L^2-cohomology

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Introductory remarks on cochains

Y a CW complex (usually not compact)

$$\mathcal{E}_n := \{n\text{-cells in } Y\}$$
$$C^n(Y) := \{f : \mathcal{E}_n \to \mathbf{R}\}$$
$$C^n_c(Y) := \{f : \mathcal{E}_n \to \mathbf{R} \mid f \text{ is finitely supported}\}$$
$$L^2C^n(Y) := \{f : \mathcal{E}_n \to \mathbf{R} \mid \sum f(\sigma)^2 < \infty\}$$

So, $C^n_c(Y) \subset L^2C^n(Y) \subset C^n(Y)$.
Taking cohomology we get 3 different answers.

- Why L^2?
- Instead of, say, L^p.
- Answer: These cohomology gps are usually ∞-dimensional vector spaces. In the L^2 case, when a group is acting, one can assign a "dimension" in $[0, \infty)$ to these cohomology spaces.

The regular representation

Γ is a countable discrete gp.

- $$L^2(\Gamma) := \{f : \Gamma \to \mathbf{R} \mid \sum f(\gamma)^2 < \infty\},$$
  where the sum is over all $\gamma \in \Gamma$.
- $L^2(\Gamma)$ is a Hilbert space with inner product:
  $$f \cdot f' := \sum_{\gamma\in\Gamma} f(\gamma)f'(\gamma).$$
- $\mathbf{R}\Gamma$ is identified with the dense subspace of $L^2(\Gamma)$ of finitely supported functions.

An orthonormal basis

- For each $\gamma \in \Gamma$,

$$e_\gamma(\gamma') := \begin{cases} 1, & \text{if } \gamma = \gamma'; \\ 0, & \text{otherwise.} \end{cases}$$

- $(e_\gamma)_{\gamma \in \Gamma}$ is an orthonormal basis for $L^2(\Gamma)$.
- Γ acts on $L^2(\Gamma)$ by left translation:

$$(\gamma \cdot f)(\gamma') := f(\gamma^{-1}\gamma').$$

This is the (left) *regular representation* of Γ.

Maps

V and V' are Hilbert spaces with orthogonal Γ-actions.

- A *map* from $V \to V'$ means a Γ-equivariant bounded linear map.
- The kernel of a map is always closed.
- The image need not be.
- A map is a *weak surjection* if its image is dense in V'.
- It is a *weak isomorphism* if, in addition, it is injective.

- If $\varphi \in \mathbf{R}G$, then $L^2(\Gamma) \to L^2(\Gamma)$ defined by $f \to f \cdot \varphi$ is a map (i.e., is Γ-equivariant and bounded).
- Similarly, if $\varphi : (\mathbf{R}\Gamma)^n \to (\mathbf{R}\Gamma)^m$ is a map of **R**Γ-modules, the induced map

$$L^2(\Gamma) \otimes_{\mathbf{R}\Gamma} \varphi : (L^2(\Gamma))^n \to L^2(\Gamma))^m$$

is bounded.

Hilbert Γ-modules

A Hilbert space with orthogonal Γ-action is a *Hilbert Γ-module* if it is isomorphic to a closed, Γ-stable subspace of a finite (orthogonal) direct sum of copies of $L^2(\Gamma)$ with the diagonal Γ-action.

Example

If F is a finite subgroup of Γ, then $L^2(\Gamma/F)$, the space of square summable functions on Γ/F, can be identified with the subspace of $L^2(\Gamma)$ consisting of the square summable functions on Γ which are constant on each coset. This subspace is clearly closed and Γ-stable; hence, $L^2(\Gamma/F)$ is a Hilbert Γ-module.

Example

(*Completed tensor product*). Suppose $\Gamma = \Gamma_1 \times \Gamma_2$ and that V_j is a Hilbert Γ_j-module, for $j = 1, 2$. The L^2-completion of the tensor product is denoted $V_1 \widehat{\otimes} V_2$. It is a Hilbert Γ-module.

Lemma

If two Hilbert Γ-modules are weakly isomorphic, then they are Γ-isometric.

