Lecture 2: L^2-Betti numbers

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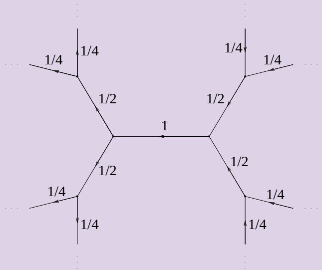
Last time

$$L^2C_i(X) := \{\varphi : \{i\text{-cells}\} \to \mathbf{R} \mid \sum \varphi(e)^2 < \infty\}$$

$$L^2\mathcal{H}_i(X) := Z_i(X)/\overline{B_i}(X)$$
$$L^2\mathcal{H}^i(X) := Z^i(X)/\overline{B^i}(X).$$

A harmonic 1-cycle



$$1 + 4\left(\frac{1}{2}\right)^2 + 8\left(\frac{1}{4}\right)^2 + \cdots = 1 + \sum 2^{n+1}\left(\frac{1}{2}\right)^{2n}$$
$$= 1 + \sum \left(\frac{1}{2}\right)^{n-1} < \infty$$

L^2 algebraic topology

(X, Y) a pair of CW-complexes. Γ acts properly and cellularly on X. Y is a Γ-stable subcx.
Reduced L^2-(co)homology groups $L^2\mathcal{H}_i(X, Y)$ are defined in the usual manner by completing of $C_i(X, Y)$. Versions of most of the Eilenberg-Steenrod Axioms hold for $L^2\mathcal{H}_*(X, Y)$.
Some standard properties.

Functorality

$f : (X_1, Y_1) \to (X_2, Y_2)$ a Γ-map. There is an induced map $f_* : L^2\mathcal{H}_i(X_1, Y_1) \to L^2\mathcal{H}_i(X_2, Y_2)$ giving a functor from pairs of Γ-complexes and Γ-homotopy classes of maps to Hilbert Γ-modules.

Properties

Exact sequence of a pair

The sequence,

$$\to L^2\mathcal{H}_i(Y) \to L^2\mathcal{H}_i(X) \to L^2\mathcal{H}_i(X,Y) \to$$

is weakly exact.

Excision

U is a Γ-stable subset of Y s.t. $Y - U$ is a subcx. Then $(X-U, Y-U) \hookrightarrow (X,Y)$ induces an iso:

$$L^2\mathcal{H}_i(X-U, Y-U) \cong L^2\mathcal{H}_i(X,Y).$$

Mayer-Vietoris sequence

$X = X_1 \cup X_2$, with X_1, X_2 Γ-stable subcxes. The M-V sequence,

$$\to L^2\mathcal{H}_i(X_1 \cap X_2) \to L^2\mathcal{H}_i(X_1) \oplus L^2\mathcal{H}_i(X_2) \to L^2\mathcal{H}_i(X) \to$$

is weakly exact.

Twisted products

H a subgp of Γ and Y is a space with H-action.
The *twisted product*:

$$\Gamma \times_H Y := (\Gamma \times Y)/H$$

where the H-action is defined by $h \cdot (g, y) = (gh^{-1}, hy)$.

It is a left Γ-space and a Γ-bundle over Γ/H. Since Γ/H is discrete, $\Gamma \times_H Y$ is a disjoint union of copies of Y, one for each element of Γ/H. If Y is an H-CW-complex, then $\Gamma \times_H Y$ is a Γ-CW-complex.

More properties

Twisted products and the induced representation

$$L^2\mathcal{H}_i(\Gamma \times_H Y) \cong \mathrm{Ind}_H^\Gamma(L^2\mathcal{H}_i(Y)).$$

Künneth Formula

$\Gamma = \Gamma_1 \times \Gamma_2$ and X_j is a Γ_j-CW-cx , $j = 1, 2$. Then $X_1 \times X_2$ is a Γ-CW-cx and

$$L^2\mathcal{H}_k(X_1 \times X_2) \cong \sum_{i+j=k} L^2\mathcal{H}_i(X_1)\widehat{\otimes} L^2\mathcal{H}_j(X_2),$$

where $\widehat{\otimes}$ denotes the completed tensor product.

