The Euler Characteristic Conjecture and the Charney-Davis Conjecture

Mike Davis

MCA
Montreal

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http://www.math.osu.edu/∼davis.12/

1. The Euler Characteristic Conjecture
   - Right-angled reflection groups
   - $\mathrm{CAT}(0)$ cube complexes
2. Haken n-manifolds
   - Orbifold Euler characteristics
3. ℓ^2-Betti numbers

The Euler Characteristic Conjecture (Hopf-Chern-Thurston)

Suppose M^{2k} is a closed aspherical manifold. Then $(-1)^k\chi(M^{2k}) \geq 0$.

- A space is *aspherical* if its universal cover is contractible.
- Hopf and Chern conjectured this for nonpositively curved closed Riemannian mflds (which are necessarily aspherical). I think their idea was that the conjecture should follow from the higher dim'l Gauss-Bonnet Thm. (It does not.) Thurston was the first to make this conjecture for aspherical mflds.
- The Euler characteristic of any closed odd dim'l mfld is $= 0$.

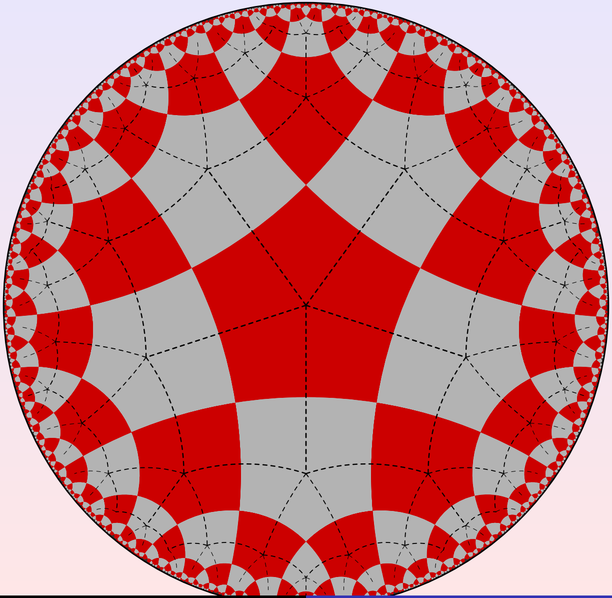
- A closed surface M^2 of genus $g > 0$ is aspherical; moreover, $\chi(M^2) = 2 - 2g \leq 0$; so, the conjecture is true for surfaces.
- The conjecture is true for a product if it is true for each factor. So, conjecture is true for a product of surfaces.
- The conjecture is true if M^{2k} is a hyperbolic mfld. Similarly, if M^{2k} is locally symmetric.

Coxeter orbifolds

- Let K be a simple polytope of dim n (later will called a "simple n-cell"). This means n facets meet at each vertex.
- $L = \partial$(dual simplicial polytope). So, L is triangulation of S^{n-1}.
- Since K is a mfld with corners, it can be given the structure of a "Coxeter orbifold" locally modeled on $\mathbb{R}^n/(\mathbb{Z}/2)^n = [0,\infty)^n$.
- K is the quotient of a $(\mathbb{Z}/2)^{[m]}$-action as a reflection gp on a mfld $M(K)$, where $[m] = \{\text{facets of } K\}$, and

$$M(K) := ((\mathbb{Z}/2)^{[m]} \times K)/\sim := \mathcal{U}((\mathbb{Z}/2)^{[m]}, K),$$

where $(g,x) \sim (g',x') \iff x = x'$ and $g^{-1}g' \in G_x$.



Orbifoldal Euler characteristic

- The orbifoldal Euler characteristic of K is obtained by weighting each face F of K by $1/$[order of its stabilizer] (a nonnegative power of 2). Thus,
  $\chi^{orb}(K) = (-1)^n \sum_{F \leq K} (-\frac{1}{2})^{\text{codim } F}$.
- Then $\chi(M(K)) = 2^m \chi^{orb}(K)$. So, $\chi(M(K))$ and $\chi^{orb}(K)$ have the same sign.
- In terms of the dual triangulation L of S^{n-1}, $\chi^{orb}(K) = \lambda(L)$, where

$$\lambda(L) = \sum_{\sigma \in \mathcal{S}(L)} \left(-\frac{1}{2}\right)^{\dim \sigma + 1} = f_L(-1/2),$$

  where $\mathcal{S}(L)$ is the poset of simplices (including $\emptyset$) and $f_L(t)$ is the f-polynomial.

Theorem

$M(K)$ is aspherical $\iff$ L is a flag complex (ie L is the clique cx of its 1-skeleton).