The proof is basically polar decomposition, i.e.,

$$(\text{invertible matrix}) = (\text{symmetric}) \cdot (\text{orthogonal})$$

Proof

Suppose $f : V_1 \to V_2$ a weak iso. Then $f^* \circ f : V_1 \to V_1$ is positive definite and $\mathrm{Im}(f^* \circ f)$ is dense. Put

$$g := \sqrt{f^* \circ f}.$$

g is self-adjoint, positive definite and $\mathrm{Im}(g) \supset \mathrm{Im}(f^* \circ f)$. Put $h := f \circ g^{-1} : \mathrm{Im}(g) \to V_2$. Then

$$\begin{aligned} hx \cdot hy &= (f^* \circ (f \circ g^{-1})x) \cdot (g^{-1}y) = (g^2 \circ g^{-1}x) \cdot (g^{-1}y) \\ &= (g \circ g^{-1}x) \cdot (g^* \circ g^{-1}y) = x \cdot y \end{aligned}$$

$h : \mathrm{Im}(g) \to \mathrm{Im}(f)$ is an isometry and hence, extends to isometry $V_1 \to V_2$. f^* and f are Γ-equivariant $\implies$ so are g and h. □

Induced representations

Suppose H a subgp of Γ and V a Hilbert H-module.

- The *induced representation*, $\mathrm{Ind}_H^\Gamma(V)$, is the completion of $\mathbf{R}\Gamma \otimes_{\mathbf{R}H} V$.
- V a Hilbert H-module $\implies$ $\mathrm{Ind}_H^\Gamma(V)$ a Hilbert Γ-module.
- For example, F a finite subgp of Γ and $\mathbf{R}$ denotes the trivial 1-dimensional representation of F, then $\mathrm{Ind}_F^\Gamma(\mathbf{R})$ is $\cong$ $L^2(\Gamma/F)$.

L^2-chains

X is a proper Γ-CW-complex with X/Γ compact $\implies$ there are only finitely many Γ-orbits of cells in X.

- $C_*(X) :=$ the usual cellular chain complex on X.
- An element of $C_i(X)$ (an *i-chain*) is a finitely supported function φ from the set of oriented i-cells in X to $\mathbb{Z}$ satisfying $\varphi(\overline{e}) = -\varphi(e)$ (where e and $\overline{e}$ denote the same cell but with opposite orientations).
- $$\begin{aligned} L^2C_i(X) :&= L^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_i(X) \\ &= \{\varphi : \{i\text{-cells}\} \to \mathbf{R} \mid \sum \varphi(e)^2 < \infty\} \end{aligned}$$

Cochains

- The definition of the space of L^2-cochains on X is the same:

$$L^2C^i(X) := L^2C_i(X).$$

- If c is an i-cell of X, then the space of L^2-chains supported on the Γ-orbit of c is $\cong L^2(\Gamma/\Gamma_c)$. Since Γ_c is finite, $L^2(\Gamma/\Gamma_c)$ is a Hilbert Γ-module.
- Since $\mathrm{Card}\{\Gamma\text{-orbits of } i\text{-cells}\} < \infty$, $L^2C_i(X)$ is the direct sum of a finite number of such subspaces
  $\implies$ it is a Hilbert Γ-module.

Boundary maps

The *boundary* $\partial_i : L^2C_i(X) \to L^2C_{i-1}(X)$ and the *coboundary* $\delta^i : L^2C^i(X) \to L^2C^{i+1}(X)$ are defined by the usual formulas. δ^i and ∂_{i+1} are the adjoints of one another.

Proposition

∂_i and δ_i are maps of Hilbert Γ-modules.

Proof.

∂_i is induced from usual $\partial_i : C_i(X) \to C_{i-1}(X)$. So, it is Γ-equivariant and bounded. $\delta_i = \partial^*_{i+1}$. □

Define subspaces of $L^2C_i(X)$:

$$Z_i(X) := \operatorname{Ker} \partial_i \qquad Z^i(X) := \operatorname{Ker} \delta^i$$
$$B_i(X) := \operatorname{Im} \partial_{i+1} \qquad B^i(X) := \operatorname{Im} \delta^{i-1}$$

the L^2-*cycles*, *-cocycles*, *-boundaries* and *-coboundaries*, resp. The corresponding quotient spaces

$$L^2H_i(X) := Z_i(X)/B_i(X)$$
$$L^2H^i(X) := Z^i(X)/B^i(X)$$

are the *unreduced* L^2*-homology* and *-cohomology groups*, resp.

Reduced (co)homology

The subspaces $B_i(X)$ and $B^i(X)$ need not be closed. So, the quotient spaces need not be Hilbert spaces. Let $\overline{B_i}(X)$ (resp., $\overline{B^i}(X)$) denote the closure of $B_i(X)$ (resp., $B^i(X)$).
The *reduced L^2-homology* and *-cohomology groups* are defined by:

$$L^2\mathcal{H}_i(X) := Z_i(X)/\overline{B_i}(X)$$
$$L^2\mathcal{H}^i(X) := Z^i(X)/\overline{B^i}(X).$$

They are Hilbert Γ-modules (each can be identified with the orthogonal complement of a closed Γ-stable subspace in a closed Γ-stable subspace of $C_i(X)$).

