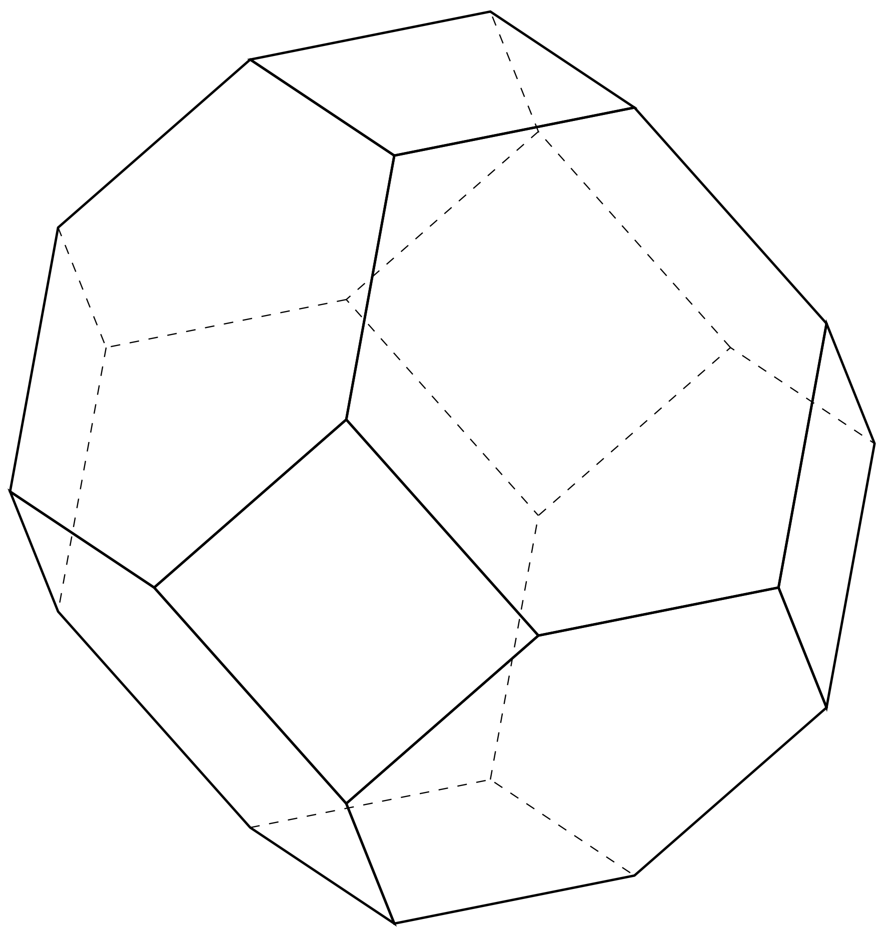
Definition. The *Coxeter polytope* (or the *Coxeter cell*) associated to (W, S) is the convex hull of Wx (a generic W-orbit).

Examples.
- If $W = \mathbf{C}_2$, the cyclic group of order 2, then C is the interval $[-x, x]$.

- If $W = \mathbf{D}_m$ is the dihedral group of order $2m$, then C is a $2m$-gon.

- If (W, S) is reducible and decomposes as $W = W_1 \times W_2$, then C_W decomposes as the product $C_W = C_{W_1} \times C_{W_2}$. In particular, if $W = (\mathbf{C}_2)^n$, then C is a product of intervals (C is a combinatorial n-cube).

- If W is the symmetric group on $n + 1$ letters (the group associated to the Coxeter diagram $\mathbf{A}_n$), then C is called the n-dimensional *permutahedron*.



The nerve of a Coxeter system (W, S).

Definition. A subset T of S is *spherical* if W_T is a finite subgroup of W.

Denote by $\mathcal{S}(W, S)$ (or simply by $\mathcal{S}$) the set of all spherical subsets of S. It is a poset: the partial order is inclusion. Let $\mathcal{S}^{(k)}$ denote the set of spherical subsets of cardinality k.

The poset $\mathcal{S}_{>\emptyset}$ of all nonempty spherical subsets is an abstract simplicial complex. (This just means that if $T \in \mathcal{S}_{>\emptyset}$ and if T' is a nonempty subset of T, then $T' \in \mathcal{S}_{>\emptyset}$.) We will also denote this simplicial complex by $L(W, S)$ (or simply by L) and call it the *nerve* of (W, S). Thus, $\operatorname{Vert} L = S$ and a nonempty subset T of S spans a simplex if and only if T is spherical. $\mathcal{S}^{(k)}$ is the set of $(k-1)$-simplices in L

Example. If W is finite, then $\mathcal{S}(W, S)$ is the power set of S and $L(W, S)$ is the full simplex on S.

Example. If W is the infinite dihedral group $\mathbf{D}_\infty$, then $L(W,S)$ is the space consisting of two points.

Example. Suppose that W is a geometric reflection group generated by the reflections across the codimension one faces of a simple convex polytope $P^n \subset \mathbb{X}^n$. Then S can be identified with the set of codimension one faces of P^n. If P^n is a spherical simplex, then $L(W,S)$ is the full simplex on S. So, suppose $\mathbb{X}^n = \mathbb{E}^n$ or $\mathbb{H}^n$. If F is a face of P^n and $T(F) \subset S$ stands for the set of codimension one faces which contain F, then $T(F)$ is spherical. Conversely, if T is a spherical subset, then the finite group W_T must have a fixed point on $\mathbb{X}^n$. (Since $\mathbb{X}^n$ is nonpositively curved, the convex hull of any W_T-orbit has a unique "center," which is fixed by W_T.) Since P^n is a strict fundamental domain for W, W_T has a fixed point on P^n and this point lies in the intersection F of the codimension one faces which belong to T, i.e., $T = T(F)$. So, we have just shown that there is a order reversing isomorphism between $L(W,S)$ and the poset of proper faces of P^n. In other words, $L(W,S)$ is the boundary complex of the dual polytope of P^n. The fact that $L(W,S)$ is a simplicial complex corresponds to the fact that P^n is a simple polytope.

Example. Suppose the Coxeter system (W,S) decomposes as

$$(W,S) = (W_1 \times W_2, S_1 \cup S_2)$$

where the elements of S_1 commute with those of S_2. A subset

$T = T_1 \cup T_2$, with $T_i \subset S_i$, is spherical if and only if T_1 and T_2 are both spherical. Hence,

$$\mathcal{S}(W,S) \cong \mathcal{S}(W_1,S_1) \times \mathcal{S}(W_2,S_2) \ .$$

Similarly, the simplicial complex $L(W,S)$ decomposes as the join

$$L(W,S) = L(W_1,S_1) * L(W_2,S_2) \ .$$

The poset of spherical cosets. A *spherical coset* is a coset of a spherical special subgroup in W. The set of all spherical cosets is denoted $W\mathcal{S}$, i.e.,

$$W\mathcal{S} := \bigcup_{T \in \mathcal{S}} W/W_T.$$

It is partially ordered by inclusion.

Exercise. $wW_T = w'W_{T'}$ iff $T = T'$ and $w^{-1}w' \in W_{T'}$.

It follows that the union in is a disjoint union. It also follows that there is well-defined projection map $W\mathcal{S} \to \mathcal{S}$ given by $wW_T \to T$ and a natural section of this projection $\mathcal{S} \hookrightarrow W\mathcal{S}$ defined by

$T \to W_T$. Moreover, W acts naturally on the poset $W\mathcal{S}$ and the quotient poset is $\mathcal{S}$.

The geometric realization of a poset. $\mathcal{P}$ a poset. A *flag* (or "chain") in $\mathcal{P}$ is a finite, nonempty, totally ordered subset.

Flag($\mathcal{P}$) := {chains in $\mathcal{P}$.

Flag($\mathcal{P}$) is an abstract simplicial complex. The corresponding cell complex (= topological space) is denoted $|\mathcal{P}|$ and called the *geometric realization* of $\mathcal{P}$.

Exercise. Suppose W is finite and that C is its associated Coxeter polytope. Let $\mathcal{F}(C)$ denote its set of faces (partially ordered by inclusion). Then the correspondence $w \to wx$ induces an isomorphism of posets, $W\mathcal{S} \cong \mathcal{F}(C)$. (In other words, a subset of W corresponds to the vertex set of a face of C if and only if it is a coset of a special subgroup of W.) It follows that the simplicial complex $|W\mathcal{S}|$ is the barycentric subdivision of C.

Remark. It follows that the combinatorial type of C doesn't depend on the choice of the generic point x.

Three definitions of Σ.

1) $\Sigma := |W\mathcal{S}|$.

2) $(W\mathcal{S})_{\leq wW_T} \cong W_T\mathcal{S}(W_T, T)$. By the Exercise, the subcomplex $|(W\mathcal{S})_{\leq wW_T}|$ of $|W\mathcal{S}|$ can be identified with a Coxeter polytope of type W_T. This defines a cell structure on Σ where each cell is a Coxeter polytope.

3) $\Sigma := \mathcal{U}(W, K)$, where $K := |\mathcal{S}|$ and where the mirror structure on K is defined by $K_s := |\mathcal{S}_{\geq\{s\}}|$.

Remarks. • If (W, S) can be realized as a group generated by reflections across the codim 1 faces of a polytope P^n in $\mathbb{E}^n$ or $\mathbb{H}^n$, then K is combinatorially isomorphic to P^n (i.e., $\{K_s\} \cong \{\text{codim 1 faces}\}$). Also, the simplicial complex $L(W, S)$ is the the dual of ∂P.

- In the Coxeter cell (or "cc") structure on Σ, the link of each vertex is $L(W, S)$. (A consequence is that if $L(W, S)$ is a triangulation of S^{n-1}, then Σ is an n-manifold.)

- The cc structure defines a PE metric on Σ (and this metric is CAT(0)).

Flag complexes

Definition. A simplicial complex L is a *flag complex* if for any finite, nonempty $T \subseteq \operatorname{Vert} L$ the following holds: T is the vertex set of a simplex in $L \iff$ any two vertices of T are connected by an edge.

Example. The boundary complex of an m-gon is a flag complex iff $m > 3$.

Exercise. If $\mathcal{P}$ is any poset, then Flag($\mathcal{P}$) is a flag complex.

In particular, the barycentric subdivision of any (regular) cell complex is a flag complex.

Corollary. *The condition of being a flag complex does not restrict the topological type of L: it can be any polyhedron.*

Definition. A Coxeter system (W, S) is *right-angled* if $m_{st} = 2$ or ∞ for any two distinct elements $s, t \in S$.

Exercise. If (W, S) is right-angled, then $L(W, S)$ is a flag complex.

Conversely suppose L is a flag complex and that $S :=$ Vert L.

Define

$$m_{st} := \begin{cases} 1, & \text{if } s = t; \\ 2, & \text{if } \{s,t\} \in \operatorname{Edge} L; \\ \infty, & \text{if } \{s,t\} \notin \operatorname{Edge} L. \end{cases}$$

The corresponding presentation gives a right-angled Coxeter system (W, S) with $L(W, S) = L$.

My favorite construction. Suppose L is a flag complex and $S := \operatorname{Vert} L$. Put

$$\square^S := [-1, 1]^S.$$

$\mathcal{S}(L) := \{T \subseteq S \mid T$ is the vertex set of a simplex$\}$.

Let $X_L \subseteq \square^S$ be the union of all faces which are parallel to $\square^T$ for some $T \in \mathcal{S}(L)$.

Examples. If $L = S^0$, then $X_L = \partial\square^2 = \partial(\text{4-gon}) \cong S^1$. If $L = \partial(\text{4-gon})$, then $X_L = T^2 \subset \partial\square^4$.

The group $(\mathbb{Z}/2)^S$ acts as a reflection group on $\square^S$. $[0,1]^S$ is a fundamental chamber. The subcomplex X_L is $(\mathbb{Z}/2)^S$-stable and $K := X_L \cap [0,1]^S$ is a fundamental chamber for X_L. Then $\operatorname{Vert} X_L = \operatorname{Vert} \square^S = \{\pm 1\}^S$ and the link of each vertex of X_L is $\cong L$. Let $p : \widetilde{X}_L \to X_L$ be the universal cover. Let W be the group of all lifts of elements of $(\mathbb{Z}/2)^S$ to $\widetilde{X}_L$. Then W acts as a reflection group on $\widetilde{X}_L$ with fundamental chamber a component of $p^{-1}(K)$ ($\cong K$). Identify an element of S with the lift of the corresponding reflection in $(\mathbb{Z}/2)^S$ fixing the appropriate face of

K. One checks that this K is the same as before and that

$$\widetilde{X_L} \cong \mathcal{U}(W, K) = \Sigma.$$

Lecture 5:

Algebraic toplogy of Σ and $\mathcal{U}$

Let X be a space and $\{X_s\}_{s \in S}$ a family of closed subspaces.

Recall that for any $T \subset S$,

$$X_T := \bigcap_{s \in T} X_s \quad \text{and} \quad X^T := \bigcup_{s \in T} X_s.$$

(W, S) a Coxeter system and

$$\mathcal{U} = \mathcal{U}(W, X) := (W \times X)/\sim .$$

Also, assume $X_T = \emptyset$ whenever $T \notin \mathcal{S}$. Our goal:

Theorem. *The following are equivalent:*

- $\mathcal{U}$ *is contractible.*
- X *is contractible and* X_T *is acyclic for all* $T \in \mathcal{S}_{>\emptyset}$.

Corollary. Σ *(*$= \mathcal{U}(W, K)$*) is contractible.*

Proof. $K_T := |\mathcal{S}_{\geq \{T\}}|$ is a cone. □

The element of longest length

Exercise. The following conditions on an element $w_0 \in W$ are equivalent.

(a) For each $u \in W$, $l(w_0) = l(u) + l(u^{-1}w_0)$.

(b) For each $r \in R$, $l(w_0) > l(rw_0)$.

Moreover, if w_0 satisfies either condition, then

(i) w_0 is unique,

(ii) w_0 exists if and only if W is finite,

(iii) $l(w_0) = \mathrm{Card}(R)$,

(iv) w_0 is an involution and

(v) $w_0 S w_0 = S$.

