$L^2$-cohomology

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Introductory remarks on cochains

$Y$ a CW complex (usually not compact)

$\mathcal{E}_n := \{n\text{-cells in } Y\}$

$C^n(Y) := \{f : \mathcal{E}_n \rightarrow \mathbb{R}\}$

$C^n_c(Y) := \{f : \mathcal{E}_n \rightarrow \mathbb{R} \mid f \text{ is finitely supported}\}$

$L^2 C^n(Y) := \{f : \mathcal{E}_n \rightarrow \mathbb{R} \mid \sum f(\sigma)^2 < \infty\}$

So, $C^n_c(Y) \subset L^2 C^n(Y) \subset C^n(Y)$.

Taking cohomology we get 3 different answers.
Why $L^2$?
Instead of, say, $L^p$.
Answer: These cohomology gps are usually $\infty$-dimensional vector spaces. In the $L^2$ case, when a group is acting, one can assign a “dimension” in $[0, \infty)$ to these cohomology spaces.
The regular representation

\[ L^2(\Gamma) := \{ f : \Gamma \to \mathbb{R} \mid \sum f(\gamma)^2 < \infty \}, \]

where the sum is over all \( \gamma \in \Gamma \).

\( L^2(\Gamma) \) is a Hilbert space with inner product:

\[ f \cdot f' := \sum_{\gamma \in \Gamma} f(\gamma)f'(\gamma). \]

\( \mathbb{R}\Gamma \) is identified with the dense subspace of \( L^2(\Gamma) \) of finitely supported functions.
An orthonormal basis

For each $\gamma \in \Gamma$,

$$e_\gamma(\gamma') := \begin{cases} 1, & \text{if } \gamma = \gamma'; \\ 0, & \text{otherwise}. \end{cases}$$

$(e_\gamma)_{\gamma \in \Gamma}$ is an orthonormal basis for $L^2(\Gamma)$.

$\Gamma$ acts on $L^2(\Gamma)$ by left translation:

$$(\gamma \cdot f)(\gamma') := f(\gamma^{-1} \gamma').$$

This is the (left) regular representation of $\Gamma$. 
Maps

$V$ and $V'$ are Hilbert spaces with orthogonal $\Gamma$-actions.

- A map from $V \to V'$ means a $\Gamma$-equivariant bounded linear map.
- The kernel of a map is always closed.
- The image need not be.
- A map is a weak surjection if its image is dense in $V'$.
- It is a weak isomorphism if, in addition, it is injective.
If $\varphi \in \mathbf{R}G$, then $L^2(\Gamma) \to L^2(\Gamma)$ defined by $f \to f \cdot \varphi$ is a map (i.e., is $\Gamma$-equivariant and bounded).

Similarly, if $\varphi : (\mathbf{R}\Gamma)^n \to (\mathbf{R}\Gamma)^m$ is a map of $\mathbf{R}\Gamma$-modules, the induced map

$$L^2(\Gamma) \otimes_{\mathbf{R}\Gamma} \varphi : (L^2(\Gamma))^n \to (L^2(\Gamma))^m$$

is bounded.
A Hilbert space with orthogonal $\Gamma$-action is a *Hilbert $\Gamma$-module* if it is isomorphic to a closed, $\Gamma$-stable subspace of a finite (orthogonal) direct sum of copies of $L^2(\Gamma)$ with the diagonal $\Gamma$-action.

**Example**

If $F$ is a finite subgroup of $\Gamma$, then $L^2(\Gamma/F)$, the space of square summable functions on $\Gamma/F$, can be identified with the subspace of $L^2(\Gamma)$ consisting of the square summable functions on $\Gamma$ which are constant on each coset. This subspace is clearly closed and $\Gamma$-stable; hence, $L^2(\Gamma/F)$ is a Hilbert $\Gamma$-module.
Example

(Completed tensor product). Suppose $\Gamma = \Gamma_1 \times \Gamma_2$ and that $V_j$ is a Hilbert $\Gamma_j$-module, for $j = 1, 2$. The $L^2$-completion of the tensor product is denoted $V_1 \hat{\otimes} V_2$. It is a Hilbert $\Gamma$-module.

