Aspherical manifolds that cannot be triangulated

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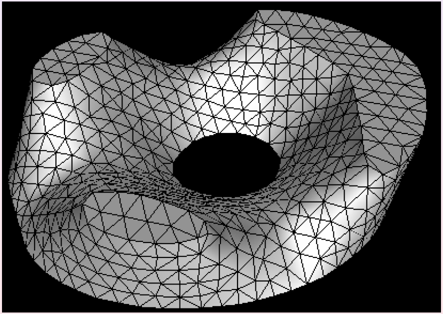
Problem

Can every manifold be triangulated? In other words, is every manifold homeomorphic to a simplicial complex?



The reason for asking this question was that at the beginning of the twentieth century one could only do algebraic topology for simplicial complexes. After the advent of singular homology this reason was no longer relevant.

Kneser's Questions - 1924

- Is a polyhedron with the local homology properties of Euclidean space, locally homeomorphic to Euclidean space?
- Is a space which is locally homeomorphic to Euclidean space, triangulable (homeomorphic to some polyhedron)?
- (Hauptvermutung). If there are two such triangulations, must they be PL equivalent?

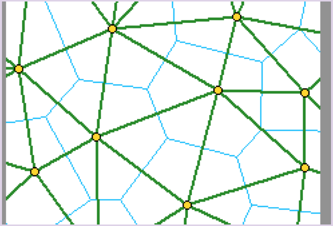
"polyhedron" $=$ space homeo to simplicial cx

Answers

- In dimensions ≤ 3 all mflds can be triangulated (due to Moise in dim 3). Also, for dim ≤ 3,
  $$\text{(homeo)} \implies \text{(PL homeo)}.$$
- Conventional wisdom: These questions were answered by Kirby-Siebenmann in dimensions ≥ 5 and by Freedman in dim 4.

Actually Kirby-Siebenmann's answer was only for certain special types of triangulations called "PL manifolds." A *PL mfld* is one in which every simplex has a "dual cell."

Dual cells



A simplicial cx L is a *PL n-manifold* if each k-simplex has a dual cell, which is PL homeomorphic to an $(n-k)$-disk. Or equivalently, if the link of each simplex is PL homeomorphic to S^{n-k-1}. (By the Poincaré Conjecture we can drop the phrase "PL" except possibly when $n-k-1=4$.)

Theorem (Kirby - Siebenmann 1969)

A topological n-mfld M^n, $n \geq 5$, admits a PL structure $\iff$ an obstruction $\Delta \in H^4(M^n; \mathbb{Z}/2)$ vanishes.

In other words, TOP/PL is the Eilenberg-MacLane space $K(\mathbb{Z}/2, 3)$.

There is also an obstruction to uniqueness in $H^3(M^n; \mathbb{Z}/2)$.

For simplicial cx L to be a topological mfld it is not necessary for it to have dual cells. However, it must be a "homology mfld." This means that L must have the same local homology groups as $\mathbb{R}^n$, i.e., $H_*(L, L - x) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - 0)$. This entails that the link of each k-simplex has the same homology as does S^{n-k-1}.

Example (Poincaré)

$\exists$ a 3-dim mfld H^3 with same homology as S^3 but $\not\cong S^3$ (because $\pi_1(H^3) \neq 1$). The suspension of H^3 is a homology 4-mfld. (However it's not a mfld in a nbhd of a suspension point.)

Question (Milnor - Seattle 1963)

Let H^3 be a homology 3-sphere with $\pi_1 \neq 1$. Is the double suspension of H^3 homeomorphic to S^5? (In particular, is the double suspension a mfld?)

The answer is yes (Edwards and Cannon $\sim$1975).

Theorem (The Double Suspension Thm)

The double suspension of any homology sphere H^n is $\cong S^{n+2}$.

Amazing Fact

Since H^3 can be triangulated, we get a triangulation of S^5 which is not equivalent to a PL triangulation.