Reduced homology of Euclidean space

Example

We know for $X = \mathbb{E}^1$ $(= \mathbf{R})$ with standard action of $\Gamma = \mathbb{Z}$ that

$$L^2\mathcal{H}_k(\mathbb{E}^1) = 0 \quad \text{for } k = 0, 1.$$

By the Künneth Formula,

$$L^2\mathcal{H}_k(\mathbb{E}^n) = 0, \quad \forall k.$$

Review of $\dim_\Gamma(\ )$

The *von Neumann dimension* of V (or its Γ-*dimension*) is defined by

$$\dim_\Gamma(V) := \mathrm{tr}_\Gamma(p_V).$$

Properties

- $\dim_\Gamma(V) \in [0, \infty)$ and $\dim_\Gamma(V) = 0$ iff $V = 0$.
- $\Gamma = \{1\} \implies \dim_\Gamma(V) = \dim_{\mathbf{R}}(V)$.
- $\dim_\Gamma(L^2(\Gamma)) = 1$.
- $\dim_\Gamma(V \oplus W) = \dim_\Gamma(V) + \dim_\Gamma(W)$.

More properties of $\dim_\Gamma(\ )$

- $f : V \to W$ a map of Hilbert Γ-modules, then

$$\begin{aligned}\dim_\Gamma(V) &= \dim_\Gamma(\operatorname{Ker} f) + \dim_\Gamma(\overline{\operatorname{Im} f}) \\ &= \dim_\Gamma(\operatorname{Ker} f) + \dim_\Gamma(\overline{\operatorname{Im} f^*}).\end{aligned}$$

- $H \subset \Gamma$ index $m \implies \dim_H(V) = m \dim_\Gamma(V)$.
- Γ finite $\implies \dim_\Gamma(V) = \frac{1}{|\Gamma|} \dim(V)$.
- $H \subset \Gamma$, W then

$$\dim_\Gamma(\operatorname{Ind}_H^\Gamma(W)) = \dim_H(W).$$

- $$\dim_{\Gamma_1 \times \Gamma_2}(V_1 \widehat{\otimes} V_2) = \dim_{\Gamma_1}(V_1) \dim_{\Gamma_2}(V_2).$$

Definition

The i^{th} L^2-*Betti number* of X is:

$$L^2 b_i(X;\Gamma) := \dim_\Gamma L^2\mathcal{H}_i(X).$$

- If X is contractible (and the Γ-action is proper and cocompact), then $L^2 b_i(X;\Gamma)$ is an invariant of Γ.
- Denote it $L^2 b_i(\Gamma)$ and call it the L^2-*Betti number of* Γ.

Properties of L^2-Betti numbers

- $L^2 b_i(X;\Gamma) = 0 \implies L^2\mathcal{H}_i(X) = 0$.
- $H \subset \Gamma$ index $m \implies L^2 b_i(X;H) = m(L^2 b_i(X;\Gamma))$.
- Künneth Formula:

$$L^2 b_k(X_1 \times X_2; \Gamma_1 \times \Gamma_2) = \sum_{i+j=k} L^2 b_i(X_1;\Gamma_1) L^2 b_j(X_2;\Gamma_2)$$

- Suppose Γ_1, Γ_2 both infinite. Then

$$L^2 b_i(\Gamma_1 * \Gamma_2) = \begin{cases} L^2 b_i(\Gamma_1) + L^2 b_i(\Gamma_2), & \text{if } i > 1, \\ L^2 b_1(\Gamma_1) + L^2 b_1(\Gamma_2) - 1 & \text{if } i = 1 \end{cases}$$

(Mayer-Vietoris sequence).

Orbihedral Euler characteristic

$$\chi^{orb}(X/\Gamma) := \sum_{\text{orbits of cells}} \frac{(-1)^{\dim c}}{|\Gamma_c|} \in \mathbf{Q},$$

where $|\Gamma_c|$ is the order of the stabilizer of the cell c.

- If Γacts freely, then $\chi^{orb}(X/\Gamma)$ is the ordinary Euler characteristic $\chi(X/\Gamma)$.
- If $H \subset \Gamma$ is index m, then $\chi^{orb}(X/H) = m\chi^{orb}(X/\Gamma)$.
- $\chi^{orb}(X_1/\Gamma_1 \times X_2/\Gamma_2) = \chi^{orb}(X_1/\Gamma_1)\chi^{orb}(X_2/\Gamma_2)$