Lemma

If two Hilbert $\Gamma$-modules are weakly isomorphic, then they are $\Gamma$-isometric.

The proof is basically polar decomposition, i.e.,

$$(\text{invertible matrix}) = (\text{symmetric}) \cdot (\text{orthogonal})$$
Proof

Suppose \( f : V_1 \to V_2 \) a weak iso. Then \( f^* \circ f : V_1 \to V_1 \) is positive definite and \( \text{Im}(f^* \circ f) \) is dense. Put

\[
g := \sqrt{f^* \circ f}.
\]

\( g \) is self-adjoint, positive definite and \( \text{Im}(g) \supset \text{Im}(f^* \circ f) \). Put \( h := f \circ g^{-1} : \text{Im}(g) \to V_2 \). Then

\[
hx \cdot hy = (f^* \circ (f \circ g^{-1})x) \cdot (g^{-1}y) = (g^2 \circ g^{-1}x) \cdot (g^{-1}y)
\]

\[
= (g \circ g^{-1}x) \cdot (g^* \circ g^{-1}y) = x \cdot y
\]

\( h : \text{Im}(g) \to \text{Im}(f) \) is an isometry and hence, extends to isometry \( V_1 \to V_2 \). \( f^* \) and \( f \) are \( \Gamma \)-equivariant \( \implies \) so are \( g \) and \( h \). \qed
Suppose $H$ a subgp of $\Gamma$ and $V$ a Hilbert $H$-module.

- The *induced representation*, $\text{Ind}_H^\Gamma(V)$, is the completion of $R\Gamma \otimes_{R^H} V$.
- $V$ a Hilbert $H$-module $\implies \text{Ind}_H^\Gamma(V)$ a Hilbert $\Gamma$-module.
- For example, $F$ a finite subgp of $\Gamma$ and $R$ denotes the trivial 1-dimensional representation of $F$, then $\text{Ind}_F^\Gamma(R)$ is $\cong L^2(\Gamma/F)$. 
\( L^2\)-chains

\( X \) is a proper \( \Gamma\)-CW-complex with \( X/\Gamma \) compact \iff there are only finitely many \( \Gamma\)-orbits of cells in \( X \).

- \( C_\ast(X) := \) the usual cellular chain complex on \( X \).
- An element of \( C_i(X) \) (an \( i\)-chain) is a finitely supported function \( \varphi \) from the set of oriented \( i\)-cells in \( X \) to \( \mathbb{Z} \) satisfying \( \varphi(\bar{e}) = -\varphi(e) \) (where \( e \) and \( \bar{e} \) denote the same cell but with opposite orientations).

\[
L^2 C_i(X) := L^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_i(X)
\]
\[
= \{ \varphi : \{ i\text{-cells} \} \to \mathbb{R} \mid \sum \varphi(e)^2 < \infty \}\]
Cochains

The definition of the space of $L^2$-cochains on $X$ is the same:

$$L^2 C^i(X) := L^2 C_i(X).$$

If $c$ is an $i$-cell of $X$, then the space of $L^2$-chains supported on the $\Gamma$-orbit of $c$ is $\cong L^2(\Gamma/\Gamma_c)$. Since $\Gamma_c$ is finite, $L^2(\Gamma/\Gamma_c)$ is a Hilbert $\Gamma$-module.

Since $\text{Card}\{\Gamma\text{-orbits of } i\text{-cells}\} < \infty$, $L^2 C_i(X)$ is the direct sum of a finite number of such subspaces $\implies$ it is a Hilbert $\Gamma$-module.
Boundary maps

The boundary $\partial_i : L^2 C_i(X) \to L^2 C_{i-1}(X)$ and the coboundary $\delta^i : L^2 C^i(X) \to L^2 C^{i+1}(X)$ are defined by the usual formulas. $\delta^i$ and $\partial_{i+1}$ are the adjoints of one another.