- $\lambda(L) = \sum(-1/2)^{\dim\sigma+1} = \chi^{orb}(K)$.
- The Euler Characteristic Conjecture for $M(K)$ is equivalent to the following.

The Charney-Davis Conjecture

Suppose L be a flag triangulation of S^{2k-1}. Then

$$(-1)^k\lambda(L) \geq 0.$$

(Alternatively, $(-1)^k\chi^{orb}(K) \geq 0$.)

Simple homotopy cells

Definition

A compact n-mfld with corners K is a *simple homotopy n-cell* if each stratum is a contractible manifold with boundary. If each stratum is a disk, then K is a *simple n-cell*.

- The dual simplicial cx to ∂ (simple homotopy n-cell) is a generalized homology $(n-1)$-sphere (abbreviated GHS^{n-1}), meaning that it is a homology mfld with the same homology as S^{n-1}. Conversely, each GHS^{n-1} has a unique "resolution" to a simple homotopy n-cell.
- The C-D Conjecture makes sense when L is a GHS.

Evidence

- Using ℓ^2-homology (of the universal cover of $M(K)$), Davis-Okun proved the Euler Char Conj (and hence, the C-D Conj) when $\dim M(K) = 4$.
- Using Stanley's "cd-index", Babson observed that the C-D Conj holds when L is the barycentric subdiv of a shellable cell structure on a sphere. (A barycentric subdiv is a flag cx.)

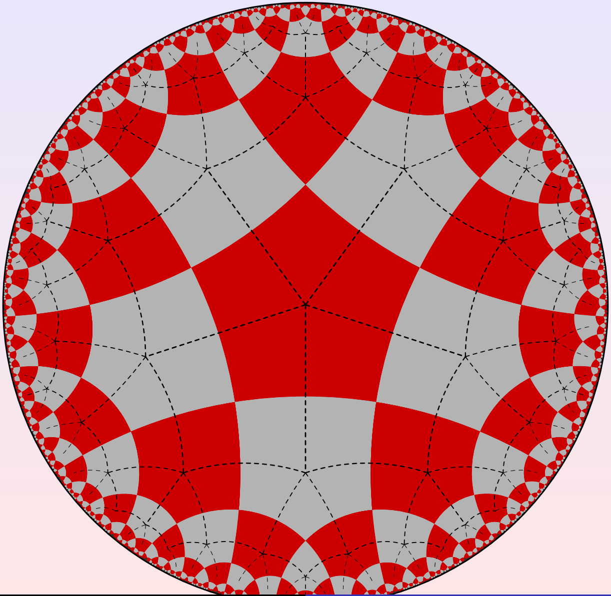
Definition

A cube cx X is a *nonpositively curved cube complex* (abbreviated NPCC) if the link of each vertex is a flag cx. The universal cover $\widetilde{X}$ of an NPCC is a CAT(0) *cube complex*.

- If L is a flag cx, then the dual cellulation of $M(K)$ is a NPCC.
- If X is a cube cx, then $\chi(X) = \sum_{\text{vertices}} \lambda(\text{Lk}(v))$. Hence,

Proposition

C-D Conj is true for every flag $\mathrm{GHS}^{2k-1} \iff$ *Euler Char Conj is true for every* M^{2k} *with structure of a* NPCC.

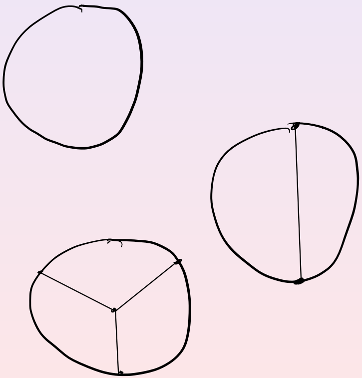


- Classical approach in surface theory: cut M^2 open along essentlal simple closed curve(s) to get a mfld with ∂. Continue cutting.
- Same approach is important in dimension 3 (folllowing Haken and Waldhausen). Cut open along "incompressible surface(s)" $N^2 \hookrightarrow M^3$, meaning $\pi_1(N^2) \to \pi_1(M^3)$ is injective. To make this work you need at least one incompressible surface to start with. A 3-manifold which admits such a incompressible surface said to be *Haken*.
- Foozwell and Rubinstein described a version of this that works in higher dimensions. Edmonds and D. - Edmonds related Euler characteristic of Haken mfld to C-D Conj.