Exercise. Suppose there is an element $w_0 \in W$ so that $l(sw_0) < l(w_0)$ for all $s \in S$. Then W is finite and w_0 is the element of longest length. (Hint: use the Exchange Condition to show that for w_0 begins with any given reduced expression $s_1 \cdots s_n$ and hence, that w_0 satisfies condition (a) in the previous exercise).

The set of letters with which an element can end.

Definition. Given $w \in W$, let $\text{In}(w) := \{s \in S \mid l(ws) < l(w)\}$

$(= \{\text{letters of } S \text{ with which a reduced expression for } w \text{ can end}\})$.

Lemma. *For any* $w \in W$, $\text{In}(w) \in \mathcal{S}$.

Proof. w can be written uniquely in the form $w = aw_0$, where a is the shortest element in $wW_{\text{In}(w)}$. It follows that w_0 is the element of longest length in $W_{\text{In}(w)}$. In particular, $W_{\text{In}(w)}$ is finite. □

An increasing union of chambers. Order the elements of W:

$1 = w_1, \ldots, w_n, \ldots$ in such a fashion that $l(w_n) \le l(w_{n+1})$. Set

$$X_n := w_n X \quad \text{and} \quad U_n := \bigcup_{i=1}^{n} X_i.$$

Lemma. $X_n \cap U_{n-1} = X^{\mathrm{In}(w_n)}$.

Theorem.

$$H_*(\mathcal{U}) \cong \bigoplus_{w \in W} H_*(X, X^{\mathrm{In}(w)}) \cong \bigoplus_{T \in \mathcal{S}} H_*(X, X^T) \otimes \mathbb{Z}(W^T)$$

where $W^T := \{w \in W \mid \mathrm{In}(w) = T\}$ *and* $\mathbb{Z}(W^T)$ *is the free abelian* *group on W^T.*

Since $H_*(\mathcal{U}) = \varinjlim H_*(U_n)$, the theorem follows from the next lemma.

Lemma.

$$H_*(U_n) \cong \bigoplus_{i=1}^{n} H_*(X, X^{\mathrm{In}(w_i)}).$$

Proof. Use the exact sequence in homology of the pair (U_n, U_{n-1}),

$n \geq 1$. By excision,

$$H_*(U_n, U_{n-1}) \cong H_*(X_n, X_n^{\mathrm{In}(w)}),$$

where $w = w_n$ and we have excised the open subset $U_n - X_n$. The right hand side is isomorphic to $H_*(X, X^T)$, where $T = \mathrm{In}(w)$. So, the sequence of the pair (U_n, U_{n-1}) can be rewritten as

$$\longrightarrow \; H_*(U_{n-1}) \; \longrightarrow \; H_*(U_n) \; \xrightarrow{f} \; H_*(X, X^T) \; \longrightarrow$$

where the map f is the composition of the excision isomorphism and the map induced by translation by w^{-1}. We want to split f.

Define an element $h_T \in \mathbb{Z}W_T$ by

$$h_T := \sum_{u \in W_T} \varepsilon(u)u$$

where $\varepsilon(u) := (-1)^{l(u)}$. Then h_T induces a chain map $C_*(X, X^T) \to C_*(W_T X)$. Similarly, $wh_T \in \mathbb{Z}W$ induces a map from $C_*(X, X^T)$ to $C_*(wW_T X)$. Since w is the longest element in wW_T, $wW_T X \subseteq U_n$. Hence, wh_T induces $H_*(X, X^T) \to H_*(U_n)$ and this map obviously splits f. Thus,

$$H_*(U_n) \cong H_*(U_{n-1}) \oplus H_*(X, X^{\mathsf{In}(w_n)}).$$

For $n = 1$, we have $w_1 = 1$, $\mathsf{In}(w_1) = \emptyset$ and $X^{\mathsf{In}(w_1)} = \emptyset$. Hence, the above formula becomes

$$H_*(U_1) = H_*(X, \emptyset) = H_*(X, X^{\mathsf{In}(w_1)}).$$

Combining these equations we get the formula of the lemma. □

Corollary. *TFAE*

- $\mathcal{U}$ *is acyclic.*

- $\forall T \in \mathcal{S}$, $\overline{H}_*(X, X^T) = 0$.

- X *is acyclic and* $\forall T \in \mathcal{S}_{>\emptyset}$, X^T *is acyclic.*

- $\forall T \in \mathcal{S}$, X_T *is acyclic.*

Definition. Given an integer $m \geq -1$, a space Y is called m-*acyclic* if $\overline{H}_i(Y) = 0$ for $-1 \leq i \leq m$. (N.B. The reduced homology of $\emptyset$ is defined to be $\mathbb{Z}$ in degree -1 and 0 in degrees ≥ 0.)

Corollary. *TFAE*

- $\mathcal{U}$ *is* m*-acyclic.*

- $\forall T \in \mathcal{S}$, $\overline{H}_i(X, X^T) = 0$, $\forall i \leq m$.

- $\forall T \in \mathcal{S}$, X_T *is* $(m - \mathrm{Card}(T))$*-acyclic.*

Corollary. $\mathcal{U}$ *is* 1*-acyclic* $\iff$

- X *is* 1*-acyclic.*

- $\forall s \in S$, X_s *is nonempty and path connected.*

- $\forall \{s,t\} \in \mathcal{S}^{(2)}$, $X_s \cap X_t \neq \emptyset$.

Simple connectivity

Theorem. *$\mathcal{U}$ is simply connected $\iff$*

- *X is simply connected.*

- *For each $s \in S$, X_s is nonempty and path connected.*

- $\forall \{s, t\} \in \mathcal{S}^{(2)}$, $X_s \cap X_t \neq \emptyset$.

Proof. ($\Longrightarrow$) Suppose $\mathcal{U}$ is simply connected. Since $p : \mathcal{U} \to X$ is a retraction, the first condition holds. Since $\mathcal{U}$ is 1-acyclic so do the other two.

($\Longleftarrow$) Let $\pi : \widetilde{\mathcal{U}} \to \mathcal{U}$ be the universal covering . We will show that if the 3 conditions hold, then π is a homeomorphism and hence, that $\mathcal{U}$ is simply connected. As usual, identify X with a subspace of $\mathcal{U}$. Since X is simply connected, π maps each path component of $\pi^{-1}(X)$ homeomorphically onto X. Choose such a component and call it $\widetilde{X}$. Define $c : X \to \mathcal{U}$ to be the inverse of the homeomorphism $\pi|_{\widetilde{X}}$. We want to lift the W-action to $\widetilde{\mathcal{U}}$. First we lift the generators. Each $s \in S$ has a unique lift $\tilde{s} : \widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}}$ fixing a given basepoint in $c(\widetilde{X_s})$. Since $\tilde{s}^2$ covers $id_{\mathcal{U}}$ and fixes a basepoint, it must be the identity on $\widetilde{\mathcal{U}}$, i.e., $(\tilde{s})^2 = 1$. Next, suppose $\{s,t\} \in \mathcal{S}^{(2)}$ (i.e., $m(s,t) \neq \infty$). By the third condition, $X_{\{s,t\}}$ $(= X_s \cap X_t)$ is nonempty, so we can choose the basepoint in $X_{\{s,t\}}$. Then $\tilde{s}\tilde{t}$ is the unique lift of st fixing the basepoint. If $m = m(s,t)$, then $(\tilde{s}\tilde{t})^m$ is a lift of the identity which fixes the basepoint; hence, $(\tilde{s}\tilde{t})^m$ must be the identity map of $\widetilde{\mathcal{U}}$. This shows that the W-action on $\mathcal{U}$ lifts to a W-action on $\widetilde{\mathcal{U}}$ (so that the projection map π is W-equivariant). (By the universal property of $\mathcal{U}$) the map $c : X \to \widetilde{\mathcal{U}}$ extends to a W-equivariant map $\tilde{c} : \mathcal{U} \to \widetilde{\mathcal{U}}$. Since c is a section of $\pi|_{\widetilde{X}}$, it is easy to see that $\tilde{c}$ is a section of q. Therefore, $\tilde{c}$ is a homeomorphism, $\pi : \widetilde{\mathcal{U}} \to \mathcal{U}$ is the trivial covering and consequently, $\mathcal{U}$ is simply connected. $\square$

Corollary. *$\mathcal{U}$ is contractible* $\iff$

- *X is contractible.*

- *$\forall T \in \mathcal{S}_{>\emptyset}$, X_T is acyclic.*

Corollary. *Σ is contractible.*

Cohomology with compact supports.

$W_n := \{w_1, \ldots, w_n\} \subset W$. Set

$$\check{U}_n := (W - W_n)X = \bigcup_{w \notin W_n} wX.$$

So, $\check{U}_n = \mathcal{U} - \mathrm{int}(U_n)$.

Theorem.

$$H_c^*(\mathcal{U}) \cong \bigoplus_{w \in W} H^*(X, X^{S-\mathrm{In}(w)}) \cong \bigoplus_{T \in \mathcal{S}} H^*(X, X^{S-T}) \otimes \mathbb{Z}(W^T).$$

Proof. We have the inverse sequence $\check{U}_1 \supset \cdots \supset \check{U}_n \supset \cdots$ and

$$H_c^*(\mathcal{U}) = \varinjlim H^*(\mathcal{U}, \check{U}_n)$$

Consider the exact sequence in cohomology of the triple $(\mathcal{U}, \check{U}_{n-1}, \check{U}_n)$:

$$\to H^*(\mathcal{U}, \check{U}_{n-1}) \to H^*(\mathcal{U}, \check{U}_n) \to H^*(\check{U}_{n-1}, \check{U}_n) \to .$$

By excision:

$$H^*(\check{U}_{n-1}, \check{U}_n) \cong H^*(wX, wX^{S-\mathsf{In}(w)}).$$

where $w := w_n$. We will show that the sequence of the triple splits so that

$$H^*(\mathcal{U}, \check{U}_n) \cong H^*(\mathcal{U}, \check{U}_{n-1}) \oplus H^*(X, X^{S-\mathsf{In}(w)})$$

and from this

$$H^*(\mathcal{U}, \check{U}_n X) \cong \bigoplus_{i=1}^{i=n} H^*(X, X^{S-\mathsf{In}(w_i)}) \ ,$$

which implies the theorem.

It remains to show that the map $H^*(\mathcal{U}, \check{U}_n) \to H^*(wX, wX^{S-\mathsf{In}(w)})$

splits. For each $T \subset S$,

$$A_T := \{w \mid w \text{ is shortest element in } W_T w\}$$

Then $A_T X \cong \mathcal{U}/W_T$. The splitting is defined via the projection map $\mathcal{U} \to A_T X$, $T = \text{In}(w)$, together with the inclusion $(wA_T X, w(A_T - 1)X) \to (\mathcal{U}, \check{U}_n)$. Combining these maps we get the splitting

$$H^*(X, X^T) \cong H^*(wA_T X, w(A_T - 1)X) \to H^*(\mathcal{U}, \check{U}_n).$$

$\square$

Corollary.

$$H_c^*(\Sigma) \cong \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes \mathbb{Z}(W^T).$$

Corollary.

$$H^*(W; \mathbb{Z}W) \cong \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes \mathbb{Z}(W^T).$$

Lecture 6:

Examples for cohomology of groups and

for aspherical manifolds

Classifying spaces. For any group π there is a connected CW complex $B\pi$ with the property that $\pi_1(B\pi) = \pi$ and $\pi_i(B\pi) = 0$ for $i > 1$. It follows that its universal cover $E\pi$ is contractible. ($B\pi$ is said to be "aspherical.") The CW complex $B\pi$ is unique up to homotopy equivalence.

Group cohomology. For any π-module M,

$$H^*(\pi; M) := H^*(B\pi; M)$$

and similarly for homology.

Cohomological dimension and geometric dimension

The *geometric dimension* of a group π, denoted $\operatorname{gd}\pi$, is the smallest integer n so that there is an n-dimensional CW model for $B\pi$. Its *cohomological dimension*, denoted $\operatorname{cd}\pi$, is the projective dimension of $\mathbb{Z}$ over $\mathbb{Z}\pi$. In other words, it is the smallest integer

n such that $\mathbb{Z}$ admits a projective resolution:

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0$$

(or ∞ if there is no such integer). This is equivalent to the following:

$$\operatorname{cd} \pi = \sup\{n \mid H^n(\pi; M) \neq 0 \text{ for some } \pi\text{-module } M\}.$$

If $B\pi$ has a model which is a finite CW complex, then

$$\operatorname{cd} \pi = \sup\{n \mid H^n(\pi; \mathbb{Z}\pi) \neq 0\}.$$

A group π *virtually* has some property if it has a subgroup π' of finite index with that property. For example, π is *virtually torsion-free* if it has a torsion-free subgroup of finite index. Given a virtually torsion-free group π, its *virtual cohomological dimension*, denoted $\operatorname{vcd}\pi$ is the cohomological dimension of any torsion-free subgroup of finite index. It is not hard to see that $\operatorname{vcd}\pi$ is well-defined.

Remark. Note that if the group π has torsion, then $\operatorname{cd}\pi = \infty$.

The reason is that the cyclic group $\mathbf{C}_m$ of order m, for $m > 1$, has nontrivial cohomology (with coefficients in $\mathbb{Z}$) in arbitrarily high degrees. So, if $\mathbf{C}_m \subseteq \pi$, then $\operatorname{cd} \pi = \infty$.

Lemma. (Selberg's Lemma) *Any finitely generated subgroup of $GL(n, \mathbb{C})$ is virtually torsion-free.*

Corollary. *Any finitely generated Coxeter group is virtually torsion-free.*

Let (W, S) be a Coxeter system. Recall $\mathcal{S} = \{\text{spherical subsets of } S\}$

and $K := |\mathcal{S}|$. Last time:

$$H^*(W; \mathbb{Z}W) \cong H_c^*(\Sigma) \cong \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes \mathbb{Z}(W^T).$$

The *nerve* L of (W, S) is the simplicial complex corresponding to

$\mathcal{S}_{>\emptyset}$. For $T \in \mathcal{S}$, let σ_T be the simplex of L with $\text{Vert}(\sigma_T) = T$.