Atiyah's Formula

The L^2-Euler characteristic

$$L^2\chi(X;\Gamma) := \sum_{i=0}^{\infty}(-1)^i L^2 b^i(X;\Gamma).$$

Theorem (Atiyah)

$$\chi^{orb}(X/\Gamma) = L^2\chi(X;\Gamma).$$

Lemma

C_* a chain complex of Hilbert Γ-modules. $\mathcal{H}_i(C_*)$ = reduced homology. Then

$$\sum_i(-1)^i \dim_\Gamma C_i = \sum_i(-1)^i \dim_\Gamma \mathcal{H}_i(C_*).$$

Proof of Lemma

Proof. Put $Z_i := \mathrm{Ker}(C_i \to C_{i-1})$, $\quad B_i := \overline{\mathrm{Im}(C_{i+1} \to C_i)}$ and

$$c_i := \dim_\Gamma(C_i), \qquad h_i := \dim_\Gamma(\mathcal{H}_i(C_*))$$
$$z_i := \dim_\Gamma(Z_i), \qquad b_i := \dim_\Gamma(B_i).$$

Weak short exact sequences:

$$0 \to Z_i \to C_i \to B_{i-1} \to 0$$
$$0 \to B_i \to Z_i \to \mathcal{H}_i \to 0.$$

So, $c_i = z_i + b_{i-1}$ and $z_i = h_i + b_i$.

$$\sum(-1)^i c_i = \sum(-1)^i(z_i + b_{i-1}) = \sum(-1)^i(h_i + b_i + b_{i-1})$$
$$= \sum(-1)^i h_i.$$

□

Proof of Atiyah's Formula.

$$c_i := \dim_\Gamma(C_i(X)) = \sum_{\text{orbits of } i\text{-cells}} \dim_\Gamma(L^2(\Gamma/\Gamma_c))$$
$$= \sum \frac{1}{|\Gamma_c|}.$$

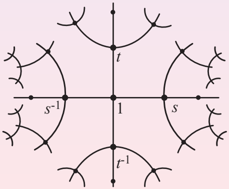
So, $\sum(-1)^i c_i = \chi^{orb}(X/\Gamma)$ and Lemma $\implies$ Formula. □

Free groups

Example

Y = a figure 8. T its universal cover (a regular 4-valent tree).
F_2 = free group of rank 2.
$L^2 b_0(T; F_2) = 0$ (because F_2 is infinite). So,

$$L^2 b_1(T; F_2) = -L^2\chi(T; F_2) = -\chi(Y) = 1.$$



Surface groups

Example

Y = closed surface of genus g (> 0), X its univ cover, $\Gamma = \pi_1(Y)$. Showed previously $L^2 b_0 = 0 = L^2 b_2$. So,

$$L^2 b_1(X;\Gamma) = -L^2\chi(X:\Gamma) = -\chi(Y) = 2g-2$$

Notation

$B\Gamma := K(\Gamma, 1)$ and $E\Gamma :=$ its univ cover.

2-dimensional groups

Example

Suppose $B\Gamma$ is a finite 2-dim cx (e.g., Γ is a small cancellation gp).

$$g = \#\{\text{1-cells}\} = \#\{\text{generators}\}$$
$$r = \#\{\text{2-cells}\} = \#\{\text{relations}\}$$

$\chi(\Gamma) = 1 - g + r$ and $L^2\chi(\Gamma) = L^2b_2(\Gamma) - L^2b_1(\Gamma)$. So,

$$r \geq g \implies \chi(\Gamma) > 0 \implies L^2b_2(\Gamma) > 0$$
$$r < g - 1 \implies \chi(\Gamma) < 0 \implies L^2b_1(\Gamma) > 0.$$

Deficiency of a finitely presented group

Definition

The *deficiency* of a presentation of Γ is $g - r = \#\{generators\} - \#\{relations\}$. The *deficiency* of a gp Γ, denoted $\operatorname{def}(\Gamma)$, is the maximum of $g - r$ over all presentations of Γ.

Let Y be presentation cx with $\chi(Y)$ minimum. Since Y can be completed to $B\Gamma$ by attaching cells of dim ≥ 3, $b_1(Y) = b_1(\Gamma)$ and $b_2(Y) \geq b_2(\Gamma)$.
So, $\operatorname{def}(\Gamma) = 1 - \chi(Y) = b_1(Y)) - b_2(Y) \leq b_1(\Gamma) - b_2(\Gamma)$.
Similarly, $\operatorname{def}(\Gamma) \leq L^2 b_1(\Gamma) - L^2 b_2(\Gamma) + 1$. So, for example, $L^2 b_1(\Gamma) = 0 \implies \operatorname{def}(\Gamma) \leq 1$.

