Lecture 4

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July 6, 2006

Coxeter groups

Let S be a finite set. A *Coxeter matrix* is a symmetric $S \times S$ matrix (m_{st}) with entries in $\mathbb{N} \cup \{\infty\}$ such that

$$m_{st} = \begin{cases} 1, & \text{if } s = t; \\ \geq 2, & \text{otherwise.} \end{cases}$$

Definition

- Set
$$W := \langle S \mid (st)^{m_{st}} = 1,\ (s,t) \in S \times S \rangle.$$
- (i.e., W is F_S/N where N is the normal subgp generated by the given relations).
- (W, S) is a *Coxeter system*.

Remarks

- It turns out that each $s \in S$ is order 2 and st is order m_{st} (rather than just dividing m_{st}).
- Suppose S is a set of involutions which generate a gp W. Then (W, S) is a Coxeter system iff the fixed point set of each $s \in S$ separates the Cayley graph of (W, S). (This is part of the reason for calling (W, S) an "abstract reflection gp.")
- If W is a geometric reflection gp generated by the set S of isometric reflections across the faces of a convex polytope in $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$, then (W, S) is a Coxeter system.

Special subgroups

For $T \subset S$, put $W_T := \langle T \rangle$. W_T is a *special subgroup*. If W_T is finite, then W_T is called *spherical* and T a *spherical subset* of S. Put

Spherical subsets and cosets

$$\mathcal{S} := \{T \subset S \mid W_T \text{ is finite}\} = \{\text{spherical subsets}\}$$

$$W\mathcal{S} := \coprod_{T \in \mathcal{S}} W/W_T = \{\text{spherical cosets}\}$$

Both are posets. Geometric realization of the second is a cx Σ on which W acts. Geometric realization K of the first is a fund domain (= "fund chamber").

The geometric realization of a poset

Given a poset $\mathcal{P}$, let

$$\mathrm{Flag}(\mathcal{P}) := \{\text{finite, nonempty, totally ordered subsets of } \mathcal{P}\}$$

It is an abstract simplicial cx. Denote the associated geometric simp cx by $|\mathcal{P}|$ and call it *geometric realization* of $\mathcal{P}$. $\mathrm{Vert}(|\mathcal{P}|) = \mathcal{P}$ and $\{p_0, \dots p_n\}$ spans an n-simplex iff (after renumbering) $p_0 < \cdots < p_n$.

Complexes L, K and Σ

Put

$$K := |\mathcal{S}| \qquad\qquad \Sigma := |W\mathcal{S}|.$$

$\mathcal{S}_{>\emptyset}$ is an abstract simp cx. The corresponding geometric cx is denoted L and called the *nerve* of (W, S). L is a finite simp cx. $\mathrm{Vert}(L) = S$ and $T \subset S$ spans a simplex $\iff$ T is spherical.

Facts about K

- K is the cone on the barycentric subdivision of L ($\emptyset \in \mathcal{S}$ is a common vertex of every simplex in K).
- For each $T \in \mathcal{S}$, put $K_T := |\mathcal{S}_{\geq T}|$ and for each $s \in S$, $K_s := K_{\{s\}}$.
- When W is a geometric reflection gp generated by reflections across the faces of a polytope, we can identify the polytope with K and the K_s with the codim 1 faces.

Alternate definition of Σ

$\Sigma = (W \times K)/\sim$, where $(w, x) \sim (w', x')$ iff $x = x'$ and $wW_{S(x)} = w'W_{S(x)}$. (Here $S(x) := \{s \in S \mid x \in K_s\}$.)

Facts about Σ

- W acts properly on Σ.
- Σ is contractible.
- Σ has a cell structure with one W-orbit of $|T|$-cells for each spherical subset T. The link of each vertex in this structure is isomorphic to L.
- So, for example, if $L \cong S^{n-1}$, then Σ is an n-mfld.