Observations. • Since K is contractible,

$$H^*(K, K^{S-T}) \cong \overline{H}^{*-1}(K^{S-T}).$$

• K^S ($:= \partial K$) is the barycentric subdivision of L and $\forall T \in \mathcal{S}$,

$K^{S-T} \sim L - \sigma_T$ (where $\sim$ means homotopy equivalent).

Theorem. $\operatorname{vcd} W = \max\{n \mid \overline{H}^{n-1}(L-\sigma_T) \neq 0,$ *for some* $T \in \mathcal{S}\}$.

Theorem. (Stallings and Swan). *If* $\operatorname{cd} \pi = 1$, *then* π *is a free group.*

Corollary. *An infinite Coxeter group W is virtually free iff $\forall T \in \mathcal{S}$, $H^*(L - \sigma_T)$ is concentrated in dimension* 0*.*

The chain complex of an n-dimensional model for $E\pi$ is a free resolution of $\mathbb{Z}$. Hence, $\mathrm{cd}\,\pi \leq \mathrm{gd}\,\pi$. When $\mathrm{cd}\,\pi \neq 2$, this inequality is an equality.

Theorem. (Eilenberg–Ganea, 1957). *If* $\mathrm{cd}\,\pi > 2$*, then* $\mathrm{cd}\,\pi = \mathrm{gd}\,\pi$*. If* $\mathrm{cd}\,\pi = 2$*, then* $\mathrm{gd}\,\pi \leq 3$*.*

A nontrivial free group F obviously satisfies $\operatorname{cd} F = \operatorname{gd} F = 1$ (since we can take BF to be a wedge of circles). So, the case $\operatorname{cd} \pi = 1$ is taken care of by the Stallings–Swan Theorem. The question of whether or not there exist groups of cohomological dimension 2 and geometric dimension 3 is known as the *Eilenberg–Ganea Problem*.

Constructing examples with Coxeter groups.

Given simplicial complex L, In Lecture 4 we saw how to use a right-angled Coxeter group to construct an aspherical cubical complex so that the link of each vertex is identified with L. Properties of L can then be translated back into properties of the Coxeter group W and its torsion-free subgroup Γ ($= \pi_1$(cubical complex)). Recall that given a flag complex L, we set $S := \mathrm{Vert}\, L$. and $\mathcal{S}(L) := \{T \subseteq S \mid T$ is the vertex set of a simplex$\}$.

If $\square^S := [-1,1]^S$, then $X_L \subseteq \square^S$ the union of all faces which are parallel to $\square^T$ for some $T \in \mathcal{S}(L)$.

Possible counterexamples to Eilenberg–Ganea. Suppose L is a flag cx s.t.

$$\dim L = 2, \quad L \text{ is acyclic} \quad \text{and} \quad \pi_1(L) \neq 1.$$

Claim. Such L exist.

Let $\Gamma := \pi_1(X_L) \subset W_L$. Then $\operatorname{cd} \Gamma = 2$ (because the number $\max\{n \mid \overline{H}^{n-1}(L - \sigma_T) \neq 0\}$ is 1) and $\operatorname{gd}\Gamma \leq 3$ (because $\dim \widetilde{X}_L = 3$). In fact, Γ acts freely on an acyclic 2-complex, namely, $\mathcal{U}(W, \partial K) \subset \widetilde{X}_L$; however this 2-complex is not simply connected. So, it seems likely that gd = 3.

Proof of Claim. Let $G :=$ the binary dodecahedral gp. *Poincaré's homology sphere* is the 3-manifold $M^3 := \mathbb{S}^3/G$. M^3 is formed by identifying opposite faces of a dodecahedron. Let L be the 2-

skeleton of M^3, i.e., the image of ∂(dodecahedron). By passing to the barycentric subdivision we can assume L is a flag cx. $\square$

In fact, Brady, Leary and Nucinkis showed that for L as above, the groups W_L are counterexamples to the natural generalization of the Eilenberg–Ganea Problem for groups with torsion.

Example. (*Bestvina, as well as, Dicks–Leary and Dranishnikov*). Suppose L is a flag triangulation of $\mathbb{R}P^2$ and $\Gamma \subset W_L$ is the torsion-free subgroup. The formula

$$H^*(W;\mathbb{Z}W) \cong H_c^*(\Sigma) \cong \bigoplus_{T\in\mathcal{S}} H^*(K, K^{S-T}) \otimes \mathbb{Z}(W^T).$$

shows that $H_c^3(\Sigma;\mathbb{Z}) = \mathbb{Z}/2$ while $H_c^*(\Sigma;\mathbb{Q}) = 0$ for $* > 2$. Hence, $\mathrm{cd}_{\mathbb{Z}}\,\Gamma = 3$ while $\mathrm{cd}_{\mathbb{Q}}\,\Gamma = 2$.

Example. (*Nonadditivity of cohomological dimnsion*). Let L_1, W_1, Σ_1 and Γ_1 be the L, W, Σ and Γ_1 of the previous example. Let L_2 be a 2-dimensional Moore space for $\mathbb{Z}/3$ (i.e., $\overline{H}_*$ is concentrated in $\dim 2$ and is $= \mathbb{Z}/3$ there). $\Gamma_2 \subset W_2$ the resulting gps and Σ_2 the cx for W_2. Then $H^3_c(\Sigma_2; \mathbb{Z}) = \mathbb{Z}/3$; so, $\mathrm{cd}_{\mathbb{Z}}\, \Gamma_2 = 3$. By the Künneth Formula, $H^6_c(\Sigma_1 \times \Sigma_2; \mathbb{Z}) = \mathbb{Z}/2 \otimes \mathbb{Z}/3 = 0$. So, $\mathrm{cd}_{\mathbb{Z}}(\Gamma_1 \times \Gamma_2) = 5 \neq \mathrm{cd}_{\mathbb{Z}}(\Gamma_1) + \mathrm{cd}_{\mathbb{Z}}(\Gamma_2)$.

Topological background: homology spheres, homology manifolds, etc. If you aren't a topologist, you can't think of a closed n-manifold with the same homology as S^n but which is not homeomorphic to S^n. Similarly, you don't know an example of a compact manifold with boundary which is contractible and $\neq D^n$ or an open contractible manifold $\neq \mathbb{R}^n$. But many such examples exist!

Definition. A closed manifold M^n is a *homology n-sphere* if $H_*(M^n) \cong H_*(S^n)$.

If M^n is a homology n-sphere ($n > 1$), then $H_1(M^n) = 0$, i.e., the abelianization of $\pi := \pi_1(M^n)$ is $= 0$. Also, by a theorem of Hopf, $H_2(\pi) = 0$. Conversely, Kervaire proved that if π is any finitely presented group satisfying the above two conditions and $n \geq 5$, then $\exists$ a homology n-sphere with $\pi_1 = \pi$.

Theorem. *(Generalized Poincaré Conjecture, Smale, Freedman and Perelman). Given a homology n-sphere M^n $(n > 1)$. Then*

$$M^n \cong S^n \iff \pi_1(M^n) = 1.$$

If C^n is a compact, contractible n-manifold, then it follows from the exact sequence of $(C, \partial C)$ and Poincaré duality that ∂C is a homology $(n-1)$-sphere.

Corollary. *Suppose C^n is a compact, contractible n-manifold ($n > 2$). Then*

$$C^n \cong D^n \iff \pi_1(\partial C) = 1.$$

Proposition. *Suppose M^n is a homology n-sphere. Then M^n is* *(topologically) the boundary of a compact, contractible* $(n+1)$-*manifold.*

Proof. Surgery theory for $n > 3$. Freedman for $n = 3$. □

Corollary. *For each* $n \geq 4$, $\exists$ *compact, contractible manifold* C^n *s.t.* $C^n \neq D^n$.

Corollary. *For each* $n \geq 4$, $\exists$ *open contractible manifold* M^n *s.t.* $M^n \neq \mathbb{R}^n$.

Simple connectivity at ∞. Suppose Y is a "reasonable" space (e.g. locally compact, locally path connected, second countable, Hausdorff) and suppose it is not compact. A *nbhd of* ∞ is the complement of a compact set. Y is 1-*ended* if every nbhd of ∞ contains a connected nbhd of ∞. It is simply connected at ∞ if, in addition, every nbhd of ∞ contains a smaller nbhd of ∞ s.t. any loop in the smaller nbhd is null-homotopic in the larger. For example, $\mathbb{R}^n$ is 1-ended for $n \geq 2$ and is simply connected at ∞

for $n \geq 3$.

Suppose $C_1 \subset C_2 \subset \cdots$ is an exhaustive sequence of compact sets in Y. This gives an inverse sequence of fundamental groups $\pi_1(Y - C_1) \leftarrow \pi_1(Y - C_2) \leftarrow \cdots$. An inverse sequence of groups $G_1 \leftarrow G_2 \leftarrow \cdots$ is *semistable* if $\exists f : \mathbb{N} \to \mathbb{N}$ s.t. the image of G_k in G_n is the same $\forall k \geq f(n)$. Y is *semistable* if $\exists$ such a semistable inverse sequence of fundamental groups. (If this holds for one exhaustive sequence then it holds for all.) So, when Y is semistable define $\pi_1^\infty(Y) := \varprojlim \pi_1(Y - C_k)$.

Fact. Suppose Y is semistable. Then

$$\text{it is simply connected at } \infty \iff \pi_1^\infty(Y) = 1.$$

Theorem. (Stallings, Freedman, Perelman). *Let M^n be an open contractible n-mfld ($n \geq 3$). Then M^n is homeomorphic to $\mathbb{R}^n$ iff it is simply connected at ∞.*

Aspherical manifolds not covered by $\mathbb{R}^n$. Suppose L is a homology $(n-1)$-sphere, $n \geq 4$, triangulated as a flag cx and that $X_L \subset \square^S$ and $\Sigma = \widetilde{X}_L$ are as before.

Theorem. Σ *is semistable.* $\pi_1^\infty(\Sigma) = 1 \iff \pi_1(L) = 1$.

If $\pi_1(L) \neq 1$, then Σ is not quite a manifold (it is a homology manifold as defined below). However, it is easy to modify the situation to make it a mfld.

Proof of Theorem. As in Lecture 5, order the elements of W:

$1 = w_1, \ldots, w_m, \ldots$ so that $l(w_m) \leq l(w_{m+1})$. Set

$$K_m := w_m K \quad \text{and} \quad U_m := \bigcup_{i=1}^{m} K_i.$$

Since $K_m \cap \partial U_{m-1}$ is an $(n-1)$-disk in ∂K_m $(= L)$,

$$\Sigma - \operatorname{int} U_m \sim \partial U_m \quad \text{and} \quad \partial U_m = \partial K_1 \sharp \cdots \sharp \partial K_m.$$

Hence, for $\pi = \pi_1(L) = \pi_1(\partial K_i)$,

$$\pi_1(\Sigma - \operatorname{int} U_m) = \pi_1(\partial U_m) = \underbrace{\pi * \cdots * \pi}_{m \text{ terms}}.$$

The map $\pi_1(\Sigma - \operatorname{int} U_m) \to \pi_1(\Sigma - \operatorname{int} U_{m-1})$ is the natural projection on the first $m-1$ factors. Since this projection is onto, Σ is semistable and $\pi_1^\infty = \varprojlim(\pi * \cdots * \pi)$ is the "projective free product" of an infinite number of copies of π. In particular, $\pi_1^\infty \neq 1$. $\square$

Modifying Σ to be a mfld. We have $\Sigma = \mathcal{U}(W, K)$, where $K = \text{Cone}(\partial K)$ and $\partial K = L$. The homology sphere ∂K bounds a contractible n-mfld C. Idea: "hollow out" each copy of K and replace it with a copy of C. Since $\partial C := \partial K$, we can define $C_s := K_s$. Then $\mathcal{U}(W, C)$ is a mfld (since there are no longer cone points), it is contractible and $\pi_1^\infty(\mathcal{U}(W, C)) = \pi_1^\infty(\Sigma)$ (since they are proper homotopy equivalent). Moreover, $M^n := \mathcal{U}(W, C)/\Gamma$ is an aspherical mfld with universal cover $\mathcal{U}(W, C)$.

Theorem. *For each $n \geq 4$, $\exists$ closed aspherical n-mflds with universal cover $\neq \mathbb{R}^n$.*

Polyhedral homology manifolds.

Definition. A space X is a *homology n-manifold* (also called a "generalized manifold") if it has the same local homology groups as $\mathbb{R}^n$, i.e., if $\forall x \in X$,

$$H_i(X, X - x) = \begin{cases} \mathbb{Z}, & \text{if } i = n; \\ 0, & \text{otherwise.} \end{cases}$$

X is a *generalized homology n-sphere* (for short, a "GHS^n ") if it is a homology n-manifold with the same homology as S^n.

Example. If M^{n-1} is a homology sphere, then its suspension $S^0 * M^{n-1}$ is a GHS^n. If $\pi_1(M^n) \neq 1$ (and $n \geq 4$), then the suspension is not a mfld.

Lemma. *Given an n-dimensional cell complex Λ, TFAE:*

- *Λ is a homology n-manifold.*

- *For each cell σ in Λ, $\mathsf{Lk}(\sigma, \Lambda)$ is a $\mathsf{GHS}^{n-\dim\sigma-1}$.*

- *For each vertex* v*,* $\mathrm{Lk}(v, \Lambda)$ *is a* GHS^{n-1}*.*

Theorem. (The Double Suspension Theorem, Cannon, Edwards). *Given a (PL) homology sphere* M^n*, its double suspension* $S^1 * M^n$ *is a topological manifold (hence,* $\cong S^{n+2}$*).*

Theorem. (Edwards, Freedman). *A polyhedral homology* n*-mfld,* $n \geq 3$*, is a topological mfld iff the link of each vertex is simply connected.*

By choosing L to be a suitable GHS^{n-1}, we get:

Corollary. (D. & Januszkiewicz). *For each* $n \geq 5$, $\exists$ *examples where* Σ *is a topological mfld* $\neq \mathbb{R}^n$. *(In particular,* $\exists$ CAT(0)-*mflds* $\neq \mathbb{R}^n$.*)*

Poincaré duality groups. A group π is *type F* if $B\pi$ has a model which is a finite complex. If π satisfies the cohomological version of this it is *type FP*.