Hodge decomposition

$\delta^{i-1}(x) \cdot y = x \cdot \partial_i(y), \qquad \forall x \in L^2C^{i-1}(X), y \in L^2C_i(X)$
$\implies$ orthogonal direct sum decompositions:

$$L^2C_i(X) = \overline{B_i}(X) \oplus Z^i(X)$$
$$L^2C_i(X) = \overline{B^i}(X) \oplus Z_i(X).$$

$$\delta^{i-1}(x) \cdot \partial_{i+1}(y) = x \cdot \partial_i \partial_{i+1}(y) = 0 \implies \overline{B_i}(X) \perp \overline{B^i}(X).$$

$$L^2C_i(X) = \overline{B_i}(X) \oplus \overline{B^i}(X) \oplus (Z_i(X) \cap Z^i(X)).$$

Both $L^2\mathcal{H}_i(X)$ and $L^2\mathcal{H}^i(X)$ are $\cong Z_i(X) \cap Z^i(X)$, the *harmonic* *i*-cycles.

Combinatorial Laplacian

Define $\Delta : L^2C_i(X) \to L^2C_i(X)$ by

$$\Delta = \delta^{i-1}\partial_i + \partial_{i+1}\delta^i.$$

Check:

$$L^2\mathcal{H}_i(X) = Z_i(X) \cap Z^i(X) = \operatorname{Ker}\Delta$$

0-dimensional cohomology

Example

A 0-cochain = a function f on $\{\text{vertices in } X\}$
e an edge from v_0 to v_1, then $\delta f(e) = v_1 - v_0$.
So, $\delta f = 0 \implies f$ is constant (provided X^1 is connected).
If Γ is infinite (so Vert(X) is infinite), the only constant function in L^2 is 0.
So, X^1 connected, Γ infinite $\implies L^2H^0(X) = 0$. Therefore, $L^2\mathcal{H}^0(X) = L^2\mathcal{H}_0(X) = 0$.
On other hand, $L^2H_0(X)$ need not $= 0$, e.g., $L^2H_0(\mathbf{R}) \neq 0$.

Top-dimensional homology

Example

- Suppose X is an n-dimensional pseudomanifold. (Each $(n-1)$-cell is contained in precisely two n-cells.) Also, suppose X is orientable and gallery connected.
- Then we can orient the n-cells s.t. the sum of all oriented n-cells is an infinite cycle, the *fundamental cycle*.
- Any n-cycle is a multiple of fund cycle.
- If Γ is infinite, this cycle is not L^2.
- Hence, $L^2H_n(X) = 0$.
- So, $L^2\mathcal{H}_n(X) = L^2\mathcal{H}^n(X) = 0$.

Universal cover of a surface

Example

$X =$ universal cover of a surface, genus > 0.

By previous two examples, $L^2\mathcal{H}_0(X) = 0$ and $L^2\mathcal{H}_2(X) = 0$.

So, $L^2\mathcal{H}_*(X)$ is concentrated in dimension 1.

von Neumann algebra $\mathcal{N}(\Gamma)$

$L^2(\Gamma)$ is an $\mathbf{R}\Gamma$-bimodule. Three equivalent definitions $\mathcal{N}(\Gamma)$:

- $\mathcal{N}(\Gamma)$ is the algebra of all maps from $L^2(\Gamma)$ to itself.
- $\mathcal{N}(\Gamma)$ is the double commutant of the right $\mathbf{R}\Gamma$-action on $L^2(\Gamma)$.
- $\mathcal{N}(\Gamma)$ is the weak closure of the algebra of operators $\mathbf{R}(\Gamma)$ acting from the right on $L^2(\Gamma)$.

The Γ-trace of a (1×1)-matrix

Given $\varphi \in \mathcal{N}(\Gamma)$, define

$$\mathrm{tr}_\Gamma(\varphi) := \varphi(e_1) \cdot e_1$$

Standard arguments show:

- $\mathrm{tr}_\Gamma(\varphi) = \mathrm{tr}_\Gamma(\varphi^*)$ (φ^* is the adjoint of φ) and
- $\mathrm{tr}_\Gamma(\varphi\psi) = \mathrm{tr}_\Gamma(\varphi)\,\mathrm{tr}_\Gamma(\psi)$.

