Lecture 3

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1. Mapping tori
2. Amenable groups
    - The Følner Condition
    - The Cheeger–Gromov Theorem
3. L^2 de Rham Theory
    - Hyperbolic manifolds
    - Kähler hyperbolic manifolds

Last time

$$L^2C_i(X) := \{\varphi : \{i\text{-cells}\} \to \mathbf{R} \mid \sum \varphi(e)^2 < \infty\}$$

$$L^2\mathcal{H}^i(X) := Z^i(X)/\overline{B^i}(X).$$

The i^{th} *L^2-Betti number* of X is:

$$L^2b_i(X;\Gamma) := \dim_\Gamma L^2\mathcal{H}^i(X).$$

If X is contractible (and the Γ-action is proper and cocompact), then

$$L^2b_i(\Gamma) := L^2b_i(X;\Gamma).$$

Bundles over S^1

Suppose $F \to M \to S^1$ is a fiber bundle with fiber F. $\widetilde{M}$ is the universal cover, $\Gamma = \pi_1(M)$.

Question

(Gromov). *Is it true that* $L^2 b_i(\widetilde{M}; \Gamma) = 0$, $\forall i$*?*

It turns out it's easier to answer a more general question about mapping tori. Suppose F a CW complex and $f : F \to F$ a self map.

Definition

$T_f := (F \times [0,1])/\sim$ is called the *mapping torus of* f, where $\sim$ is defined by $(x,0) \sim (f(x),1)$.

Lück's Theorem

- There is a canonical map $T_f \to S^1$.
- If f is a homeomorphism, $T_f \to S^1$ is a fiber bundle.
- If f is a homotopy equivalence, $T_f \to S^1$ is a fibration with fiber F.

Suppose canonical epimorphism $\pi_1(T_f) \to \mathbb{Z}$ factors as $\varphi \circ \psi$ where $\psi : \pi_1(T_f) \to \Gamma$ and $\varphi : \Gamma \to \mathbb{Z}$ are both onto (e.g. Γ could $= \pi_1(T_f)$). Let $\widetilde{T}_f \to T_f$ be the covering space corresponding to ψ.

Theorem

$$L^2 b_i(\widetilde{T_f}; \Gamma) = 0, \ \forall i.$$

Observations

- By Cellular Approx Theorem, f is homotopic to a *cellular map* (i.e., $f(F^i) \subset F^i$). So, let us assume this.
- Denote number of i-cells in F by $c_i(F)$. Then T_f has a CW structure with

$$c_i(T_f) = c_{i-1}(F) + c_i(F).$$

- Let $\Gamma_n := \varphi^{-1}(n\mathbb{Z}) \subset \Gamma$. So, $\widetilde{T}_f/\Gamma_n \to T_f$ is an n-fold covering.
- Exercise: There is a homotopy equivalence $T_{f^n} \to \widetilde{T}_f/\Gamma_n$.

Proof of Lück's Theorem

Desired formula

$$L^2 b_i(\widetilde{T}_f;\Gamma) = 0$$

Proof.

$$L^2 b_i(\widetilde{T}_f;\Gamma_n) \leq \dim_{\Gamma_n}(L^2 C_i(\widetilde{T}_f)) = c_i(T_{f_n}) = c_{i-1}(F) + c_i(F).$$

By multiplicativity of the $L^2 b_i$, $L^2 b_i(\widetilde{T}_f;\Gamma) = \frac{1}{n} L^2 b_i(\widetilde{T}_f;\Gamma_n)$. So,

$$L^2 b_i(\widetilde{T}_f;\Gamma) \leq \frac{c_{i-1}(F) + c_i(F)}{n}$$

Taking the limit as $n \to \infty$, we get $L^2 b_i(\widetilde{T}_f;\Gamma) = 0$. □

Amenable groups

A *mean* on a gp G is a linear map, $M : L^\infty(G) \to \mathbf{R}$, s.t.

- $M(1) = 1$ (where $1 : G \to \mathbf{R}$ is the constant function 1).
- M is G-invariant (i.e., $M(g\varphi) = M(\varphi)$, $\forall g \in G$).
- $\varphi \geq 0 \implies M(\varphi) \geq 0$.