Milnor's Questions - Seattle 1963

• Let M^3 be a homology 3-sphere with $\pi_1 \neq 1$. Is the double suspension of M^3 homeomorphic to S^5?
• Is simple homotopy type a topological invariant?
• Can rational Pontrjagin classes be defined as topological invariants?
• (*Hauptvermutung*). If two PL-manifolds are homeomorphic, does it follow that they are PL-homeomorphic?
• Can topological manifolds be triangulated?
• The Poincaré hypothesis in dimensions 3, 4.
• (*The annulus conjecture*). Is the region bounded by two locally flat n-spheres in $(n+1)$-space necessarily homeomorphic to $S^n \times [0, 1]$?

It follows from Freedman's work in 1981 that there are 4-mflds which do not admit a PL structure. From later work of Casson it follows that these cannot be triangulated (as homology mflds).

By Kirby - Siebenmann, in dimensions ≥ 5, $\exists$ mflds which do not admit a PL structures. However, because of the Double Suspension Thm, the possibility remained that all mflds of dim ≥ 5 could be triangulated. In the late 1970s Galewski - Stern analyzed the situation. They showed that all mflds of dim ≥ 5 could be triangulated $\iff$ $\exists$ a homology 3-sphere with certain properties.

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In 2013 this question about homology 3-spheres was resolved by Manolescu by using gauge theory. He showed no such homology 3-spheres exist. So, $\exists$ nontriangulable M^n for $n \geq 5$.

Definition

A space is *aspherical* if its universal cover is contractible.

Why aspherical mflds?

At one point the only examples of closed aspherical mflds came from differential geometry or from Lie groups; hence, were smooth mflds (and hence, could be triangulated). In 1991 Januszkiewicz and I applied Gromov's hyperbolization technique to Freedman's 4-mflds to show the existence of nontriangulable aspherical 4-mflds. The product of one of these with a torus also does not have a PL structure. (However, by the Double Suspension Thm, any such product with a positive dim'l torus can be triangulated.)

Gromov's hyperbolization procedure showed that you could convert simplicial complexes (hence, triangulated mflds) into aspherical ones. To get nontriangulable aspherical 4-mflds Januszkiewicz and I had to use Freedman's work. To convert the Galewski - Stern mflds into nontriangulable aspherical mflds of dim ≥ 5 you need a further trick: relative hyperbolization. Using this, we show that these Galewski - Stern mflds can be chosen to be aspherical, at least for dim ≥ 6.

1. Introduction

2. Dimension 4
   - Rokhlin's Theorem and the μ-invariant
   - Freedman's E_8-manifold
   - Hyperbolization

3. Dimensions > 4
   - Kirby - Siebenmann
   - Galewski - Stern $+$ Manolescu
   - Relative hyperbolization

Polyhedral homology mflds

Definitions

A simplicial cx L^n is a *polyhedral homology n-mfld* (a PHM for short) if for each k-simplex σ, $\mathrm{Lk}(\sigma, L)$ has the same homology as S^{n-k-1}. L^n is a *PL mfld* if $\forall \sigma \in L$, $\mathrm{Lk}(\sigma, L)$ is PL homeomorphic to $S^{n-\dim\sigma-1}$.

For any closed orientable mfld M^{4k}, cup product defines a nondegenerate symmetric bilinear form on

$$H^{2k}(M^{4k})/\text{torsion}.$$

In dimension 4 this form is even if $w_2 = 0$.

Fact

If B is an even, nondegenerate, symmetric bilinear form over $\mathbb{Z}$, then its signature, $\sigma(B)$, is divisible by 8.

Theorem (Rokhlin 1952)

If M^4 is a closed PL 4-mfld, with $w_1 = 0$ and $w_2 = 0$, then

$$\sigma(M^4) \equiv 0 \mod 16.$$

Fact

If H^3 is a homology 3-sphere, then $H^3 = \partial W^4$, where W^4 is a PL mfld with even intersection form.

The μ-invariant

Define

$$\mu(H^3) = \frac{\sigma(W^4)}{8} \in \mathbb{Z}/2.$$

This defines a homomorphism $\mu : \Theta_3^H \to \mathbb{Z}/2$, where Θ_3^H is the group of homology cobordism classes of homology 3-spheres.