Two formulas for $\chi^{orb}(\Sigma/W)$

Recall

Orbihedral Euler characteristic

$$\chi^{orb}(X/\Gamma) := \sum_{\text{orbits of cells}} \frac{(-1)^{\dim c}}{|\Gamma_c|} \in \mathbb{Q},$$

where $|\Gamma_c|$ is the order of the stabilizer of the cell c.

First formula

$$\chi(W) = \chi^{orb}(\Sigma/W) = \sum_{T \in \mathcal{S}} \frac{1 - \chi(\mathrm{Lk}(\sigma_T, L))}{|W_T|}.$$

Here σ_T is the simplex of L corresponding to T and $\mathrm{Lk}(\sigma_T, L)$ stands for the "link of σ_T in L."

Proof.

- Each W-orbit of simplices intersects the fundamental chamber K in a single simplex.
- Stabilizer of $\sigma \supset W_T$ iff $\sigma \subset K_T$. Moreover, Stab $\supsetneq W_T$ iff $\sigma \subset \partial K_T$ $(:= |\mathcal{S}_{>T}|)$.
- ∂K_T is barycentric subdiv of $\mathrm{Lk}(\sigma_T, L)$ so they have the same Euler char.
- So, contribution of simplices with stabilizer W_T to χ^{orb} is the product of $[\chi(K_T) - \chi(\partial K_T)]$ with $1/\operatorname{Card}(W_T)$.
- Since K_T is a cone, $\chi(K_T) = 1$. Therefore, $\chi^{orb}(\Sigma/W) = \sum(1 - \chi(\mathrm{Lk}(\sigma_T, L)))/|W_T|$.

□

Second formula

$$\chi^{orb}(\Sigma/W) = \sum_{T \in \mathcal{S}} \frac{(-1)^{|T|}}{|W_T|}.$$

Proof.

Use other cell structure. $\forall T \in \mathcal{S}$, there is a W-orbit of cells. The stabilizer of such a cell is W_T and $\dim(\text{cell}) = |T|$. The formula follows. □

The f-polynomial of a simplicial complex

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- L a finite simplicial complex
- Denote by $f_i(L)$ (or f_i) the number of i-simplices of L.
- Put $f_{-1}(L) = 1$ (the empty simplex).
- $(f_{-1}, f_0, \dots, f_{\dim L})$ is the *f-vector*.
- The *f-polynomial* is

$$f(t) := \sum_{i=-1}^{\dim L} f_i t^{i+1}.$$

- Suppose (W, S) is a right-angled (i.e., $m_{st} = 2$ or ∞).
- $T \in \mathcal{S} \implies W_T = (\mathbb{Z}_2)^T$. So, $|W_T| = 2^{|T|} = 2^{\dim \sigma_T + 1}$. Therefore,

$$\chi^{orb}(\Sigma/W) = \sum_{T \in \mathcal{S}} \frac{(-1)^{|T|}}{|W_T|} = \sum_{i=0}^{\dim L+1} \frac{(-1)^i}{2^i} f_{i-1}$$
$$= f(-\frac{1}{2}).$$

Flag complexes

Definition

A simplicial cx L is a *flag complex* if a subset $T \subset \text{Vert}(L)$ spans a simplex iff any pair of elements in T are connected by an edge.

Example

Suppose L is a triangulation of S^1 into m edges (an m-gon). Then L is a flag cx iff $m > 3$.

Nerve of a right-angled Coxeter gp

- If W is right-angled, then its nerve L is a flag cx.
- Conversely, if L is a flag cx, there is a right-angled W with nerve L. Namely, put $S := \mathrm{Vert}(L)$ and

$$m_{st} = \begin{cases} 1, & \text{if } s = t, \\ 2, & \text{if } \{s,t\} \text{ is an edge}, \\ \infty, & \text{otherwise.} \end{cases}$$

The Flag Complex Conjecture

Euler Characteristic Conjecture

Suppose Γ acts properly and cocompactly on contractible $\widetilde{M}^{2k}$ (i.e., $\widetilde{M}^{2k}/\Gamma$ is an aspherical orbifold). Then

$$(-1)^k \chi^{orb}(\widetilde{M}^{2k}/\Gamma) \geq 0.$$

For a right-angled Coxeter gp W, $\chi^{orb}(\Sigma/W) = f(-\frac{1}{2})$, so this becomes:

Conjecture

If L is a flag triangulation of S^{2k-1}, then $(-1)^k f_L(-\frac{1}{2}) \geq 0$.