Definition. A group π is an n-dimensional *Poincaré duality group* (a "PD^n-*group*" for short) if it is type FP and

$$H^i(\pi;\mathbb{Z}\pi) \cong \begin{cases} 0, & \text{if } i \neq n, \\ \mathbb{Z}, & \text{if } i = n. \end{cases}$$

A virtually torsion-free group G is said to be a *virtual Poincaré* *duality group* (for short, a “VPD^n-*group*”) if it contains a finite index subgroup which is a PD^n-group.

Definition. (W, S) is *type* HM^n if L is a GHS^{n-1}. (This is equivalent to the condition that Σ be a homology n-manifold.)

Of course, if (W, S) is type HM^n, then it is a VPD^n-group. We will see below that the converse is essentially true. First, recall our formula:

$$H^*(\Gamma; \mathbb{Z}\Gamma) \cong H_c^*(\Sigma) \cong \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes \mathbb{Z}(W^T),$$

where $\Gamma \subset W$ is a torsion-free subgroup of finite index. If the homology on the RHS is concentrated in dimension n and is $\cong \mathbb{Z}$ in that dimension, then two things must happen:

- W^T is a singleton, for some $T \in \mathcal{S}$ and

- $H^n(K, K^{S-T}) \cong \mathbb{Z}$ and is 0 elsewhere and $\forall T' \in \mathcal{S}$, $T' \neq T$,

  $H^*(K, K^{S-T'}) = 0$.

Exercise. $W^T = \{w_0\} \iff$

(a) w_0 is the longest element in W_T and (b) $W = W_T \times W_{S-T}$.

Theorem. *Suppose W does not split off a nontrivial finite factor.*

Then TFAE

- W *is a* VPD^n*-group.*

- L *has the same cohomology as* S^{n-1} *and* $\forall T \in \mathcal{S}$, $T \neq \emptyset$, $L - \sigma_T$ *is acyclic.*

- L *is a* GHS^{n-1} *(i.e.,* (W, S) *is type* HM^n*).*

Corollary. *The condition that* (W, S) *is type* HM^n *does not depend on the choice of fundamental generating set* S

From this we eventually get the following:

Corollary. (Charney & D.) *Coxeter groups of type* HM^n *are rigid, i.e., if* S *and* S' *are two subsets of* W *such that* (W, S) *and* (W, S') *are both Coxeter systems, then* S' *is conjugate to* S*.*

Lecture 7:

The reflection group trick,

the Euler Characteristic Conjecture

The reflection group trick. The main consequence of this trick is the following:

Theorem. *Given a group π of type F, $\exists$ a closed aspherical mfld M such that $\pi_1(M)$ retracts onto π.*

The construction in a nutshell. Assume, as we may, that $B\pi$ is a finite polyhedron. "Thicken" $B\pi$ to X, a compact PL manifold with boundary. (For example, if $n > 2(\dim B\pi)$, piecewise linearly embed $B\pi$ in some triangulation of $\mathbb{R}^n$ and then take X to be a regular neighborhood of $B\pi$ in $\mathbb{R}^n$.) X is homotopy equivalent to $B\pi$ (it collapses onto it). Let (W, S) be a Coxeter system whose nerve L is a triangulation of ∂X. (We could take L to be a flag triangulation of ∂X as a flag complex and (W, S) to be the associated right-angled Coxeter system.) For each $s \in S$, let X_s denote the geometric realization of $\mathcal{S}_{\geq\{s\}}$ (regarded as a subset of ∂X). In other words, X_s is the closed star of the vertex s of

L in the barycentric subdivision of L. As usual, $\mathcal{U} := \mathcal{U}(W, X)$. Since L is a PL triangulation of ∂X, it is easy to see that $\mathcal{U}$ is a manifold with a proper, locally linear W-action. Let $\Gamma \subset W$ be any torsion-free subgroup of finite index. Define M to be the quotient space

$$M := \mathcal{U}/\Gamma.$$

Since Γ acts properly and freely on $\mathcal{U}$, the quotient map $\mathcal{U} \to M$

is a covering projection; hence, M is a mfld.

Lemma. *$\mathcal{U}$ is aspherical.*

Proof. Order W as before and set

$$X_m := w_m X \quad \text{and} \quad U_m := \bigcup_{k=1}^{m} X_k.$$

Since $K_m \cap \partial U_{m-1}$ is an $(n-1)$-disk in ∂X_m $(= L)$, $U_m \sim \bigvee X_k$, which is aspherical. Since $\pi_i(\mathcal{U}) = \varinjlim \pi_i(U_m)$, $\mathcal{U}$ is aspherical (& $\pi_1(\mathcal{U}) =$ free product of an infinite number of copies of π). $\square$

Theorem. *Given a group π of type F, the mfld M satisfies:*

- *M is a closed apherical manifold and*

- *M retracts onto $B\pi$.*

Proof. M is aspherical since it is covered by $\mathcal{U}$. M is compact since $X = \mathcal{U}/W$ is compact and since Γ is finite index in W.

Since X can be identified with a subspace of M, the orbit map $\mathcal{U} \to X$ induces the retraction $M \to X \sim B\pi$. $\square$

Fundamental groups of aspherical manifolds. For a long time almost all examples of closed aspherical mflds came from Lie groups, so it was thought that the class of their fundamental groups was fairly restrictive. On the other hand, it was known that there were many interesting aspherical complexes not related to Lie groups.

Given a pair of integers (p, q), define the *Baumslag–Solitar group* $BS(p, q)$ to be the 1-relator group defined by the presentation:

$$BS(p, q) := \langle a, b \mid ab^p a^{-1} = b^q \rangle.$$

Theorem. (Lyndon). *If* Γ *is a 1-relator gp and the relation is not a proper power, then the presentation 2-cx for* Γ *is a* $B\Gamma$*.*

Corollary. *The presentation complex for* $BS(p, q)$ *is aspherical* *(i.e., $BS(p,q)$ is type F and its* gd $= 2$*).*

Every 2-dimensional polyhedron can be embedded in $\mathbb{R}^5$. Although it is not true that every 2-complex can be embedded in $\mathbb{R}^4$, every finite 2-dimensional CW complex can be thickened to a compact 4-manifold. Thus, for each Baumslag–Solitar group $\pi = BS(p,q)$, $B\pi$ can be thickened to a compact aspherical n-mfld with boundary for any $n \geq 4$.

Recall that π is *residually finite* if $\forall$ two elements $g_1, g_2 \in \pi$, $\exists$ a homomorphism φ to some finite group F s. t. $\varphi(g_1) \neq \varphi(g_2)$.

Example. (*Not residually finite*, Mess). The Baumslag–Solitar group $\pi = BS(2,3)$ is not residually finite. Since π is not residually finite, neither is any group which retracts onto it. Hence, for each $n \geq 4$, there are closed aspherical n-mflds whose fundamental groups are not residually finite.

Example. (*Infinitely divisible abelian subgroups*, Mess). This time $\pi := BS(1,2)$. The centralizer of b in this group is isomorphic to a copy of the dyadic rationals. Hence, for each $n \geq 4$, there are closed aspherical n-manifold whose fundamental groups contain an infinitely divisible abelian group.

Example. (*Unsolvable word probem*, Weinberger). There are examples of finitely presented groups π with unsolvable word problem such that $B\pi$ is a finite 2-complex. Any group which retracts onto such a group also has unsolvable word problem.

So, for each $n \geq 4$, there are closed aspherical n-mflds whose fundamental group have unsolvable word problem.

Nonsmoothable aspherical manifolds.

Theorem. (D. & Hausmann). *In each dimension* ≥ 13*, there are closed aspherical mflds not homotopy equivalent to a smooth mflds.*

Sketch of proof. If $M := \mathcal{U}/\Gamma$ is a smooth mfld, then its tangent bundle, TM, restricted to X is TX. Find a thickening X of T^k such that the "Spivak normal fiber space" of X does not lift to linear vector bundle. The existence of such follows from calculations of homotopy groups of various classifying spaces BO, BPL and BG. □

PD^n**-groups which are not finitely presented.** A famous question in topology:

Question. *Is every* PD^n*-group* $= \pi_1$*(closed aspherical mfld)?*

We shall use the reflection group trick to show that the answer is no, but for a cheap reason: PD^n-groups need not be finitely presented while fundamental groups of closed mflds are. So, we should change the question by modifying "PD^n-group" by the phrase "finitely presented." Here are the details:

Fact. (Kirby-Siebenmann). Any compact topological mfld is homotopy equivalent to a finite cx (and hence, has finitely presented π_1).

Recall that for π to be a PD^n-*group* means that it is type FP and

$$H^i(\pi;\mathbb{Z}\pi) \cong \begin{cases} 0, & \text{if } i \neq n, \\ \mathbb{Z}, & \text{if } i = n. \end{cases}$$

The FP condition is weaker than being type F – it does not imply that π is finitely presented. One way to prove that π is FP

is to show that it is type FH, i.e., that acts freely on an acyclic complex with finite quotient (if we change acyclic to contractible we have the definition of type F.) Wall showed

type FH + finitely presented $\iff$ type F. Bestvina–Brady showed that nonfinitely presented groups of type FH exist.

Theorem. (Bestvina–Brady). $\exists$ *finite* 2*-complex* Y *and a regular covering space* $\widetilde{Y} \to Y$ *with gp of deck transformations* π *s.t.*

- $\widetilde{Y}$ *is acyclic.*

- π *is not finitely presented.*

Corollary. *For each* $n \geq 4$, $\exists$ PD^n*-groups which are not finitely presented.*

Proof. A slight variation of the reflection group trick. As before, thicken Y to X. Set $\mathcal{U} = \mathcal{U}(W, X)$ and $M = \mathcal{U}/\Gamma$. The covering space $\widetilde{Y} \to Y$ corresponds to a covering space $p : \widetilde{X} \to X$. There is an induced (infinitely generated) Coxeter system $(\widetilde{W}, \widetilde{S})$ with $\widetilde{S} := p^{-1}(S)$ and an epimorphism $\varphi : \widetilde{W} \to W$. Put $\widetilde{\mathcal{U}} := \mathcal{U}(\widetilde{W}, \widetilde{X})$ and $\widetilde{\Gamma} := \varphi^{-1}(\Gamma)$. The gp of deck transformations of $\widetilde{\mathcal{U}} \to M$ is $\widetilde{\Gamma} \rtimes \pi$. Since this gp retracts onto π, it is not finitely presented. Using the theorem from Lecture 5, it is not difficult to show that $\widetilde{\mathcal{U}}$ is an acyclic n-mfld. ($\widetilde{W}$ acts on it with fundamental chamber $\widetilde{X}$.) Hence, $\widetilde{\Gamma} \rtimes \pi$ is a PD^n-group. $\square$

The Euler Characteristic Conjecture.

Conjecture. (Chern, Hopf, Thurston). *Suppose M^{2n} is a closed aspherical $2n$-dimensional mfld. Then $(-1)^n\chi(M^{2n}) \geq 0$. (Here χ denotes the Euler characteristic.)*

Remark. In odd dimensions, by Poincaré duality, $\chi(M^{2n+1}) = 0$.

Remark. Hopf and Chern conjectured this for Riemannian mflds of nonpositive sectional curvature. It makes sense more generally for mflds which are nonpositively curved in the sense of having metrics which are locally CAT(0).

Suppose that Y is a cell complex and that G is a discrete group acting properly, cellularly and cocompactly on Y. The quotient space Y/G is an "orbihedron" (an "orbifold" when Y is a mfld.) If H a normal subgroup of G acting freely on Y and $Z = Y/H$, then $Z/(G/H)$ and Y/G are the same orbihedron; furthermore, Z is an "orbihedral covering space" of Y/G.

Definition. The *orbihedral Euler characteristic* of Y/G is the rational number defined by the formula:

$$\chi^{orb}(Y/G) = \sum_{\sigma} \frac{(-1)^{\dim \sigma}}{|G_{\sigma}|},$$

where σ ranges over a set of representatives for the G-orbits of cells.

Suppose H is a subgroup of finite index in G. The main property of the orbihedral Euler characteristic is that it is multiplicative with respect to coverings, i.e.,

$$\chi^{orb}(Y/H) = [G : H] \ \chi^{orb}(Y/G).$$

We can expand the Euler Characteristic Conjecture as follows:

Conjecture. *Suppose X^{2n} is a closed, aspherical orbifold of dimension $2n$. Then $(-1)^n \chi^{orb}(X^{2n}) \geq 0$.*

Example. Suppose $G = W_L$, the right-angled Coxeter group with nerve L, and that Σ is the cubical complex of Lecture 4. There is one W_L-orbit of cubes for each element $T \in \mathcal{S}$; the dimension of a corresponding cube is $\mathrm{Card}(T)$; its stabilizer $\cong (\mathbb{Z}/2)^T$. Hence,

$$\begin{aligned}
\chi^{orb}(\Sigma/W_L) &= \sum_{T \in \mathcal{S}} \frac{(-1)^{\mathrm{Card}(T)}}{2^{\mathrm{Card}(T)}} \\
&= 1 + \sum_{\sigma \in L} \left(-\frac{1}{2}\right)^{\dim \sigma + 1} \\
&= 1 + \sum_{i=0}^{\dim L} \left(-\frac{1}{2}\right)^{i+1} f_i.
\end{aligned}$$

where f_i is the number of i-simplices in L. If L is a flag triangulation S^{2n-1}, Σ a $2n$-manifold. This leads to the following:

Conjecture. (The Flag Complex Conjecture, Charney–D). *If* L *is a flag triangulation of* S^{2n-1}*, then* $(-1)^n\kappa(L) \geq 0$*, where*

$$\kappa(L) := 1 + \sum_{i=0}^{\dim L} \left(-\frac{1}{2}\right)^{i+1} f_i.$$

Recall:

Definition. A simplicial complex L is a *flag complex* if for any finite, nonempty $T \subseteq \operatorname{Vert} L$ the following holds: T is the vertex set of a simplex in $L \iff$ any two vertices of T are connected by an edge.