Associated to the Dynkin diagram E_8 there is a nondegenerate even, symmetric form which is positive definite of rank 8 (hence, $\sigma = 8$).

By a process calle "plumbing," one can use the E_8 diagram to to build a smooth 4-mfld with bdry and intersection form given by E_8.

$$Q(E_8) := \text{the } E_8 \text{ plumbing.}$$

$\partial Q(E_8) = H^3$, Poincaré's homology 3-sphere. $\sigma(Q(E_8) = 8$. Let $X^4 := Q(E_8) \cup c(H^3)$. It is a PHM of signature 8.

Theorem (Freedman 1982)

$H^3 = \partial C^4$, where C^4 is a top contractible mfld. Put $M^4 = Q(E_8) \cup C^4$, the "E_8-manifold".

By Rokhlin's Thm, M^4 does not have a PL structure.

Fact

Any triangulation of a 4-mfld is automatically PL. (Pf: By the Poincaré Conj, the link of any vertex is PL homeomorphic to S^3.) So, Freedman's M^4 is not triangulable.

Hyperbolization (Gromov)

A *hyperbolization procedure* is a functor $\mathfrak{h}$ from {simplicial complexes} to {locally CAT(0) spaces} together with a map $f : \mathfrak{h}(K) \to K$ with the following properties:

- $\mathfrak{h}$ preseves local structure: $\forall \sigma \in K$, $\mathrm{Lk}(\mathfrak{h}(\sigma)) \cong \mathrm{Lk}(\sigma)$ (Lk means "link".) In particular, if K is a mfld (or a PHM), then so is $\mathfrak{h}(K)$.
- f^* is a split injection on cohomology.
- When K is a mfld, f pulls back stable tangent bundle to stable tangent bundle. So, f^* pulls back characteristic classes of K to those of $\mathfrak{h}(K)$.

(D - Januszkiewicz)

- Apply $\mathfrak{h}$ to the E_8 homology mfld X^4.
- Resolve it to $N^4 = (\mathfrak{h}(X^4) - \text{nbhd of cone pt}) \cup C^4$.
- Then N^4 is aspherical and not triangulable.

Theorem (DJ 1991)

$\exists$ closed aspherical 4-mflds that cannot be triangulated. For $n \geq 5$, $\exists$ closed aspherical n-mflds which are not homotopy equivalent to PL mflds.

Proof of 2nd sentence.

$N^4 \times T^k$ is not PL. □

Remark

By Double Suspension Thm, for $k > 0$, $N^4 \times T^k \cong X^4 \times T^k$ (where X^4 is the PHM). So, $N^4 \times T^k$ can be triangulated.

Theorem (Kirby - Siebenmann 1969)

A top n-mfld, $n \geq 5$, admits a PL structure $\iff$ an obstruction $\Delta \in H^4(M^n; \mathbb{Z}/2)$ vanishes.

Polyhedral Mfld Characterization Theorem

Theorem (Edwards 1978 + Perelman)

A PHM (of dim > 2) is a top mfld $\iff$ the link of each vertex is simply connected.

Example

The double suspension of a homology sphere, with $\pi_1 \neq 1$.

Such triangulations are not PL.

In the early seventies such considerations led Siebenmann to ask if all mflds could actually be triangulated (before the Double Suspension Thm or Freedman's E_8 4-mfld were known).

I will describe highlights of a theory worked out in the 1970 s by several people, Siebenmann, Matumoto, most notably Galewski - Stern (with important contributions by others, eg, Cohen, Sullivan, Martin, Maunder).

Suppose X is a PHM. Let $\lambda \in H^4(X; \Theta_3^H)$ be the cohomology class which associates to the "dual cell" (actually dual cone) of a codim 4 simplex σ, the class of $\mathrm{Lk}(\sigma)$ in Θ_3^H (where Θ_3^H is the group of homology cobordism classes of homology 3-spheres). λ is the obstruction to finding an "acyclic resolution" of X by a PL manifold.