**Proposition**

$\partial_i$ and $\delta_i$ are maps of Hilbert $\Gamma$-modules.

**Proof.**

$\partial_i$ is induced from usual $\partial_i : C_i(X) \to C_{i-1}(X)$. So, it is $\Gamma$-equivariant and bounded. $\delta_i = \partial_{i+1}^*$. 
Define subspaces of $L^2 C_i(X)$:

$$Z_i(X) := \text{Ker } \partial_i$$

$$B_i(X) := \text{Im } \partial_{i+1}$$

$$Z^i(X) := \text{Ker } \delta^i$$

$$B^i(X) := \text{Im } \delta^{i-1}$$

the $L^2$-cycles, -cocycles, -boundaries and -coboundaries, resp. The corresponding quotient spaces

$$L^2 H_i(X) := Z_i(X)/B_i(X)$$

$$L^2 H^i(X) := Z^i(X)/B^i(X)$$

are the *unreduced* $L^2$-homology and -cohomology groups, resp.
Reduced (co)homology

The subspaces $B_i(X)$ and $B^i(X)$ need not be closed. So, the quotient spaces need not be Hilbert spaces. Let $\overline{B}_i(X)$ (resp., $\overline{B}^i(X)$) denote the closure of $B_i(X)$ (resp., $B^i(X)$).

The *reduced $L^2$-homology* and *-cohomology groups* are defined by:

\[
L^2H_i(X) := Z_i(X)/\overline{B}_i(X)
\]
\[
L^2H^i(X) := Z^i(X)/\overline{B}^i(X).
\]

They are Hilbert $\Gamma$-modules (each can be identified with the orthogonal complement of a closed $\Gamma$-stable subspace in a closed $\Gamma$-stable subspace of $C_i(X)$).
Hodge decomposition

\[ \delta^{i-1}(x) \cdot y = x \cdot \partial_i(y), \quad \forall x \in L^2C^{i-1}(X), y \in L^2C_i(X) \]

\[ \implies \text{orthogonal direct sum decompositions:} \]

\[ L^2C_i(X) = \overline{B}_i(X) \oplus Z^i(X) \]

\[ L^2C_i(X) = \overline{B}^i(X) \oplus Z_i(X). \]

\[ \delta^{i-1}(x) \cdot \partial_{i+1}(y) = x \cdot \partial_i \partial_{i+1}(y) = 0 \implies \overline{B}_i(X) \perp \overline{B}^i(X). \]

\[ L^2C_i(X) = \overline{B}_i(X) \oplus \overline{B}^i(X) \oplus (Z_i(X) \cap Z^i(X)). \]

Both \( L^2\mathcal{H}_i(X) \) and \( L^2\mathcal{H}^i(X) \) are \( \cong Z_i(X) \cap Z^i(X) \), the harmonic \( i \)-cycles.
Define $\Delta : L^2 C_i(X) \to L^2 C_i(X)$ by

$$\Delta = \delta^{i-1} \partial_i + \partial_{i+1} \delta^i.$$ 

Check:

$$L^2 \mathcal{H}_i(X) = Z_i(X) \cap Z^i(X) = \text{Ker} \Delta$$
0-dimensional cohomology

**Example**

A 0-cochain = a function \( f \) on \{vertices in \( X \}\)
e an edge from \( v_0 \) to \( v_1 \), then \( \delta f(e) = v_1 - v_0 \).
So, \( \delta f = 0 \implies f \) is constant (provided \( X^1 \) is connected).
If \( \Gamma \) is infinite (so \( \text{Vert}(X) \) is infinite), the only constant function in \( L^2 \) is 0.
So, \( X^1 \) connected, \( \Gamma \) infinite \( \implies L^2H^0(X) = 0 \). Therefore,
\( L^2\mathcal{H}^0(X) = L^2\mathcal{H}_0(X) = 0 \).
On other hand, \( L^2H_0(X) \) need not = 0, e.g., \( L^2H_0(\mathbb{R}) \neq 0 \).
Example