Boundary patterns

- M^n a mfld with ∂. A *boundary pattern* is a partition of ∂M into $(n-1)$-mflds with ∂ (called *facets*) so that the intersection of any m of them is empty or an $(n-m)$-mfld with ∂ (= a *stratum* or *face*). That is, M is a mfld with corners.
- The boundary pattern is *useful* if every 0-gon, bigon or triangle which is null-homotopic in M is null homotopic and "standard" in ∂M. We might also want to require the induced ∂ pattern on each face to be useful (*really useful*?)
- M a mfld with ∂ pattern. $F \subset M$ a hypersurface transverse to faces of ∂M. Let $M \odot F$ be result of cutting open along F. So, $M \odot F$ is a mfld with ∂ pattern.

0- 2- and 3-gon



There is a notion of hypersurface, $F \subset M$, being "essential" (stronger than $\pi_1(F) \to \pi_1(M)$ being injective) s.t.
(M has really useful ∂ pattern) $\implies$
($M \odot F$ has really useful ∂ pattern).

With hindsight we have the following:

Definition

A simple homotopy n-cell K is a *homotopy Haken cell* if its dual simplicial cx is a flag cx.

Def'n of Foozwell-Rubinstein uses induction on dimension.)

- Suppose M is an n-mfld with ∂ pattern. A *hierarchy* for M is a sequence:

$$(M_0, F_0) \cdots (M_i, F_i) \cdots (M_{k+1}, \emptyset)$$

where $F_i \subset M_i$ is an essential ∂ pattern, $M_{i+1} = M_i \odot F_i$, and M_{k+1} is a disjoint union of homotopy Haken cells.

- M is (generalized) *Haken* if it admits a hierarchy.

Theorem (Foozwell)

Haken $\implies$ aspherical.

Example

Suppose a homotopy Haken cell K is given its structure of a right-angled orbifold. Then the resulting reflection mfld $M(K)$ is Haken. Hypersurfaces are the reflecting hypersurfaces.

- If M is a mfld with ∂ pattern, then it has structure of a right-angled orbifold by "silvering" each facet. The local gp at each facet is $\mathbb{Z}/2$ and at each codim k face the local gp is $(\mathbb{Z}/2)^k$.
- The orbifold structure on $M \odot F$ is obtained by silvering F, ie, by assigning the local gp $\mathbb{Z}/2$ to each pure stratum corresponding to F.

Lemma

$$\chi^{orb}(M \odot F) = \chi^{orb}(M).$$

Proof.

If M is closed, $\chi^{orb}(M \odot F) = \chi(M, F) + (2)(\frac{1}{2})\chi(F) = \chi(M, F) + \chi(F) = \chi(M)$. □

- Suppose M be a Haken manifold. Let $c_1, \dots, c_m$ be the homotopy Haken cells at the end of a hierarchy. and L_i be the simplicial complexes dual to ∂c_i. Then
- $\chi(M) = \sum \chi^{orb}(c_i) = \sum \lambda(L_i)$.

Theorem (D-Edmonds)

C-D Conj for all $\mathrm{GHS}^{2k-1} \implies$ *Euler Char. Conj for* $2k$*-dim Haken mflds.*

- Suppose π acts properly, cocompactly on contractible mfld $\widetilde{M}$ or CW complex $\widetilde{X}$ with quotient orbifold M or orbihedron X. One can then define a Hilbert space, called the ℓ^2-(co)homology of $\widetilde{X}$ in degree k. It has a "von Neumann dimension" $\in [0, \infty)$ wrt π, denoted $b_k^{(2)}(\pi)$.
- The ℓ^2-cohomology somehow interpolates between ordinary cohomology of $\widetilde{X}$ and the compactly supported cohomology.
- ℓ^2 Euler char: $\chi^{(2)}(\pi) = \sum(-1)^k b_k^{(2)}(\pi)$.
- Atiyah's Formula: $\chi^{(2)}(\pi) = \chi^{orb}(X)$ (or $\chi^{orb}(M)$.)

Dodzuik-Singer Conjecture

Singer Conjecture

If M^n is a closed aspherical mfld (orbifold), then $b_k^{(2)}(\pi) = 0$ for $k \neq n/2$.

Singer Conj. $\implies$ Euler Char. Conj

Theorem (D-Okun)

Singer Conj is true for right-angled reflection gps when $\dim M = 4$.

Corollary

C-D Conj is true in dim 3 (eg for flag triangulations of S^3).

Corollary

Euler Char Conj is true for 4-mflds which are NPCC

Corollary

Euler Char Conj is true for Haken 4-mflds.

Problem

Prove the Singer Conj. for Haken 4-maflds.

Thank you