Definition

G is *amenable* if it admits a mean.

There is a more workable condition.

The Følner Condition

Let Γ be a finitely generated gp. Λ its Cayley graph w.r.t. some finite set of generators.
Suppose $F \subset \Gamma$.

$$\partial F := \{g \in F \mid \exists \text{ an edge of } \Lambda \text{ connecting } g \text{ to an element } \notin F\}$$

The Følner Condition

$\forall \varepsilon > 0$, $\exists$ a finite subset $F \subset \Gamma$ s.t.

$$\frac{|\partial F|}{|F|} < \varepsilon.$$

Theorem

A fin gen Γ is amenable $\iff$ Følner Condition.

Example

- $\mathbb{Z}$ is amenable.
- Finite gps, abelian gps and solvable gps are all amenable.

The Cheeger–Gromov Theorem

Since $L^2C^i(X) \subset C^i(X;\mathbf{R})$, there are canonical maps, $\mathrm{can} : L^2H^i(X) \to H^i(X;\mathbf{R})$ and $\mathrm{can} : L^2\mathcal{H}^i(X) \to H^i(X;\mathbf{R})$. (The second takes a harmonic cocycle to an ordinary one.)

Theorem

Suppose Γ is an infinite amenable gp. Then $\mathrm{can} : L^2\mathcal{H}^i(X) \to H^i(X;\mathbf{R})$ is injective.

Corollary

- *If X is contractible, then $L^2\mathcal{H}^i(X) = 0$, $\forall i$.*
- $L^2 b_i(\Gamma) = 0, \forall i$. *($\Gamma$ infinite amenable.)*

Corollary

Γ infinite amenable acting (not necessarily cocompactly) on a contractible X (with uniform geometry). Then $L^2 b_i(\Gamma) = 0$, $\forall i$.

Corollary

Γ contains an infinite normal amenable subgp A. Then $\chi(\Gamma) = 0$ (i.e., $\chi^{orb}(X/\Gamma) = 0$).

Example

Assume $S^1 \to Y \to B$ an S^1-bundle, B aspherical, $\Gamma = \pi_1(Y)$. Then

$$1 \to \mathbb{Z} \to \Gamma \to \pi_1(B) \to 1.$$

So, $L^2 b_i(\Gamma) = 0$, $\forall i$.

Sketch of Eckmann's proof of Cheeger–Gromov Thm

- Put $K := \mathrm{Ker}(L^2\mathcal{H}^i(X) \to H^i(X;\mathbf{R}))$.
  Idea: Use Følner Condition to show $\dim_\Gamma K = 0$.
- Γ is countable. Følner Condition $\implies$ $\exists$ an exhaustion $F_1 \subset F_2 \subset \cdots$ s.t

$$\bigcup_{j=1}^{\infty} F_j = \Gamma \quad \text{and} \quad \lim_{j\to\infty} \frac{|\partial F_j|}{|F_j|} = 0.$$

- D = fund domain = $\bigcup$ closed cells, 1 cell in each Γ-orbit. Put

$$X_j := \bigcup F_j D \quad \text{and} \quad \partial X_j := \text{its bdry in } X$$

- Let $P : L^2C^i(X) \to L^2\mathcal{H}^i(X)$ be orthogonal proj and π_j composition of P with inclusion $L^2C^i(X_j) \hookrightarrow L^2C^i(X)$.

Proof

- $\dim_{\mathbf{R}} \pi_j(K) = \sum_{gc \subset F_j D} \pi_j(gc) \cdot gc = |F_j| \sum_{c \subset D} P(c) \cdot c = |F_j| \dim_\Gamma K$. So,

$$\dim_\Gamma K = \frac{dim_{\mathbf{R}} \pi_j(K)}{|F_j|}$$

- Estimate: $\dim_{\mathbf{R}} \pi_j(K) \leq \dim_{\mathbf{R}}(C^i(\partial X_j; \mathbf{R}) \leq |\partial F_j| \alpha_i$, where $\alpha_i = \#(i\text{-cells in } D)$. So,