Poincaré duality

Theorem

X^n an n-mfld, then $L^2 b_i(X^n;\Gamma) = L^2 b_{n-i}(X^n;\Gamma)$.

There is a nonsingular pairing:

$$L^2\mathcal{H}^i(X) \otimes L^2\mathcal{H}^{n-i}(X) \to \mathbf{R},$$

defined by $\alpha \otimes \beta \to \langle \alpha \cup \beta, [X] \rangle$.
Point is the cup product of 2 L^2-classes is L^1, $[X]$ is a bounded class and you can evaluate an L^1-cohomology class on a bounded homology class.

Remark

Suppose X is the univ cover of a Poincaré duality cx. Same argument shows $L^2\mathcal{H}^i(X) \cong L^2\mathcal{H}^{n-i}(X)$.

Example

Suppose Γ is a PD^2-gp (i.e., the fund gp of a 2-dim PD cx whose univ cover X is contractible). This implies Γ is infinite. So, $L^2b_0 = 0$. By Poincaré duality $L^2b_2 = 0$. So, $\chi(\Gamma) = \chi(X/\Gamma) = -L^2b_1(X;\Gamma) \leq 0$. So,

$$b_1(\Gamma) - 2 = -b_0(\Gamma) + b_1(\Gamma) - b_2(\Gamma) \geq 0.$$

So, $b_1(\Gamma) = \mathrm{rk}(\Gamma^{ab}) \geq 2$. (This fact was important in proof that PD^2-gps are surface gps.)

The Euler Characteristic Conjecture

A space Y is *aspherical* if its univ cover is contractible.

Example

A complete Riemannian mfld M of nonpositive sectional curvature is aspherical. (Pf: $\exp : T_x\widetilde{M} \to \widetilde{M}$ is a diffeomorphism.)

Conjecture

If M^{2k} is a closed aspherical mfld, then $(-1)^k\chi(M^{2k}) \geq 0$.

- In nonpositively curved context this is called the Chern–Hopf Conj or Hopf Conj.
- Conj doesn't follow from the Gauss–Bonnet Theorem.

Euler Char Conj

- For odd-dimensional mflds, $\chi = 0$. (Pf: Poincaré duality).
- Conj true for surfaces: M^2 is aspherical iff $\chi(M^2) \leq 0$. (Pf: $\chi = 0 \iff$ univ cover $= \mathbb{E}^2$. $\chi < 0 \iff$ univ cover $= \mathbb{H}^2$.)
- Conj true for product of surfaces: if M^{2k} is product of k surfaces of nonpositive Euler char, then $(-1)^k M^{2k} \geq 0$ (because χ is multiplicative for products).
- True for closed hyperbolic mflds and other locally symmetric mflds.

Other versions

Conjecture

Suppose Γ *acts properly and cocompactly on contractible* $\widetilde{M}^{2k}$ *(i.e.,* $\widetilde{M}^{2k}/\Gamma$ *is an aspherical orbifold). Then*

$$(-1)^k \chi^{orb}(\widetilde{M}^{2k}/\Gamma) \geq 0.$$

Conjecture

Suppose Γ *is a* PD^{2k}*-gp. Then* $(-1)^k \chi(\Gamma) \geq 0$.

The Dodziuk–Singer Conjecture

Conjecture

$\widetilde{M}^n$ a contractible mfld with cocompact proper Γ-action. Then

$$L^2 b^i(\widetilde{M}^n; \Gamma) = 0, \quad \forall i \neq \frac{n}{2}.$$

If n is odd, this means all L^2-Betti numbers are 0.

Theorem

Singer Conj. $\Longrightarrow$ Euler Char. Conj.

Proof.

Suppose $n = 2k$, $\Gamma = \pi_1(M^n)$. Singer Conj $\implies$ only $L^2 b_k \neq 0$. Atiyah's Formula gives:

$$(-1)^k L^2 b_k(\widetilde{M}^{2k}; \Gamma) = \chi^{orb}(\widetilde{M}^{2k}/\Gamma).$$

$$\text{So,} \qquad (-1)^k \chi^{orb}(\widetilde{M}^{2k}/\Gamma) \geq 0.$$

□