The Chern-Gauss-Bonnet Theorem.

Theorem. (Chern, Gauss, Bonnet). *Suppose M^{2n} is a closed, $2n$-dimensional Riemannian manifold. Then*

$$\chi(M^{2n}) = \int \kappa$$

where κ is a certain $2n$-form on M^{2n} called the "Euler form". (κ is a constant multiple of the Pfaffian of the curvature).

The theorem was proved in dimension two by Gauss and Bonnet; in this case κ is just the Gaussian curvature (times $1/2\pi$). Versions of the higher dimensional result were proved by Poincaré, Hopf and Allendoerfer–Weil. The "correct" differential geometric proof in higher dimensions is due to Chern.

Remark. The naive idea for proving the Euler Characteristic Conjecture in the nonpositively curved case is to show that the condition on the sectional curvature forces the Chern-Gauss-

Bonnet integrand κ to have the correct sign, i.e., $(-1)^n\kappa \geq 0$. In dimension 2, κ is just the Gaussian curvature (up to a positive constant), so this naive idea works. As shown by Chern (who attributes the result to Milnor) the naive idea also works in dimension 4. Later Geroch showed that in dimensions ≥ 6 the naive idea does not work.

Theorem. (The Combinatorial Gauss-Bonnet Theorem). *If X is a finite PE cell complex, then*

$$\chi(X) = \sum_{v \in Vert(X)} \kappa_v.$$

where $\kappa_v = \kappa(\mathsf{Lk}(v, X))$ is a function of the piecewise spherical cell complex $\mathsf{Lk}(v, X)$.

When X is a cubical cell cx the local contribution of a link L is given by the familiar formula:

$$\kappa(L) := 1 + \sum_{i=0}^{\dim L} \left(-\frac{1}{2}\right)^{i+1} f_i.$$

Lemma. (Gromov's Lemma). *A cubical cell cx X is nonpositively curved $\iff$ $\forall v \in \mathrm{Vert}(X)$, $\mathrm{Lk}(v, X)$ is a flag cx.*

Corollary. *Euler Characteristic Conj for nonposiively curved cubical complexes $\iff$ Flag Cx Conj.*

(So, the Flag Cx Conj is the local pointwise version of the Euler

Char. Conj. and, in contrast with Geroch's result, it should hold in all odd dimensions.)

Lecture 8:

Growth series, buildings, Hecke algebras

Growth series. Let G be a gp and S a set of generators. $l : G \to \mathbb{N}$ is word length. Define a power series $f(t)$ (the *growth series* of G) by $f(t) := \sum_{g \in G} t^{l(g)}$. If G is finite, $f(t)$ is a polynomial. Under favorable circumstances (for example, when G is an "automatic group"), it is known that $f(t)$ is a rational function of t. One of the first results along this line was the proof of the rationality in the case of a Coxeter group W. We give the proof below. In the case of a Coxeter gp, it is a possible to define the growth series as a power series $W(\mathbf{t})$ in a certain vector $\mathbf{t}$ of indeterminates. Again, it is a rational function of $\mathbf{t}$.

Rationality. As usual, (W, S) is a Coxeter system. Suppose given an index set I and a function $i : S \to I$ s.t. $i(s) = i(s')$ whenever s and s' are conjugate in W. Let $\mathbf{t} := (t_i)_{i \in I}$ stand for an I-tuple of indeterminates and let $\mathbf{t}^{-1} := (t_i^{-1})_{i \in I}$. Write t_s instead of $t_{i(s)}$. If $s_1 \cdots s_l$ is a reduced expression for w, define t_w

to be the monomial $t_w := t_{s_1} \cdots t_{s_l}$. Similarly, define a monomial in the $(t_i)^{-1}$ by $t_w^{-1} := (t_{s_1})^{-1} \cdots (t_{s_l})^{-1}$.

Lemma. *t_w is independent of the choice of reduced expression for w.*

Proof. This follows from Tits' solution to the word problem. Indeed, two reduced expressions for w differ by a sequence of elementary M-operations of type (II). Such an operation replaces an alternating subword $ss' \cdots$ of length $m_{ss'}$ by the alternating word $s's \cdots$ of the same length but in the other order. If $m_{ss'}$ is even, s and s' occur the same number of times in these subwords, so the monomial t_w stays the same. If $m_{ss'}$ is odd, such an operation changes the number of occurrences of s and s' in the reduced expression. However, when $m_{ss'}$ is odd, s and s' are conjugate in the dihedral subgroup which they generate and so, *a fortiori*, are conjugate in W. Thus, $i(s) = i(s')$ and t_w again remains unchanged. □

The *growth series* of W is the power series in $\mathbf{t}$ defined by

$$W(\mathbf{t}) := \sum_{w \in W} t_w.$$

For any subset X of W, define

$$X(\mathbf{t}) := \sum_{w \in X} t_w.$$

For any subset T of S, we have the the special subgroup W_T (a subset of W) and its growth series $W_T(\mathbf{t})$. Note that if $T \subset S$ is a spherical subset, then $W_T(\mathbf{t})$ is a polynomial in $\mathbf{t}$.

Lemma. *Suppose W is finite, w_S is its element of longest length and $t_S := t_{w_S}$. Then $W(\mathbf{t}) = t_S W(\mathbf{t}^{-1})$.*

Proof. For any $w \in W$, $l(w_S w) = l(w_S) - l(w)$. So, concantenation of a reduced expression for $w_S w$ with one for w^{-1} gives a reduced expression for w_S. Hence, $t_S = t_{w_S w} t_{w^{-1}}$. If $s_1 \cdots s_l$ is a reduced expression for w, then $s_l \cdots s_1$ is a reduced expression for w^{-1}; so, $t_{w^{-1}} = t_w$ and therefore, $t_{w_S w} = t_S t_w^{-1}$. This gives

$$W(\mathbf{t}) = \sum_{w_S w \in W} t_{w_S w} = \sum_{w \in W} t_S t_w^{-1} = t_S W(\mathbf{t}^{-1}).$$

□

For each $T \subset S$, let $B_T := \{w \in W \mid l(wt) = l(w) + 1, \forall t \in T\}$.

B_T is a set of representatives for W/W_T.

Exercise. For each $T \subset S$, $W(\mathbf{t}) = B_T(\mathbf{t}) W_T(\mathbf{t})$.

Given a finite set T, define $\varepsilon(T) := (-1)^{|T|}$.

Exercise. (Möbius Inversion). Suppose f, g are two functions from the power set of a finite set S to an abelian gp s.t. for any $T \subseteq S$,

$$f(T) = \sum_{U \subseteq T} g(U).$$

Then for any $T \subseteq S$,

$$g(T) = \sum_{U \subseteq T} \varepsilon(T - U) f(U).$$

Recall $W^T := \{w \in W \mid \mathrm{In}(w) = T\}$.

Exercise. For any $T \subseteq S$,

$$W^T(\mathbf{t}) = W(\mathbf{t}) \sum_{U \subseteq T} \frac{\varepsilon(T-U)}{W_{S-U}(\mathbf{t})}.$$

Corollary. • *Suppose W is finite. Then*

$$t_S = W(\mathbf{t}) \sum_{T \subseteq S} \frac{\varepsilon(\sigma)}{W_T(\mathbf{t})}.$$

- *If W is infinite, then*

$$0 = \sum_{T \subseteq S} \frac{\varepsilon(T)}{W_T(\mathbf{t})}.$$

Corollary. (Rationality of growth series). $W(\mathbf{t}) = f(\mathbf{t})/g(\mathbf{t})$, *where* $f, g \in \mathbb{Z}[\mathbf{t}]$.

Two more formulas:

$$\frac{1}{W(\mathbf{t}^{-1})} = \sum_{T \in \mathcal{S}} \frac{\varepsilon(T)}{W_T(\mathbf{t})} \qquad \text{(Steinberg)}$$

$$\frac{1}{W(\mathbf{t})} = \sum_{T \in \mathcal{S}} \frac{1 - \chi(\mathsf{Lk}(\sigma_T, L))}{W_T(\mathbf{t})} \qquad \text{(Charney-D.)}$$

Corollary. $\frac{1}{W(\mathbf{1})} = \chi^{orb}(\Sigma/W)$, *where* $\mathbf{1}$ *denotes the constant* I*-tuple with all entries* $= 1$*.*

Definition. Let $\delta = \pm 1$. The rational function $W(\mathbf{t})$ is δ-*reciprocal* if $W(\mathbf{t}^{-1}) = \delta W(\mathbf{t})$.

Corollary. (Charney-D.) *Suppose* W *is type* HM^n *(i.e.,* L *is a* GHS^{n-1}*). Then* $W(\mathbf{t})$ *is* $(-1)^n$*-reciprocal*

Subexponential growth. If I is a singleton, write t for $\mathbf{t}$. So, $W(t)$ is a power series in one variable. Denote its radius of convergence by ρ.

Let G be a finitely generated gp, S a set of generators with $S = S^{-1}$. Take the word metric on G and let $b_n :=$ Card(ball of radius n).

G has *exponential growth* if $\lim \frac{\log b_n}{n} \neq 0$.

Definition. (The Følner Condition). $\forall A \subset G$ ($=$ Vert(Cay(G, S))), put $\partial A := \{a \in A \mid \exists s \in S$ s.t. $as \notin A\}$. G is *amenable* if $\forall \epsilon > 0, \exists$ a finite $A \subset G$ with Card$(\partial A) < \epsilon$ Card(A).

Proposition. *For (W, S), TFAE.*

(i) W *is amenable.*

(ii) $W \not\supseteq F_2$*, the free group on two generators.*

(iii) $\nexists$ *a finite index subgroup* Γ *and a surjection* $\Gamma \twoheadrightarrow F_2$ *(i.e.,* W *does not virtually map onto* F_2*).*

(iv) W *is virtually abelian.*

(v) (W, S) *decomposes as* $(W_0 \times W_1, S_0 \cup S_1)$ *where* W_1 *is finite and* W_0 *is a cocompact Euclidean reflection group.*

(vi) $\rho = 1$.

(vii) W *has subexponential growth.*

Proof. (i) $\implies$ (ii) is a standard fact.

(ii) $\implies$ (iii) is obvious.

(iii) $\implies$ (iv). (Margulis–Vinberg, Gonciulea).W is not virtually abelian $\implies$ subgroup $W \supset \Gamma \twoheadrightarrow F_2$.

(iv) $\implies$ (v). Σ has a $CAT(0)$ metric (Moussong). This implies any abelian subgroup of W is finitely generated. So, W is virtually free abelian. Suppose $\exists$ a rank n free abelian subgroup of finite index. Then W is a virtual PD^n-group. By Lecture 5,

W decomposes as in (v), where the complex Σ_0 for (W_0, S_0) is a $CAT(0)$ homology n-manifold. By the Flat Torus Theorem, the "min set " of the free abelian subgroup on Σ_0 is isometric to $\mathbf{R}^n$. Hence, $\Sigma_0 = \mathbf{R}^n$ and W_0 acts as an isometric reflection group on it.

(v) $\Longrightarrow$ (vi). Since a Euclidean reflection group is virtually free abelian, it has polynomial growth and therefore, the radius of convergence of its growth series is 1.

(vi) $\Longrightarrow$ (vii) is obvious.

(vii) $\Longrightarrow$ (i) by the Følner condition for amenability. $\square$

Buildings. A *building* consists of the following data:

- a set Φ,
- a Coxeter system (W, S),
- a collection of equivalence relations on Φ indexed by S.
- a function $\delta : \Phi \times \Phi \to W$.

This data must satisfy certain additional conditions explained below. First condition: $\forall s \in S$, each s-equivalence class contains at least two elements. The elements of Φ are *chambers*. Given $s \in S$, two chambers φ and φ' are *s-equivalent* if they are equivalent via the equivalence relation corresponding to s. If, in addition, $\varphi \neq \varphi'$, they are *s-adjacent*. A *gallery* is a sequence $(\varphi_0, \ldots, \varphi_n)$ of adjacent chambers; its *type* is the word $(s_1, \ldots, s_n)$ in the letters of S, where φ_{i-1} and φ_i are s_i-adjacent.

Given $T \subset S$, $(\varphi_0, \ldots, \varphi_n)$ is a T-*gallery* if each $s_i \in T$. The gallery is *reduced* if $w = s_1 \cdots s_n$ is a reduced expression.

Second condition for Φ to be a building: $\exists$ a W-*valued distance function* $\delta : \Phi \times \Phi \to W$. This means that there is a reduced gallery of type $(s_1, \ldots, s_n)$ from φ to φ' if and only if $s_1 \cdots s_n$ is a reduced expression for $\delta(\varphi, \varphi')$.

The s-*mirror* of a chamber φ is the s-equivalence class containing

φ. More generally, given a subset $T \subset S$, the T-*residue* of φ is the T-gallery connected component containing φ. Each such T-residue is naturally a building with associated Coxeter system (W_T, T). The residue is *spherical* if T is a spherical.

Example. A Coxeter gp W is itself a bldg. Put $\Phi := W$, define w and w' to be s-equivalent iff they lie in the same left coset of $\langle s \rangle$ $(= W_{\{s\}})$ and define $\delta(w, w') := w^{-1}w'$.

The *geometric realization* of Φ is defined by:

$$|\Phi| := (\Phi \times K)/\sim,$$

where $\sim$ is the equivalence relation defined by $(\varphi, x) \sim (\varphi', x') \iff x = x'$ and φ, φ' are in the same $S(x)$-residue. (Recall $S(x) := \{s \in S \mid x \in K_s\}$.)