$$\dim_\Gamma K \leq \frac{|\partial F_j|}{|F_j|} \alpha_i \to 0.$$

□

Review of classical theory

- M a smooth closed mfld.
- $\Omega^p(M)$ the vector space of smooth p-forms.
- $d : \Omega^p \to \Omega^{p+1}$, the exterior differential
- The de Rham cochain cx:

$$\cdots \to \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \to \cdots$$

- The corresponding cohomology gps: $H^*_{\mathrm{dR}}(M)$.
- If M has a smooth triangulation, integration of p -forms over p-simplices gives an iso:

$$H^p_{\mathrm{dR}}(M) \to H^p(M; \mathbf{R}) \text{ (or } H^p_{\mathrm{sing}}(M; \mathbf{R})).$$

Inner product

Suppose $\dim M = n$.

- $\exists$ iso, $\Lambda^p(\mathbf{R}^n) \xrightarrow{\cong} \Lambda^{n-p}(\mathbf{R}^n)$.
- inducing *Hodge star operator* $* : \Omega^p(M) \xrightarrow{\cong} \Omega^{n-p}(M)$, (ignoring the $\pm$ signs).
- Define inner product on $\Omega^p(M)$ by

$$\omega \cdot \eta := \int_M \omega \wedge *\eta.$$

- $d^* : \Omega^p(M) \to \Omega^{p-1}(M)$, the adjoint of d.
- The *Laplacian*, $\Delta := dd^* + d^*d : \Omega^p \to \Omega^p$.

L^2 version

- Suppose $\widetilde{M}$ is smooth mfld with proper, cocompact, smooth Γ-action (e.g. $\widetilde{M} \to M$ is regular covering with deck transformations $= \Gamma$).
- $L^2\Omega^p(\widetilde{M})$ the Hilbert space completion of $\Omega^p_c(\widetilde{M})$.
- As before, we get reduced cohomology gps:

$$L^2\mathcal{H}^p_{\mathrm{dR}}(\widetilde{M}) := \mathrm{Ker}(d)/\overline{\mathrm{Im}(d)}$$

  It is a Hilbert space with orthogonal Γ-action.
- $L^2\mathcal{H}^p_{\mathrm{dR}}(\widetilde{M})$ is $\cong$ the space of square integrable harmonic p-forms.

Dodziuk's Theorem

Theorem (Dodziuk)

$$L^2\mathcal{H}^p_{dR}(\widetilde{M}) \cong L^2\mathcal{H}^p(\widetilde{M}).$$

Singer Conj for $\mathbb{H}^n$

Theorem

$L^2\mathcal{H}^q(\mathbb{H}^n) = 0, \quad \forall q \neq \frac{n}{2}.$

- We will sketch Dodziuk's proof of this.
- All that it uses is that we have a "rotationally symmetric" metric on a mfld M^n diffeomorphic to $\mathbf{R}^n$.
- This means that in polar coordinates metric has the form

$$ds^2 = dr^2 + f(r)^2 d\theta^2$$

where $d\theta$ is the standard round metric on S^{n-1}, r is the Euclidean distance to origin and $f(r)$ satisfies:

$$f(0) = 0, \quad f'(0) = 1, \quad f(r) > 0, \quad \lim_{r\to\infty} f(r) = \infty.$$

Sketch of proof.

- Start with harmonic q-form. Write it in terms of functions of (r, θ). Do some work to conclude:
- $L^2\mathcal{H}^q(M) \neq 0 \implies \int_1^\infty f^{n-2q-1}(r)dr < \infty$.
- By Poincaré duality, $L^2\mathcal{H}^q(M) \cong L^2\mathcal{H}^{n-q}(M)$. So, $L^2\mathcal{H}^{n-q}(M) \neq 0 \implies \int_1^\infty f^{-n+2q-1}(r)dr < \infty$.
- So, both exponents must give convergent integrals. If $n = 2q$, both exponents $= -1$ and we get the condition $\int_1^\infty \frac{1}{f(r)}dr < \infty$.
- $(n - 2q - 1)(-n + 2q - 1) = 1 - (n - 2q)^2$. So, if $n - 2q = \pm 1$, one exponent is 0 and integral diverges. Otherwise, exponents have different signs, so one integral diverges.