- Suppose $X$ is an $n$-dimensional pseudomanifold. (Each $(n - 1)$-cell is contained in precisely two $n$-cells.) Also, suppose $X$ is orientable and gallery connected.
- Then we can orient the $n$-cells s.t. the sum of all oriented $n$-cells is an infinite cycle, the *fundamental cycle*.
- Any $n$-cycle is a multiple of fund cycle.
- If $\Gamma$ is infinite, this cycle is not $L^2$.
- Hence, $L^2 H_n(X) = 0$.
- So, $L^2 \mathcal{H}_n(X) = L^2 \mathcal{H}^n(X) = 0$. 
Universal cover of a surface

Example

$X =$ universal cover of a surface, genus $> 0$.

By previous two examples, $L^2\mathcal{H}_0(X) = 0$ and $L^2\mathcal{H}_2(X) = 0$.

So, $L^2\mathcal{H}_\ast(X)$ is concentrated in dimension 1.
von Neumann algebra $\mathcal{N}(\Gamma)$

$L^2(\Gamma)$ is an $\mathbb{R}\Gamma$-bimodule. Three equivalent definitions $\mathcal{N}(\Gamma)$:

- $\mathcal{N}(\Gamma)$ is the algebra of all maps from $L^2(\Gamma)$ to itself.
- $\mathcal{N}(\Gamma)$ is the double commutant of the right $\mathbb{R}\Gamma$-action on $L^2(\Gamma)$.
- $\mathcal{N}(\Gamma)$ is the weak closure of the algebra of operators $\mathbb{R}(\Gamma)$ acting from the right on $L^2(\Gamma)$. 

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$L^2$-cohomology
Given \( \varphi \in \mathcal{N}(\Gamma) \), define

\[
\text{tr}_\Gamma(\varphi) := \varphi(e_1) \cdot e_1
\]

Standard arguments show:

- \( \text{tr}_\Gamma(\varphi) = \text{tr}_\Gamma(\varphi^*) \) (\( \varphi^* \) is the adjoint of \( \varphi \)) and
- \( \text{tr}_\Gamma(\varphi \psi) = \text{tr}_\Gamma(\varphi) \text{tr}_\Gamma(\psi) \).
The Γ-trace of a \((n \times n)\)-matrix

Given \(n \in \mathbb{N}\), \(L^2(\Gamma)^n = \) orthogonal direct sum of \(n\) copies of \(L^2(\Gamma)\).
\(M_n(\mathcal{N}(\Gamma)) = \{(n \times n)\)-matrices with coefficients in \(\mathcal{N}(\Gamma)\}\). Given \(\Phi = (\varphi_{ij}) \in M_n(\mathcal{N}(\Gamma))\), define

\[
\text{tr}_\Gamma(\Phi) := \sum_{i=1}^{n} \text{tr}_\Gamma(\varphi_{ii}).
\]

\(\text{tr}_\Gamma(\ )\) has the usual properties. Suppose \(\Phi, \Psi \in M_n(\mathcal{N}(\Gamma))\). Then
Properties of $\text{tr}_\Gamma(\cdot)$

- $\text{tr}_\Gamma(\Phi) = \text{tr}_\Gamma(\Phi^*)$.
- $\text{tr}_\Gamma(\Phi\Psi) = \text{tr}_\Gamma(\Psi\Phi)$.
- $\Phi$ is self-adjoint and idempotent. Then $\text{tr}_\Gamma(\Phi) \geq 0$, with equality iff $\Phi = 0$.
- Given any Hilbert $\Gamma$-module $V$ isomorphic to $L^2(\Gamma)^n$ and any self-map $\Phi$ of $V$, one can define $\text{tr}_\Gamma(\Phi)$. 