Example. (*Projective planes*). Let k be a field with q elements. Φ is the set of complete flags of subspaces in k^3, i.e.,

$\Phi := \{(V^1, V^2) \mid V^1 \subset V^2\}$. W is the dihedral gp of order 6 and $S = \{s_1, s_2\}$. Two flags are s_i-*equivalent*, $i = 1, 2$, if they share a common i-dimensional subspace. K is the cone on an interval. The usual (spherical) geometric realization of Φ is a certain bipartite graph obtained by gluing together copies of this interval. . At a vertex of a given edge there are precisely q other adjacent edges. Our version $|\Phi|$ is the cone on this graph

Example. (*Trees*). W is the infinite dihedral group (s.t. $S =$

$\{s_1, s_2\}$). Any tree is bipartite. Suppose T is a tree with vertices labeled by S. Also suppose no vertex of T is of valence 1. Let $\Phi := \text{Edge}(T)$. For $i = 1, 2$, call two edges s_i-*equivalent* if they meet at a vertex of type s_i. An $\{s_i\}$-residue is the set of edges in the star of a vertex of type s_i. A gallery in Φ corresponds to an edge path in T. The type of the gallery is the word obtained by taking the types of the vertices crossed by the corresponding edge path. This word is reduced if and only if the edge path does not backtrack. Given two edges φ, φ' of T, there is a (unique) minimal gallery connecting them. The corresponding word represents an element of $w \in W$ and $\delta(\varphi, \varphi') := w$. Not surprisingly, the geometric realization of Φ is T.

A building Φ of type (W, S) has *finite thickness* if for each $s \in S$, each s-equivalence class is finite. If Φ has finite thickness, then it follows from the existence of a W-valued distance function that each of its spherical residues is finite and hence, that $|\Phi|$ is a locally finite cell cx.

Let us say that Φ is *regular* if for each $s \in S$, the s-equivalence classes have constant cardinality. When finite, we denote this number by $q_s + 1$. It can be shown that if s and s' are conjugate in W, then $q_s = q_{s'}$. Let I be the set of conjugacy classes of elements in S. Then for any regular building Φ, the integers q_s

define an I-tuple $\mathbf{q}$ called the *thickness* of Φ. For example, if $\Phi = W$, then $\mathbf{q} = \mathbf{1}$.

Example. (*Regular right-angled buildings*). For any right-angled Coxeter system (W, S) and any S-tuple $\mathbf{q} = (q_s)_{s \in S}$ of positive integers, there is a regular building Φ of type (W, S) with thickness $\mathbf{q}$. In the case where W is the infinite dihedral group this is well-known: the building is a (bipartite) tree with edge set Φ, it is "regular" in the sense that for each $s \in S$ there are exactly

$q_s + 1$ edges meeting at each vertex of type s.

In the general case, the construction goes as follows. For each $s \in S$, choose a finite group Γ_s with $\mathrm{Card}(\Gamma_s) = q_s + 1$ and let Γ be the "graph product" of the $(\Gamma_s)_{s \in S}$ where the graph is the 1-skeleton of L. In other words, Γ is the quotient of the free product of the $(\Gamma_s)_{s \in S}$ by the normal subgroup generated by all commutators $[g_s, g_t]$ with $g_s \in \Gamma_s$, $g_t \in \Gamma_t$ and $m_{st} = 2$. We get a bldg $\Phi = \Gamma$ with two elements $g, g' \in \Gamma$ in an s-equivalent iff they determine same coset in Γ/Γ_s.

Relationship to growth series.

Proposition. (Serre). *Suppose* $\Gamma \subset \mathrm{Aut}(\Phi)$ *is a discrete subgp which acts transitively on* Φ. *Then*

$$\chi^{orb}(|\Phi|/\Gamma) = \frac{1}{W(\mathbf{q})}.$$

Hecke algebras

$\mathbb{R}^{(W)}$ is the $\mathbb{R}$-vector space on W with basis $(e_w)_{w\in W}$ (i.e., $\mathbb{R}^{(W)} :=$ {finitely supported functions $\mathbb{R} \to W$}). $\mathbb{R}W$ means $\mathbb{R}^{(W)}$ with its structure as the group algebra of W. A "Hecke algebra" is a certain deformation of $\mathbb{R}W$.

As before, $i : S \to I$ is a function such that $i(s) = i(s')$ whenever s and s' are conjugate and given an I-tuple $\mathbf{q} = (q_i)_{i\in I} \in A^I$, write q_s for $q_{i(s)}$.

Proposition. (Exercise in Bourbaki). *Given* $\mathbf{q} \in \mathbb{R}^I$, $\exists!$ *algebra structure on* $\mathbb{R}^{(W)}$ *s.t.*

$$e_s e_w = \begin{cases} e_{sw}, & \text{if } l(sw) > l(w); \\ q_s e_{sw} + (q_s - 1)e_w, & \text{if } l(sw) < l(w), \end{cases}$$

for all $s \in S$ *and* $w \in W$.

We will use the notation $\mathbb{R}_{\mathbf{q}}W$ to denote this algebra and call it the *Hecke algebra* of W associated to the *multiparameter* $\mathbf{q}$.

Note $\mathbb{R}_{\mathbf{1}}W = \mathbb{R}W$.

Exercise. The following formulas hold in $\mathbb{R}_{\mathbf{q}}W$:

- $\forall u, v \in W$ with $l(uv) = l(u) + l(v)$, $e_u e_v = e_{uv}$.
- $\forall s \in S$, $e_s^2 = (q_s - 1)e_s + q_s$.
- (Artin relations). For any two distinct elements $s, t \in S$ with

$$m_{st} \neq \infty,\ \underbrace{e_s e_t \ldots}_{m_{st}} = \underbrace{e_t e_s \ldots}_{m_{st}}\ .$$

The relationship of Hecke algebras to buildings. Suppose a bldg Φ "comes from a (B, N)-pair." This means that $\exists$ a chamber-transitive gp G, that the stabilizer B of a given chamber φ acts simply transitively on $\{$"apartments" containing $\varphi\}$. Hence, $\Phi \cong G/B$ and $W \cong B\backslash G/B$. Pullback G/B and $B\backslash G/B$ to subalgebras L and H, respectively, of compactly supported continuous functions on G (with multiplication given by convolution). Since G/B is discrete, we can think of L as finitely supported functions on G/B and $H = L^B$.

Lemma. *Suppose, as above, that Φ comes from a (B, N) pair. Let $\mathbf{q}$ be its thickness vector. Then $H \cong \mathbb{R}_{\mathbf{q}}W$.*

Lecture 9:

Background on L^2-cohomology

Hilbert modules and von Neumann algebras. Γ a countable discrete gp. $L^2(\Gamma)$ the vector space of square-summable, real-valued functions on Γ:

$$L^2(\Gamma) := \{f : \Gamma \to \mathbb{R} \mid \sum f(\gamma)^2 < \infty\},$$

where the sum is over all $\gamma \in \Gamma$. $L^2(\Gamma)$ is a Hilbert space with inner product:

$$f \cdot f' := \sum_{\gamma \in \Gamma} f(\gamma) f'(\gamma).$$

The group algebra $\mathbb{R}\Gamma$ is identified with the dense subspace of $L^2(\Gamma)$ consisting of the functions with finite support. For each $\gamma \in \Gamma$,

$$e_\gamma(\gamma') := \begin{cases} 1, & \text{if } \gamma = \gamma'; \\ 0, & \text{otherwise.} \end{cases}$$

So, $(e_\gamma)_{\gamma \in \Gamma}$ is an orthonormal basis for $L^2(\Gamma)$. There is an left-action of Γ on $L^2(\Gamma)$ defined by left translation, i.e.,

$$(\gamma \cdot f)(\gamma') := f(\gamma\gamma').$$

This action is the (left) *regular representation* of Γ.

Suppose V and V' are Hilbert spaces with orthogonal Γ-actions. A *map* from V to V' means a Γ-equivariant bounded linear map $f : V \to V'$. The kernel of a map is always closed; however, the image need not be. The map is a *weak surjection* if its image is dense in V'; it is a *weak isomorphism* if, in addition, it is injective.

A sequence of maps

$$\cdots \longrightarrow V \xrightarrow{f} V' \xrightarrow{g} V'' \cdots$$

is *weakly exact* at V' if $\overline{\mathrm{Im}\, f} = \mathrm{Ker}\, g$.

Hilbert Γ-modules. A Hilbert space with orthogonal Γ-action is a *Hilbert* Γ*-module* if it is isomorphic to a closed Γ-stable subspace of a finite (orthogonal) direct sum of copies of $L^2(\Gamma)$ with the diagonal Γ-action.

Example. If F is a finite subgroup of Γ, then $L^2(\Gamma/F)$, the space of square summable functions on Γ/F, can be identified with the subspace of $L^2(\Gamma)$ consisting of the square summable functions on Γ which are constant on each coset. This subspace is clearly closed and Γ-stable; hence, $L^2(\Gamma/F)$ is a Hilbert Γ-module.

Example. (*Completed tensor product*) Suppose $\Gamma = \Gamma_1 \times \Gamma_2$ and that V_j is a Hilbert Γ_j-module, for $j = 1, 2$. The L^2-completion of the tensor product is denoted $V_1 \widehat{\otimes} V_2$. It is a Hilbert Γ-module.

Lemma. *If two Hilbert* Γ*-modules are weakly isomorphic, then they are* Γ*-isometric.*

Induced representations. Suppose H is a subgroup of Γ and that V is a Hilbert H-module. The *induced representation*, $\mathrm{Ind}_H^\Gamma(V)$, is the completion of $\mathbb{R}\Gamma \otimes_{\mathbb{R}H} V$. It is easy to see that if V is a Hilbert H-module, then $\mathrm{Ind}_H^\Gamma(V)$ is a Hilbert Γ-module. For example, if F is a finite subgroup of Γ and $\mathbb{R}$ denotes the trivial 1-dimensional representation of F, then $\mathrm{Ind}_F^\Gamma(\mathbb{R})$ can be identified with $L^2(\Gamma/F)$.

The von Neumann algebra $\mathcal{N}(\Gamma)$. $L^2(\Gamma)$ is an $\mathbb{R}\Gamma$-bimodule. Here are three equivalent definitions of the *von Neumann algebra* $\mathcal{N}(\Gamma)$ associated to the group algebra $\mathbb{R}\Gamma$.

- $\mathcal{N}(\Gamma)$ is the algebra of all maps from $L^2(\Gamma)$ to itself.

- $\mathcal{N}(\Gamma)$ is the double commutant of the right $\mathbb{R}\Gamma$-action on $L^2(\Gamma)$.

- $\mathcal{N}(\Gamma)$ is the weak closure of the algebra of operators $\mathbb{R}(\Gamma)$ acting from the right on $L^2(\Gamma)$.

The Γ-trace. The Γ-*trace* of an element $\varphi \in \mathcal{N}(\Gamma)$ defined by

$$\mathrm{tr}_\Gamma(\varphi) := \varphi(e_1) \cdot e_1$$

(where $e_1 \in L^2(\Gamma)$ is the basis element corresponding to $1 \in \Gamma$).

Standard arguments show:

- $\mathrm{tr}_\Gamma(\varphi) = \mathrm{tr}_\Gamma(\varphi^*)$ (where φ^* is the adjoint of φ) and

- $\mathrm{tr}_\Gamma(\varphi\psi) = \mathrm{tr}_\Gamma(\varphi)\,\mathrm{tr}_\Gamma(\psi)$.

Given $n \in \mathbb{N}$, let $L^2(\Gamma)^n$ denote the orthogonal direct sum of n copies of $L^2(\Gamma)$ and let $M_n(\mathcal{N}(\Gamma))$ denote the set of $n \times n$ matrices with coefficients in $\mathcal{N}(\Gamma)$. Given $\Phi = (\varphi_{ij}) \in M_n(\mathcal{N}(\Gamma))$, define

$$\mathrm{tr}_\Gamma(\Phi) := \sum_{i=1}^{n} \mathrm{tr}_\Gamma(\varphi_{ii}).$$

$\mathrm{tr}_\Gamma(\ )$ has the usual properties. Suppose $\Phi, \Psi \in M_n(\mathcal{N}(\Gamma))$.

Then

- $\mathrm{tr}_\Gamma(\Phi) = \mathrm{tr}_\Gamma(\Phi^*)$.

- $\mathrm{tr}_\Gamma(\Phi \circ \Psi) = \mathrm{tr}_\Gamma(\Psi \circ \Phi)$.

- Suppose Φ is self-adjoint and idempotent. Then $\text{tr}_\Gamma(\Phi) \geq 0$ with equality if and only if $\Phi = 0$.

Similarly, given any Hilbert Γ-module V isomorphic to $L^2(\Gamma)^n$ and any self-map Φ of V, one can define $\text{tr}_\Gamma(\Phi)$.

von Neumann dimension. Let V be a Hilbert Γ-module. Choose an embedding of V as a closed, Γ-stable subspace of $L^2(\Gamma)^n$ for some $n \in \mathbb{N}$. Let $p_V : L^2(\Gamma)^n \to L^2(\Gamma)^n$ denote orthogonal projection onto V. The *von Neumann dimension* of V (also called its Γ-*dimension*) is denoted $\dim_\Gamma(V)$ and defined by

$$\dim_\Gamma(V) := \mathrm{tr}_\Gamma(p_V).$$

If $E \subset L^2(\Gamma)$ is a not neccessarily closed Γ-stable subspace of $L^2(\Gamma)^n$, then put $\dim_\Gamma(E) := \dim_\Gamma(\overline{E})$.

We list some standard properties of $\dim_\Gamma(V)$:

- $\dim_\Gamma(V) \in [0, \infty)$.

- $\dim_\Gamma(V) = 0$ if and only if $V = 0$.

- If Γ is the trivial group (so that the Hilbert space V is finite dimensional), then $\dim_\Gamma(V) = \dim(V)$.