□

Kähler manifolds

- M a complex n-mfld ($\dim_{\mathbf{R}}(M) = 2n$) with Hermitian metric.
- The imaginary part of Hermitian metric is a nondegenerate 2-form ω.
- M is a *Kähler mfld* if ω is closed.

Example

$\mathbf{C}P^N$ is a Kähler mfld. A smooth projective variety $M \subset \mathbf{C}P^N$ is a closed Kähler mfld.

The Hard Lefschetz Theorem

M a Kähler mfld with Kähler 2-form ω. Put

$$L = \wedge\, \omega : \Omega^p(M) \to \Omega^{p+2}(M).$$

Theorem

Suppose M is a closed Kähler n-mfld with Kähler class $\alpha := [\omega] \in H^2(M)$. Put $\ell := \wedge\, \alpha : H^*(M) \to H^{*+2}(M)$. Then

$$\ell^{n-i} := \ell \circ \cdots \circ \ell : H^p(M) \to H^{2n-p}(M)$$

is an isomorphism.

Sketch of proof

$L^{n-p} : \Omega^p \to \Omega^{2n-p}$ is an iso (because this holds pointwise, i.e., $\bigwedge^p(T_xM) \to \bigwedge^{2n-p}(T_xM)$ is iso.)

Key Fact: L takes harmonic forms to harmonic forms:

Suppose $\mathcal{H}^p(M) \subset \Omega^p(M)$ denotes the harmonic p-forms, i.e., $\mathcal{H}^p(M) := \operatorname{Ker} \Delta$, then L takes $\mathcal{H}^p(M)$ to $\mathcal{H}^{p+2}(M)$.

Hodge theory $\implies \mathcal{H}^p(M) = H^p(M)$. □

The L^2 Lefschetz Theorem

Theorem

M a Kähler n-mfld, then $L^{n-p} : L^2\Omega^p(M) \to L^2\Omega^{2n-p}(M)$ is iso and takes $L^2\mathcal{H}^p(M)$ to $L^2\mathcal{H}^{2n-p}(M)$.

Definition

Suppose (M, ω) is a closed Kahler mfld, $(\widetilde{M}, \tilde{\omega})$ its univ cover. M is *Kähler hyperbolic* if $\tilde{\omega} = d(\text{bounded})$, i.e., $\exists$ a bounded 1-form η s.t. $\tilde{\omega} = d(\eta)$.

Gromov's Theorem

Theorem (Gromov)

Suppose M is Kähler hyperbolic; $\pi = \pi_1(M)$. Then

- $\forall p \neq n$, $L^2\mathcal{H}^p(\widetilde{M}) = 0$ *and* $L^2 b_p(\widetilde{M};\pi) = 0$.
- $L^2\mathcal{H}^n(\widetilde{M}) \neq 0$ *and* $L^2 b_n(\widetilde{M};\pi) \neq 0$
- $(-1)^n \chi(M) > 0$.

Proof (of first part).

Suppose $\lambda \in L^2\Omega^p(\widetilde{M})$ is closed. We show $L^{n-p}(\lambda)$ represents 0 in cohomology . Note $\lambda \wedge$ (bounded) is also L^2. We have:

$$d(\lambda \wedge \eta) = (\lambda \wedge d\eta) \pm (d\lambda \wedge \eta) = \lambda \wedge \tilde{\omega} = L(\lambda)$$

So, $L(\lambda)$ represents 0 in cohomology. □

Theorem (Gromov)

M a closed Kähler mfld which is negatively curved as Riem mfld $\implies$ M is Kähler hyperbolic.

Example

Here are some other examples of Kähler mflds which are Kähler hyperbolic:

- $\pi_1(M)$ is word hyperbolic and $\pi_2(M) = 0$.
- M is a submfld of a Kähler hyperbolic mfld.
- $\widetilde{M}$ is a Hermitian symmetric space of noncompact type with no Euclidean factor.