$V$ a Hilbert $\Gamma$-module. Choose an embedding of $V$ as a closed, $\Gamma$-stable subspace of $L^2(\Gamma)^n$ for some $n \in \mathbb{N}$.

$p_V : L^2(\Gamma)^n \rightarrow L^2(\Gamma)^n$ is orthogonal projection onto $V$. The *von Neumann dimension* of $V$ (or its $\Gamma$-*dimension*) is defined by

$$\dim_\Gamma(V) := \text{tr}_\Gamma(p_V).$$

If $E \subset L^2(\Gamma)$ is a not necessarily closed $\Gamma$-stable subspace of $L^2(\Gamma)^n$, put $\dim_\Gamma(E) := \dim_\Gamma(\overline{E})$. 
Properties of $\dim_{\Gamma}(\ )$

- $\dim_{\Gamma}(\ )$ is well-defined.
- $\dim_{\Gamma}(V) \in [0, \infty)$.
- $\dim_{\Gamma}(V) = 0$ iff $V = 0$.
- If $\Gamma = \{1\}$ (s.t. $\dim_{\mathbb{R}} V < \infty$), then $\dim_{\Gamma}(V) = \dim_{\mathbb{R}}(V)$.
- $\dim_{\Gamma}(L^{2}(\Gamma)) = 1$.
- $\dim_{\Gamma}(V \oplus W) = \dim_{\Gamma}(V) + \dim_{\Gamma}(W)$.
- If $f : V \rightarrow W$ is a map of Hilbert $\Gamma$-modules, then
  $$\dim_{\Gamma}(V) = \dim_{\Gamma}(\ker f) + \dim_{\Gamma}(\text{Im } f).$$
More properties

- $f : V \to W$ a map, $f^* : W \to V$ its adjoint, then $\ker f$ and $\text{Im} \, f^*$ are orthogonal complements in $V$. Hence,

$$\dim_\Gamma(V) = \dim_\Gamma(\ker f) + \dim_\Gamma(\text{Im} \, f^*).$$

So, $\dim_\Gamma(\text{Im} \, f) = \dim_\Gamma(\text{Im} \, f^*)$.

- If $0 \to V_n \to \cdots \to V_0 \to 0$ a weak exact sequence, then

$$\sum_{i=0}^{n} (-1)^i \dim_\Gamma(V_i) = 0.$$

- $H$ a subgroup of finite index $m$ in $\Gamma$ implies

$$\dim_\mathcal{H}(V) = m \dim_\Gamma(V).$$
and more properties

- If $\Gamma$ is finite, then $\dim_{\Gamma}(V) = \frac{1}{|\Gamma|} \dim(V)$.
- If $H$ is a subgroup of $\Gamma$, and $W$ is a Hilbert $H$-module, then
  \[ \dim_{\Gamma}(\text{Ind}^\Gamma_H(W)) = \dim_H(W). \]
- If $F$ is a finite subgroup of $\Gamma$, then $\dim_{\Gamma}(L^2(\Gamma/F)) = \frac{1}{|F|}$.
- If $\Gamma = \Gamma_1 \times \Gamma_2$, and $V_j$ is a Hilbert $\Gamma_j$-module for $j = 1, 2$, then
  \[ \dim_{\Gamma}(V_1 \widehat{\otimes} V_2) = \dim_{\Gamma_1}(V_1) \dim_{\Gamma_2}(V_2). \]
Lecture 1: the basics
Hilbert $\Gamma$-modules
$L^2$-(co)homology
von Neumann dimension

Next time

$L^2$-Betti numbers
Define $L^2 b_i(X; \Gamma) := \dim_{\Gamma}(L^2 H^i(X))$. 