Consider the coefficient sequence:

$$0 \to \operatorname{Ker}\mu \longrightarrow \Theta_3^H \xrightarrow{\mu} \mathbb{Z}/2 \to 0.$$

Fact 1

When X is a top mfld, μ_* takes $\lambda \in H^4(X;\Theta_3^H)$ to the Kirby-Siebenmann obstruction $\Delta \in H^4(X;\mathbb{Z}/2)$.

Fact 2

If M is a top mfld, then the obstruction to triangulation is $\beta(\Delta) \in H^5(M;\operatorname{Ker}\mu)$, where β is the Bockstein associated to the above coefficient sequence.

Theorem (Galewski-Stern $\sim$ 1980)

In dim $n > 4$, $\exists$ nontriangulable M^n iff the sequence $0 \to \operatorname{Ker} \mu \to \Theta_3^H \to \mathbb{Z}/2 \to 0$ does not split, ie, $\iff$ $\nexists$ a homology 3-sphere H^3 with $\mu(H^3) \neq 0$ and $H^3 \# H^3 = 0$ in Θ_3^H.

Theorem (Manolescu 2013)

The sequence does not split.

Galewski-Stern mflds

It suffices to consider the Bockstein associated to

$$0 \to \mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \to 0.$$

This Bockstein is Sq^1 (the first Steenrod square).

$\exists$ a (nonorientable) PHM P^5 with bdry st

- int P^5 is a top mfld (this uses Edwards' Thm).
- $\Delta(P^5) = \mu_*(\lambda(P^5)) \neq 0$ in $H^4(P^5; \mathbb{Z}/2)$ and $\Delta(\partial P^5) = 0$.
- $Sq^1(\Delta) \neq 0$ in $H^5(P^5, \partial P^5; \mathbb{Z}/2)$.
- $\exists$ a PHM bordism U from ∂P^5 to a PL mfld V^4, and V^4 is bdry of PL mfld W^5.
- $M^5 := P^5 \cup U \cup W^5$ is not triangulable.

Relative hyperbolization (D - Januszkiewicz - Weinberger, 2001)

Let $(M, \partial M)$ be a triangulated mfld with bdry. Put

$$\mathcal{H}(M, \partial M) := \mathfrak{h}(M \cup c(\partial M)) - (\text{nbhd of cone point})$$

Key properties

- $\mathcal{H}(M, \partial M)$ is mfld with bdry; its bdry is ∂M.
- $\pi_1(\partial M) \to \pi_1(\mathcal{H}(M, \partial M))$ is injective.
- $\mathcal{H}(M, \partial M)$ is aspherical iff ∂M is aspherical.

Corollary (DJW)

If an aspherical mfld bounds a triangulable mfld, then it bounds an aspherical mfld.

Corollary (DJW)

If an aspherical mfld V bounds a triangulable mfld W, then it bounds an aspherical mfld W'.

Proof.

$W' = \mathcal{H}(W, V)$. □

GS mflds in dimensions ≥ 6

Put $P^6 := P^5 \times S^1$. Since $\Delta(\partial P^6) = 0$, ∂P^6 admits a PL structure (by Kirby-Siebenmann).

Put $M^6 = P^6 \cup U \cup W$, where U is the mapping cylinder of a (necessarily non-PL) homeomorphism from ∂P^6 to a PL mfld V^5 and W is a PL 6-mfld bounded by V^5.

Theorem (D-Fowler-Lafont)

In each dim $n \geq 6$, $\exists$ an aspherical mfld N^n that cannot be triangulated.

Proof.

Start with $\mathfrak{h}(P^6)$. Then $\mathfrak{h}(\partial P^6)$ is homeomorphic to a PL mfld V^5. Let U be the mapping cylinder of a homeomorphism $V^5 \to \mathfrak{h}(\partial P^6)$. V^5 is bdry of a PL 6-mfld W. Put

$$N^6 := \mathfrak{h}(P^6) \cup U \cup \mathcal{H}(W, V).$$

We check immediately that

- N^6 is aspherical.
- $\Delta(N^6) \neq 0$ and $Sq^1(\Delta(N^6)) \neq 0$.

So, N^6 cannot be triangulated. □

Thank you.