The Γ-trace of a $(n \times n)$-matrix

Given $n \in \mathbb{N}$, $L^2(\Gamma)^n$ = orthogonal direct sum of n copies of $L^2(\Gamma)$. $M_n(\mathcal{N}(\Gamma)) = \{(n \times n)\text{-matrices with coefficients in } \mathcal{N}(\Gamma)\}$. Given $\Phi = (\varphi_{ij}) \in M_n(\mathcal{N}(\Gamma))$, define

$$\mathrm{tr}_\Gamma(\Phi) := \sum_{i=1}^{n} \mathrm{tr}_\Gamma(\varphi_{ii}).$$

$\mathrm{tr}_\Gamma(\;)$ has the usual properties. Suppose $\Phi, \Psi \in M_n(\mathcal{N}(\Gamma))$. Then

Properties of $\mathrm{tr}_\Gamma(\ )$

- $\mathrm{tr}_\Gamma(\Phi) = \mathrm{tr}_\Gamma(\Phi^*)$.
- $\mathrm{tr}_\Gamma(\Phi\Psi) = \mathrm{tr}_\Gamma(\Psi\Phi)$.
- Φ is self-adjoint and idempotent. Then $\mathrm{tr}_\Gamma(\Phi) \geq 0$, with equality iff $\Phi = 0$.
- Given any Hilbert Γ-module V isomorphic to $L^2(\Gamma)^n$ and any self-map Φ of V, one can define $\mathrm{tr}_\Gamma(\Phi)$.

Γ-dimension

V a Hilbert Γ-module. Choose an embedding of V as a closed, Γ-stable subspace of $L^2(\Gamma)^n$ for some $n \in \mathbb{N}$.

$p_V : L^2(\Gamma)^n \to L^2(\Gamma)^n$ is orthogonal projection onto V. The *von Neumann dimension* of V (or its Γ-*dimension*) is defined by

$$\dim_\Gamma(V) := \mathrm{tr}_\Gamma(p_V).$$

If $E \subset L^2(\Gamma)$ is a not neccessarily closed Γ-stable subspace of $L^2(\Gamma)^n$, put $\dim_\Gamma(E) := \dim_\Gamma(\overline{E})$.

Properties of $\dim_\Gamma(\ )$

- $\dim_\Gamma(\ )$ is well-defined.
- $\dim_\Gamma(V) \in [0, \infty)$.
- $\dim_\Gamma(V) = 0$ iff $V = 0$.
- If $\Gamma = \{1\}$ (s.t. $\dim_{\mathbf{R}} V < \infty$), then $\dim_\Gamma(V) = \dim_{\mathbf{R}}(V)$.
- $\dim_\Gamma(L^2(\Gamma)) = 1$.
- $\dim_\Gamma(V \oplus W) = \dim_\Gamma(V) + \dim_\Gamma(W)$.
- If $f : V \to W$ is a map of Hilbert Γ-modules, then

$$\dim_\Gamma(V) = \dim_\Gamma(\operatorname{Ker} f) + \dim_\Gamma(\overline{\operatorname{Im} f}).$$

More properties

- $f : V \to W$ a map, $f^* : W \to V$ its adjoint, then $\operatorname{Ker} f$ and $\overline{\operatorname{Im} f^*}$ are orthogonal complements in V. Hence,

$$\dim_\Gamma(V) = \dim_\Gamma(\operatorname{Ker} f) + \dim_\Gamma(\overline{\operatorname{Im} f^*}).$$

  So, $\dim_\Gamma(\overline{\operatorname{Im} f}) = \dim_\Gamma(\overline{\operatorname{Im} f^*})$.
- If $0 \to V_n \to \cdots \to V_0 \to 0$ a weak exact sequence, then

$$\sum_{i=0}^{n} (-1)^i \dim_\Gamma(V_i) = 0.$$

- H a subgroup of finite index m in $\Gamma \implies \dim_H(V) = m \dim_\Gamma(V)$.

and more properties

- Γ finite $\implies \dim_\Gamma(V) = \frac{1}{|\Gamma|}\dim(V)$.
- H a subgp of Γ, W a Hilbert H-module, then

$$\dim_\Gamma(\mathrm{Ind}_H^\Gamma(W)) = \dim_H(W).$$

- F a finite subgp of $\Gamma \implies \dim_\Gamma(L^2(\Gamma/F)) = \frac{1}{|F|}$.
- $\Gamma = \Gamma_1 \times \Gamma_2$, V_j is a Hilbert Γ_j-module for $j = 1, 2$. Then

$$\dim_\Gamma(V_1 \widehat{\otimes} V_2) = \dim_{\Gamma_1}(V_1)\dim_{\Gamma_2}(V_2).$$

Next time

L^2-Betti numbers

Define $L^2 b_i(X;\Gamma) := \dim_\Gamma(L^2\mathcal{H}^i(X))$.