- $\dim_\Gamma(L^2(\Gamma)) = 1$.

- $\dim_\Gamma(V \oplus W) = \dim_\Gamma(V) + \dim_\Gamma(W)$.

- If $f : V \to W$ is a map of Hilbert Γ-modules, then

$$\dim_\Gamma(V) = \dim_\Gamma(\mathrm{Ker}\, f) + \dim_\Gamma(\overline{\mathrm{Im}\, f}).$$

- If $f : V \to W$ is a map of Hilbert Γ-modules and $f^* : W \to V$ its adjoint, then $\mathrm{Ker}\, f$ and $\overline{\mathrm{Im}\, f^*}$ are orthogonal complements in V. Hence,

$$\dim_\Gamma(V) = \dim_\Gamma(\mathrm{Ker}\, f) + \dim_\Gamma(\overline{\mathrm{Im}\, f^*}).$$

So, $\dim_\Gamma(\overline{\mathrm{Im}\, f}) = \dim_\Gamma(\overline{\mathrm{Im}\, f^*})$.

- If $0 \to V_n \to \cdots \to V_0 \to 0$ is a weak exact sequence of Hilbert Γ-modules, then

$$\sum_{i=0}^{n} (-1)^i \dim_\Gamma(V_i) = 0.$$

- If H is a subgroup of finite index m in Γ, then

  $\dim_H(V) = m \dim_\Gamma(V)$.

- If Γ is finite, then $\dim_\Gamma(V) = \frac{1}{|\Gamma|} \dim(V)$.

- If H is a subgroup of Γ and W is a Hilbert H-module, then

$$\dim_\Gamma(\mathrm{Ind}_H^\Gamma(W)) = \dim_H(W).$$

- If F is a finite subgroup of Γ, then $\dim_\Gamma(L^2(\Gamma/F)) = \frac{1}{|F|}$.

- Suppose $\Gamma = \Gamma_1 \times \Gamma_2$ and that V_j is a Hilbert Γ_j-module for $j = 1, 2$. Then

$$\dim_\Gamma(V_1 \widehat{\otimes} V_2) = \dim_{\Gamma_1}(V_1) \dim_{\Gamma_2}(V_2).$$

L^2-**chains**. Suppose X is a proper Γ-CW-complex and X/Γ is compact. This implies that there are only finitely many Γ-orbits of cells in X.

$$C_*(X) := \text{ the usual cellular chain complex on } X.$$

An element of $C_i(X)$ (a i-*chain*) is a finitely supported function φ from the set of oriented i-cells in X to $\mathbb{Z}$ satisfying $\varphi(\overline{e}) = \varphi(e)$ (where e and $\overline{e}$ denote the same cell but with opposite orientations). Set

$$L^2C_i(X) := L^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_i(X).$$

An element of $L^2C_i(X)$ is an L^2-*chain*; it is an infinite chain with square summable coefficients. The above formula means that $L^2C_i(X)$ is the group of Γ-equivariant chains with coefficients in the Γ-module $L^2(\Gamma)$. The definition of the space of L^2-cochains on X is exactly the same, i.e., $L^2C^i(X) := L^2C_i(X)$.

If c is an i-cell of X, then the space of L^2-chains which are supported on the Γ-orbit of c can be identified with $L^2(\Gamma/\Gamma_c)$.

Since Γ_c is finite, $L^2(\Gamma/\Gamma_c)$ is a Hilbert Γ-module. Since there are a finite number of Γ-orbits of i-cells, $C_i(X)$ is the direct sum of a finite number of such subspaces; hence, it is a Hilbert Γ-module.

Unreduced and reduced L^2-homology. Define the boundary $\partial_i : L^2C_i(X) \to L^2C_{i-1}(X)$ and the coboundary $\delta^i : L^2C^i(X) \to L^2C^{i+1}(X)$ by the usual formulas. They are maps of Hilbert Γ-modules. δ^i and ∂_{i+1} are the adjoints of one another. Define subspaces of $L^2C_i(X)$:

$$Z_i(X) := \mathrm{Ker}\,\partial_i \qquad Z^i(X) := \mathrm{Ker}\,\delta^i$$

$$B_i(X) := \mathrm{Im}\,\partial_{i+1} \qquad B^i(X) := \mathrm{Im}\,\delta^{i-1}$$

the L^2-*cycles*, -*cocycles*, -*boundaries* and -*coboundaries*, respectively. The corresponding quotient spaces

$$L^2H_i(X) := Z_i(X)/B_i(X)$$

$$L^2H^i(X) := Z^i(X)/B^i(X)$$

are the *unreduced* L^2*-homology* and *-cohomology groups*, respectively. (In other words, $L^2H_i(X)$ is the ordinary equivariant homology of X with coefficients in $L^2(\Gamma)$, i.e., $L^2H_i(X) = H_i^{\Gamma}(X; L^2(\Gamma))$.) Since the subspaces $B_i(X)$ and $B^i(X)$ need not be closed, these quotient spaces need not be isomorphic to Hilbert spaces. Let $\overline{B_i}(X)$ (resp., $\overline{B^i}(X)$) denote the closure of $B_i(X)$ (resp., $B^i(X)$). The *reduced* L^2*-homology* and

-*cohomology groups* are defined by:

$$L^2\mathcal{H}_i(X) := Z_i(X)/\overline{B_i}(X)$$

$$L^2\mathcal{H}^i(X) := Z^i(X)/\overline{B^i}(X).$$

They are Hilbert Γ-modules (since each can be identified with the orthogonal complement of a closed Γ-stable subspace in a closed Γ-stable subspace of $C_i(X)$).

Hodge decomposition. Since $\delta^{i-1}(x) \cdot y = x \cdot \partial_i(y)$, $\forall x \in$

$L^2C^{i-1}(X)$ and $y \in L^2C_i(X)$, we have orthogonal direct sum decompositions:

$$L^2C_i(X) = \overline{B_i}(X) \oplus Z^i(X)$$

$$L^2C_i(X) = \overline{B^i}(X) \oplus Z_i(X).$$

Since $\delta^{i-1}(x) \cdot \partial_{i+1}(y) = x \cdot \partial_i \partial_{i+1}(y) = 0$, the subspaces $\overline{B_i}(X)$ and $\overline{B^i}(X)$ are orthogonal. This gives the *Hodge decomposition*:

$$C_i(X) = \overline{B_i}(X) \oplus \overline{B^i}(X) \oplus (Z_i(X) \cap Z^i(X)).$$

It follows that the reduced L^2-homology and L^2-cohomology groups can both be identified with the subspace $Z_i(X) \cap Z^i(X)$. We denote this intersection again by $L^2\mathcal{H}_i(X)$ and call it the subspace of *harmonic* i-cycles. Thus, an i-chain is harmonic if and only if it is simultaneously a cycle and a cocycle.

The *combinatorial Laplacian* $\Delta : C_i(X) \to C_i(X)$ is defined by

$$\Delta = \delta^{i-1}\partial_i + \partial_{i+1}\delta^i.$$

One checks that $L^2\mathcal{H}_i(X) = \mathsf{Ker}\,\Delta$.

L^2 **algebraic topology.** Suppose (X, Y) is a pair of Γ-CW-complexes. The reduced L^2-(co)homology groups $L^2\mathcal{H}_i(X, Y)$ are defined in the usual manner. Versions of most of the Eilenberg-Steenrod Axioms hold for $L^2\mathcal{H}_*(X, Y)$. We list some standard properties.

Functoriality. For $i = 1, 2$, suppose (X_i, Y_i) is a pair of Γ-CW-complexes and that $f\colon (X_1, Y_1) \to (X_2, Y_2)$ is a Γ-map. Then there is an induced map $f_* : L^2\mathcal{H}_i(X_1, Y_1) \to L^2\mathcal{H}_i(X_2, Y_2)$ giving a functor from pairs of Γ-complexes to Hilbert Γ-modules. Moreover, if $f' : (X_1, Y_1) \to (X_2, Y_2)$ is another Γ-map which is homotopic to f (not necessarily Γ-homotopic), then $f_* = f'_*$.

Exact sequence of a pair. The sequence of a pair (X, Y),

$$\to L^2\mathcal{H}_i(Y) \to L^2\mathcal{H}_i(X) \to L^2\mathcal{H}_i(X, Y) \to$$

is weakly exact.

Excision. Suppose that (X, Y) is a pair of Γ-CW-complexes and that U is a Γ-stable subset of Y such that $Y - U$ is a subcomplex. Then the inclusion $(X - U, Y - U) \to (X, Y)$ induces an isomorphism:

$$L^2\mathcal{H}_i(X - U, Y - U) \cong L^2\mathcal{H}_i(X, Y).$$

Mayer-Vietoris sequences. Suppose $X = X_1 \cup X_2$, where X_1

and X_2 are Γ-stable subcomplexes of X. Then $X_1 \cap X_2$ is also Γ-stable and the Mayer-Vietoris sequence,

$$\to L^2\mathcal{H}_i(X_1 \cap X_2) \to L^2\mathcal{H}_i(X_1) \oplus \mathcal{H}_i(X_2) \to L^2\mathcal{H}_i(X) \to$$

is weakly exact.

Twisted products and the induced representation. Suppose that H is a subgroup of Γ and that Y is a space on which H acts. The *twisted product*, $\Gamma \times_H Y$, is the quotient space of $\Gamma \times Y$ by the H-action defined by $h(g,y) = (gh^{-1}, hy)$. It is a left Γ-space and a Γ-bundle over Γ/H. Since Γ/H is discrete, $\Gamma \times_H Y$ is a disjoint union of copies of Y, one for each element of Γ/H. If Y is an H-CW-complex, then $\Gamma \times_H Y$ is a Γ-CW-complex and the following formula obviously holds:

$$L^2\mathcal{H}_i(\Gamma \times_H Y) \cong \mathrm{Ind}_H^\Gamma(L^2\mathcal{H}_i(Y)).$$

Künneth Formula. Suppose $\Gamma = \Gamma_1 \times \Gamma_2$ and that X_j is a Γ_j-

CW-complex for $j = 1, 2$. Then $X_1 \times X_2$ is a Γ-CW-complex and

$$L^2\mathcal{H}_k(X_1 \times X_2) \cong \sum_{i+j=k} L^2\mathcal{H}_i(X_1)\widehat{\otimes} L^2\mathcal{H}_j(X_2),$$

where $\widehat{\otimes}$ denotes the completed tensor product.

(Co)homology in dimension $\mathbf{0}$. An element of $C_0(X)$ is an L^2 function on the set of vertices of X; it is a 0-cocycle if and only if it takes the same value on the endpoints of each edge. Hence, if X is connected, any 0-cocycle is constant. If, in addition, Γ is infinite (so that the 1-skeleton of X is infinite), then this constant must be 0. So, when X is connected and Γ is infinite, $L^2H^0(X) = L^2\mathcal{H}^0(X) = 0$. Hence, $L^2\mathcal{H}_0(X) = 0$.

The unreduced homology $L^2H_0(X)$ need not be 0. For example, if $X = \mathbb{R}$, cellulated as the union of intervals $[n, n+1]$, and $\Gamma = \mathbf{C}_\infty$, then any vertex of $\mathbb{R}$ is an L^2-0-cycle which is not L^2-

boundary. (A vertex bounds a half-line which can be thought of as an infinite 1-chain but this 1-chain is not square summable.) In fact, if Γ is infinite, then $L^2H_0(X) = 0$ if and only if Γ is not amenable.

The top-dimensional homology of a pseudomanifold. Suppose that an n-dimensional, regular Γ- cell complex X is a pseudomanifold. This means that each $(n-1)$-cell is contained in precisely two n-cells. If a component of the complement of the $(n-2)$-skeleton is not orientable, then it does not support a nonzero n-cycle (with coefficients in $\mathbb{R}$). If such a component is orientable, then any n-cycle supported on it is a constant multiple of the n-cycle with all coefficients are $+1$. If the component has an infinite number of n-cells, then this n-cycle does not have square summable coefficients. Hence, if each component of the complement of the $(n-2)$-skeleton is either infinite or nonorientable, then $L^2H_n(X) = 0$. In particular, if the complement of the $(n-2)$-skeleton is connected and if Γ is infinite, then $L^2H_n(X) = 0$.

L^2-**Betti numbers and Euler characteristics** Define the i^{th} L^2-*Betti number* of X by

$$L^2b^i(X;\Gamma) := \dim_\Gamma L^2\mathcal{H}^i(X).$$

The L^2-*Euler characteristic* of X is defined by

$$L^2\chi(X;\Gamma) := \sum_{i=0}^{\infty} (-1)^i L^2 b^i(X;\Gamma).$$

As before,

$$\chi^{orb}(X/\Gamma) := \sum_{\text{orbits of cells}} \frac{(-1)^{\dim c}}{|\Gamma_c|},$$

where $|\Gamma_c|$ denotes the order of the stabilizer of the cell c. We note that if the Γ-action is free, then $\chi^{orb}(X/\Gamma)$ is just the ordinary Euler characteristic $\chi(X/\Gamma)$.

Atiyah's Formula.

Theorem. (Atiyah). $\chi^{orb}(X/\Gamma) = L^2 b^i(X;\Gamma)$.

Conjecture. (The Dodziuk–Singer Conjecture). *Suppose* $X = \widetilde{M}^n$ *a contractible mfld. Then*

$$L^2 b^i(\widetilde{M}^n;\Gamma) = 0, \quad \forall i \neq \frac{n}{2}.$$

Observation. Singer Conj. $\Longrightarrow$ Euler Char. Conj.

Proof. Suppose $n = 2k$. Singer Conj. $\Longrightarrow$ only $L^2 b_k \neq 0$.

Atiyah's Formula gives: $(-1)^k L^2 b^i(\widetilde{M}^{2k};\Gamma) = \chi^{orb}(\widetilde{M}^{2k}/\Gamma)$. So, $(-1)^k \chi^{orb}(\widetilde{M}^{2k}/\Gamma) \geq 0$. $\square$

A version of Singer Conj for Coxeter groups. Suppose L is a triangulation of a GHS^{n-1} as a flag cx and W the associated right-angled Coxeter gp.