Theorem

- *(Stanley) Flag Cx Conj is true for L is the barycentric subdivision of the boundary of any convex polytope.*
- *(Davis – Okun) It is true for L ia triangulation of a rational homology 3-sphere as a flag complex.*

Right-angled Coxeter gps

L a flag cx, W_L associated RA Coxeter gp, Σ_L the complex.

- Suppose L a finite flag cx and L' a full subcx.
- We get a functor from such pairs (L, L') to Hilbert W_L-modules: $(L, L') \to L^2\mathcal{H}_i(\Sigma_L, W_L \times_{W_{L'}} \Sigma_{L'})$.
- Put $\beta_i(L, L') := L^2 b_i(\Sigma_L, W_L \times_{W_{L'}} \Sigma_{L'})$.

RA Singer Conj

Suppose L is a flag cx, W_L the associated right-angled (= RA) Coxeter gp, Σ_L associated contractible cx.

Conjecture

Suppose L a triangulation of S^{n-1}. (So, Σ is an n-mfld.) Then

$$\beta_i(L) := L^2 b_i(W_L) = 0, \quad \forall i \neq \frac{n}{2}.$$

This $\implies$ Flag Cx Conj when $n = 2k$, i.e., $(-1)^k f(-\frac{1}{2}) = L^2 b_k(W_L) \geq 0$.

Theorem

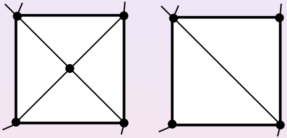
(Davis – Okun). *RA Singer Conj is true for $n \leq 4$.*

Sketch of Proof

- $n = 1$. Then $L = S^0$, $W_L =$ infinite dihedral gp, $\Sigma_L = \mathbf{R}$.
- $n = 2$. Then $L \cong S^1$, $\Sigma_L =$ univ cover of surface with $g > 0$.
- $n = 3$. Then $L \cong S^2$. Use Andreev's Theorem (on realizing convex polytopes with prescribed dihedral angles in $\mathbb{H}^3$). Then use Mayer-Vietoris sequence.
- $n = 4$. So, $L \cong S^3$. Inductive arguments of [D – Okun] shows: (RA Singer for $n - 1$) $\implies$ (RA Singer for n), when n is even.

□

Details for $n = 3$



a full 4-cycle   not full

Andreev's Thm $\implies$ that if L does not contain a full 4-cycle, then L is dual to a right-angled convex polytope in $\mathbb{H}^3$. So, $W_L \subset \mathrm{Isom}(\mathbb{H}^3)$. $\implies$ $L^2\mathcal{H}^*(\Sigma_L) = 0$.

- In general, we can cut L apart along full 4-cycles to get decomposition of W_L as amalgamated product.
- Univ cover for a full 4-cycle gives $\mathbf{D}_\infty \times \mathbf{D}_\infty$ on $\mathbf{R}^2$. $L^2\mathcal{H}^*(\mathbf{R}^2) = 0$; so, MV sequence gives the result.

Growth series

- Γ a gp and S a set of generators.
- $l : \Gamma \to \mathbb{N}$ is word length
- The *growth series* of Γ is power series $f(t) := \sum_{g \in \Gamma} t^{l(g)}$.
- Γ finite $\implies$ $f(t)$ a polynomial
- Often (e.g., when G is an "automatic group") $f(t)$ is a rational function.
- Prototypical example: growth series of a Coxeter gp.

Series of a Coxeter gp

- (W, S) a Coxeter system
- $W(t) := \sum_{w \in W} t^{l(w)}$
- For $X \subset W$, put $X(t) := \sum_{w \in X} t^{l(w)}$.

Lemma

Suppose W finite, w_S its element of longest length and $m_S := l(w_S)$. Then $W(t) = t^{m_S} W(t^{-1})$.

Proof.

Fact: for any $w \in W$, $l(w_S w) = l(w_S) - l(w)$. This gives

$$W(t) = \sum_{w_S w \in W} t^{l(w_S w)} = \sum_{w \in W} t^{m_S} t^{-l(w)} = t^{m_S} W(t^{-1}).$$

□

- $\forall w \in W$, $\mathrm{In}(w) := \{s \in S \mid l(ws) < l(w)\}$
- Fact: $\mathrm{In}(w)$ is a spherical subset of S.

- For $T \subset S$, put

$$B_T := \{w \in W \mid l(wt) = l(w) + 1, \forall t \in T\}$$
$$W^T := \{w \in W \mid \mathrm{In}(w) = T\}$$

- $w \in B_T \iff$ it is the shortest element in wW_T. So, B_T is a set of representatives for W/W_T.
- $B_{S-T} = \bigcup_{U \subset T} W^U$
- So, $B_{S-T}(t) = \sum_{U \supset T} W^U(t)$.

Exercise

$\forall\ T \subset S$, $W(t) = B_T(t)W_T(t)$.

Exercise (Möbius Inversion)

Suppose f, g are functions from the power set of a finite set S to an abelian gp s.t. $\forall\ T \subset S$,

$$f(T) = \sum_{U \subset T} g(U).$$

Then $\forall\ T \subset S$,

$$g(T) = \sum_{U \subset T} (-1)^{|T-U|} f(U).$$

Apply this with $f(T) = B_{S-T}(t)$, $g(T) = W^T(t)$ to get:

Lemma

$$W^T(t) = W(t) \sum_{U \supset T} (-1)^{|U-T|} \frac{t^{m_U}}{W_{S-U}(t)}$$

Corollary

- *If $|W| < \infty$, then*

$$t^{m_S} = W(t) \sum_{T \subset S} \frac{(-1)^{|T|}}{W_T(t)}.$$

- *If $|W| = \infty$, then*

$$0 = \sum_{T \subset S} \frac{\varepsilon(T)}{W_T(t)}.$$

Rationality

With a little more work we get:

Theorem

$$\frac{1}{W(t^{-1})} = \sum_{T \in \mathcal{S}} \frac{(-1)^{|T|}}{W_T(t)}$$

Corollary

$W(t) = f(t)/g(t)$, where $f, g \in \mathbb{Z}[t]$.

h-polynomial and RA growth series

- Recall f_i = number of i-simplices of L.
- $f_L(t) := \sum_{i=0}^{n} f_{i-1} t^i$, where $n := \dim L + 1$.
- Define
$$h_L(t) := (1-t)^n f_L\left(\frac{t}{1-t}\right).$$
- Suppose W is RA and W_T is finite. Then $W_T(t) = (1+t)^{|T|}$. Hence,

$$\frac{1}{W_T(t)} = \left(\frac{1}{1+t}\right)^{|T|} \quad \text{and} \quad \frac{1}{W_T(t^{-1})} = \left(\frac{t}{1+t}\right)^{|T|}$$

Theorem

(W, S) right-angled with nerve L. Then

$$\frac{1}{W(t)} = \frac{h_L(-t)}{(1+t)^n}.$$

Proof.

$$\frac{1}{W(t)} = \sum_{T \in \mathcal{S}} \frac{(-1)^{|T|}}{W_T(t^{-1})} = \sum_{T \in \mathcal{S}} \left(\frac{-t}{1+t}\right)^{|T|}$$
$$= f_L\left(\frac{-t}{1+t}\right) = \frac{h_L(-t)}{(1+t)^n}.$$

□