Conjecture. $L^2 b_i(\Sigma; W) = 0, \ \forall i \neq \frac{n}{2}$ *and for* $n = 2k$,

$$L^2 b_k(\Sigma; W) = (-1)^k \chi^{orb}(\Sigma/W) := (-1)^k \kappa(L) \geq 0.$$

Recall

$$\kappa(L) := 1 + \sum_{i=0}^{\dim L} \left(-\frac{1}{2}\right)^{i+1} f_i.$$

Lecture 10:

Weighted L^2-cohomology

joint work with

Jan Dymara, Tadeusz Januszkiewicz and Boris Okun

(W,S) a Coxeter system, Σ the associated cx. Recall

$i : S \to I$	the index function
$\mathbf{q} \in \mathbb{R}^I$	the multiparameter
$\mathbb{R}_{\mathbf{q}}W$	the Hecke algebra
$W(\mathbf{t})$	the growth series

$$\varepsilon_w := (-1)^{l(w)} \quad \text{and} \quad q_w := q_{s_1} \cdots q_{s_k},$$

whenever $s_1 \cdots s_k$ is a reduced expression for w.

From now on assume $\mathbf{q} \in (0, \infty)^I$. $\mathcal{R}$ is the region of convergence of $W(\mathbf{t})$ and $\mathcal{R}^{-1} := \{\mathbf{q} \mid \mathbf{q}^{-1} \in \mathcal{R}\}$.

Goal: Define $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$. It has the following

Properties:

- It $\mathbf{q} = \mathbf{1}$, then it is ordinary $L^2\mathcal{H}^*(\Sigma)$.

- There are "weighted L^2-Betti numbers," $b^i_{\mathbf{q}}(\Sigma)$ (= "von Neumann dim of $L^2\mathcal{H}^i(\Sigma)$ w.r.t. the Hecke algebra").

- The "weighted L^2-Euler characteristic," $\chi_{\mathbf{q}}(\Sigma)$ is $= 1/W(\mathbf{q})$.

- If Φ is a bldg with chamber transitive automorphism gp G and thickness $\mathbf{q}$, then the $b^i_{\mathbf{q}}(\Sigma)$ are the L^2-Betti numbers of Φ w.r.t. G.

- If $\mathbf{q} \in \mathcal{R}$, then $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ vanishes except in dimension 0 (like ordinary cohomology).

- If $\mathbf{q} \in \mathcal{R}^{-1}$, then $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ "looks like" cohomology with compact supports.

An inner product on $\mathbb{R}^{(W)}$.

$$\mathbb{R}^{(W)} := \{\text{finitely supported functions } W \to \mathbb{R}\}$$

$$\langle e_v, e_w \rangle_{\mathbf{q}} := q_w \delta_{vw}$$

$$L^2_{\mathbf{q}}(W) := \text{completion of } \mathbb{R}^{(W)}.$$

The anti-involution $*$. $e_w \to e_{w^{-1}}$ extends to a linear endomorphism $*$ of $\mathbb{R}_{\mathbf{q}}W$, i.e.,

$$\left(\sum a_w e_w\right)^* := \sum a_{w^{-1}} e_w.$$

The next proposition shows that the algebra of operators $\mathbb{R}_{\mathbf{q}}W$ on $L^2_{\mathbf{q}}(W)$ satisfies the necessary conditions to be completable to a von Neumann algebra of operators. Its proof is a series of straightforward computations.

Proposition. *The inner product defined above and map* $*$ *give* $\mathbb{R}_{\mathbf{q}}W$ *the structure of a "Hilbert algebra," i.e., the following properties hold:*

- $(xy)^* = y^* x^*$,

- $\langle x, y \rangle_{\mathbf{q}} = \langle y^*, x^* \rangle_{\mathbf{q}}$,

- $\langle xy, z \rangle_{\mathbf{q}} = \langle y, x^* z \rangle_{\mathbf{q}}$,

- *for any* $x \in \mathbb{R}_{\mathbf{q}}W$, *left translation by* x, $L_x : \mathbb{R}_{\mathbf{q}}W \to \mathbb{R}_{\mathbf{q}}W$, *defined by* $L_x(y) = xy$, *is continuous,*

- *the products* xy *over all* $x, y \in \mathbb{R}_{\mathbf{q}}W$ *are dense in* $\mathbb{R}_{\mathbf{q}}W$.

The Hecke-von Neumann algebra.

$\mathcal{N}_{\mathbf{q}} =$ a completion of $\mathbb{R}_{\mathbf{q}}W$

$:= \{\mathbb{R}_{\mathbf{q}}W$-equivariant bounded linear operators on $L^2_{\mathbf{q}}(W)\}$

von Neumann trace. For $\phi \in \mathcal{N}_{\mathbf{q}}$, set

$$\mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(\phi) = \langle \phi(e_1), e_1 \rangle_{\mathbf{q}}.$$

For $\Phi = (\phi_{ij}) \in M_m(\mathcal{N}_{\mathbf{q}})$, set

$$\mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(\Phi) = \sum \mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(\phi_{ii}).$$

von Neumann dimension. Given a $\mathbb{R}_{\mathbf{q}}W$-stable, closed subspace $V \subset \oplus L^2_{\mathbf{q}}(W)$, let $p_V : \oplus L^2_{\mathbf{q}}(W) \to \oplus L^2_{\mathbf{q}}(W)$ be orthogonal projection onto V. Define

$$\dim_{\mathcal{N}_{\mathbf{q}}} V = \mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(p_V) \in [0, \infty).$$

Idempotents in $\mathcal{N}_{\mathbf{q}}$. For $T \subset S$, define

$$a_T := \frac{1}{W_T(\mathbf{q})} \sum_{w \in W_T} e_w,$$

$$h_T := \frac{1}{W_T(\mathbf{q}^{-1})} \sum_{w \in W_T} \varepsilon_w q_w^{-1} e_w.$$

Exercise.

- $a_T \in \mathcal{N}_{\mathbf{q}} \iff \mathbf{q} \in \mathcal{R}_T$ and $(a_T)^2 = a_T$.

- $h_T \in \mathcal{N}_{\mathbf{q}} \iff \mathbf{q} \in \mathcal{R}_T^{-1}$ and $(h_T)^2 = h_T$,

where $\mathcal{R}_T =$ region of convergence of $W_T(\mathbf{t})$.

Some subspaces of $L^2_{\mathbf{q}}(W)$. $\forall s \in S$, define

$$A_s := \operatorname{Im} a_s \quad \text{and} \quad H_s := \operatorname{Im} h_s.$$

Exercise. A_s and H_s are orthogonal complements.

$\forall T \subset S$, define

$$A_T := \bigcap_{s \in T} A_s \quad \text{and} \quad H_T := \bigcap_{s \in T} H_s.$$

Exercise. $A_T = \operatorname{Im} a_T$ if $\mathbf{q} \in \mathcal{R}_T$ and is 0 otherwise. $H_T = \operatorname{Im} h_T$ if $\mathbf{q} \in \mathcal{R}_T^{-1}$ and is 0 otherwise.

Define

$$D_T := \left(\sum_{U \supsetneq T} A_{S-U} \right)^{\perp} \bigcap A_{S-T}.$$

Decomposition Theorem. *We have the following direct sum decompositions of $\mathcal{N}_{\mathbf{q}}$-modules.*

- *If $\mathbf{q} \in \mathcal{R}$, then*

$$L^2_{\mathbf{q}} = \overline{\bigoplus_{T \in \mathcal{S}} D^T}.$$

- *If* $\mathbf{q} \in \mathcal{R}^{-1}$, *then*

$$L^2_{\mathbf{q}} = \overline{\bigoplus_{T \in \mathcal{S}} D^{S-T}}.$$

Recall $\mathcal{U} = \mathcal{U}(W, X) := (W \times X)/\sim$.

Cellular cochains.

$$\mathcal{E}_k := \{k\text{-cells in } \mathcal{U}\}$$

$$C^k(\mathcal{U}) := \{k\text{-cochains on } \mathcal{U}\}$$

$$:= \{\text{functions on } \mathcal{E}_k\}$$

$$= \{ \sum_{\text{infinite}} a_\sigma \ \sigma \mid \sigma \in \mathcal{E}_k\}$$

$$C^k_c(\mathcal{U}) := \{ \sum_{\text{finite}} a_\sigma \ \sigma\}$$

How about weighted $L^2_{\mathbf{q}}$-cochains?

Given $\sigma \in \mathcal{E}_k$, let $d(\sigma)$ be the shortest $w \in W$ s.t. $w^{-1}\sigma \subset X$.

Define an inner product on $C_c^k(\mathcal{U})$ by $\langle \sigma, \tau \rangle_{\mathbf{q}} := q_{d(\sigma)} \delta_{\sigma\tau}$.

$$L^2_{\mathbf{q}} C^k(\mathcal{U}) := \text{ the completion of } C_c^k(\mathcal{U})$$

$L^2_{\mathbf{q}} C^*(\mathcal{U})$ is a $\mathcal{N}_{\mathbf{q}}$-module and $\delta : L^2_{\mathbf{q}} C^k(\mathcal{U}) \to L^2_{\mathbf{q}} C^{k+1}(\mathcal{U})$ is a map of $\mathcal{N}_{\mathbf{q}}$-modules.

$$L^2_{\mathbf{q}}\mathcal{H}^k(\mathcal{U}) \qquad \text{reduced } L^2_{\mathbf{q}}\text{-cohomology} \quad := \mathrm{Ker}\,\delta/\overline{\mathrm{Im}\,\delta}$$

$$b^k_{\mathbf{q}}(\mathcal{U}) := \dim L^2_{\mathbf{q}}\mathcal{H}^k(\mathcal{U})$$

$$\chi_{\mathbf{q}}(\mathcal{U}) := \sum (-1)^k b^k_{\mathbf{q}}(\mathcal{U})$$

Theorem. (Dymara). $\chi_{\mathbf{q}}(\Sigma) = \frac{1}{W(\mathbf{q})}$

Theorem. (Dymara). *If* $\mathbf{q} \in \mathcal{R}$, *then* $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ *is concentrated in dimension* 0*.*

Theorem. *The* $b^k_{\mathbf{q}}(\mathcal{U})$ *are continuous in* $\mathbf{q}$.

Theorem. (Dymara) *Suppose* Φ *is a building of type* (W, S) *with a chamber transitive automorphism group* G. *Then its* L^2*-Betti number (with respect to* G*),* $L^2b^k(\Phi; G)$*, is equal to* $b^k_{\mathbf{q}}(\Sigma)$.

Here $\mathbf{q}$ is the "thickness" of the building. For buildings only integral values of $\mathbf{q}$ matter!

Main Theorem.

- *If* $\mathbf{q} \in \mathcal{R}$*, then*

$$L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U}) = \overline{\bigoplus_{T \in \mathcal{S}} H^*(X, X^T) \otimes D^T}.$$

- *If* $\mathbf{q} \in \mathcal{R}^{-1}$*, then*

$$L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U}) = \overline{\bigoplus_{T \in \mathcal{S}} H^*(X, X^{S-T}) \otimes D^{S-T}}.$$

This is almost an immediate consequence of the Decomposition

Theorem, which we recall:

Decomposition Theorem. • *If* $\mathbf{q} \in \mathcal{R}$, *then*

$$L^2_{\mathbf{q}} = \overline{\bigoplus_{T \in \mathcal{S}} D^T}.$$

• *If* $\mathbf{q} \in \mathcal{R}^{-1}$, *then*

$$L^2_{\mathbf{q}} = \overline{\bigoplus_{T \in \mathcal{S}} D^{S-T}}.$$

Recall that in Lecture 5 we proved:

$$H_k(\mathcal{U}) = \bigoplus_{T \in \mathcal{S}} H_k(X, X^T) \otimes \mathbb{Z}(W^T)$$

$$H_c^k(\mathcal{U}) = \bigoplus_{T \in \mathcal{S}} H^k(X, X^{S-T}) \otimes \mathbb{Z}(W^T)$$

Corollary.

- *If* $\mathbf{q} \in \mathcal{R}$, *then* $H_k(\mathcal{U};\mathbb{R}) \to L^2_{\mathbf{q}}\mathcal{H}_k(\mathcal{U})$ *is injective with dense image.*

- *If* $\mathbf{q} \in \mathcal{R}^{-1}$, *then* $H^k_c(\mathcal{U};\mathbb{R}) \to L^2_{\mathbf{q}}\mathcal{H}^k(\mathcal{U})$ *is injective with dense image.*

Example. Suppose W is type HM^n (i.e., , L is a GHS^{n-1}).

- If $\mathbf{q} \in \mathcal{R}$, then $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ is concentrated in dimension 0.

- If $\mathbf{q} \in \mathcal{R}^{-1}$, then $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ is concentrated in dimension n.

Example. Suppose $n = 2$ and L is a circle. $\mathbf{q} = q \in (0, \infty)$.

$$\rho = \text{the radius of convergence of } W(t).$$

Then $L^2_q\mathcal{H}^*(\Sigma)$ is concentrated in dimension

$$\begin{cases} 0, & \text{if } q \le \rho; \\ 1, & \text{if } \rho < q < 1/\rho; \\ 2, & \text{if } q \ge \rho. \end{cases}$$

Question. *What happens in the intermediate range,* $\mathbf{q} \notin \mathcal{R} \cup \mathcal{R}^{-1}$*?*

Conjecture. (A version of the Singer Conjecture). *Suppose* W *is type* HM^n. *Then for* $\mathbf{q} \leq \mathbf{1}$,

$$L^2_{\mathbf{q}}\mathcal{H}^k(\Sigma) = 0 \quad \textit{for } k > \frac{n}{2}.$$

Similarly, for $\mathbf{q} \geq \mathbf{1}$,

$$L^2_{\mathbf{q}}\mathcal{H}^k(\Sigma) = 0 \quad \textit{for } k < \frac{n}{2}.$$