Lecture 5

Mike Davis

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1 Hecke – von Neumann algebras
   • Hecke algebras
   • von Neumann algebra version
   • Idempotents

2 $L_q^2$-cohomology
joint work with
Jan Dymara, Tadeusz Januszkiewicz and Boris Okun

Recall

\((W, S)\) is a Coxeter system, \(\Sigma\) the associated cx.
\(W(t) = \) the growth series, \(\rho = \) its radius of convergence

Goal

Define \(L^2_q H^*(\Sigma), q \in (0, \infty)\).

- Change the definition of the inner product. \(W\) will no longer act orthogonally.
- Change the von Neumann algebra.
Properties of $L^2_q \mathcal{H}^*(\Sigma)$

- When $q = 1$, it is ordinary $L^2 \mathcal{H}^*(\Sigma)$.
- There are “weighted $L^2$-Betti numbers,” $L^2_q b_i(\Sigma)$ (= “von Neumann dim of $L^2_q \mathcal{H}^i(\Sigma)$ w.r.t. a Hecke algebra”).
- The “weighted $L^2$-Euler characteristic,” $\chi_q(\Sigma)$ is $= 1/W(q)$.
- If $\Phi$ is a bldg with chamber transitive automorphism gp $G$ and “thickness” $q$, then the $L^2_q b_i(\Sigma)$ are the $L^2$-Betti numbers of $\Phi$ w.r.t. $G$.
- If $q < \rho$, then $L^2_q \mathcal{H}^*(\Sigma)$ vanishes except in dimension 0 (like ordinary cohomology).
- If $q > \rho^{-1}$, then $L^2_q \mathcal{H}^*(\Sigma)$ “looks like” cohomology with compact supports.
Definitions

\[ \mathcal{R}(W) := \{ \text{finitely supported functions } \mathcal{R} \rightarrow W \} \]
\[ := \text{the } \mathcal{R}\text{-vector space on } W \text{ with basis } (e_w)_{w \in W} \]
\[ \mathcal{R}W := \mathcal{R}(W) \text{ with its structure as the gp algebra} \]

A “Hecke algebra” is a deformation of \( \mathcal{R}W \) depending on \( q \).

Proposition (Exercise in Bourbaki)

\[ \exists! \text{ algebra structure on } \mathcal{R}(W) \text{ s.t.} \]
\[ e_s e_w = \begin{cases} 
    e_{sw}, & \text{if } l(sw) > l(w); \\
    q e_{sw} + (q - 1) e_w, & \text{if } l(sw) < l(w), 
\end{cases} \]
\[ \forall s \in S \text{ and } w \in W. \]
Definition

We denote this algebra \( R_q W \) and call it the \textit{Hecke algebra} of \( W \) associated to the \textit{parameter} \( q \).

Remark

Given \( i : S \to I \) s.t. \( i(s) = i(s') \) whenever \( s, s' \) are conjugate and \( I \)-tuple \( q \), one can define the Hecke algebra with multiparameter \( q \). Similarly, given an \( I \)-tuple \( t \) of indeterminates there is a well-defined growth series \( W(t) \).
An inner product on $\mathbf{R}(W)$

On the standard basis define

$$\langle e_v, e_w \rangle_q := \begin{cases} q^{l(w)}, & \text{if } v = w, \\ 0, & \text{if } v \neq w. \end{cases}$$

$L^2_q(W) :=$ completion of $\mathbf{R}(W)$.

The anti-involution $\ast$

$e_w \rightarrow e_{w^{-1}}$ extends to a linear endomorphism $\ast$ of $\mathbf{R}_q W$, i.e.,

$$\left( \sum a_w e_w \right)^\ast := \sum a_{w^{-1}} e_w.$$
$\mathbb{R}_q W$ is an algebra of operators on $L^2_q(W)$. (Actually, there are two algebras – multiplication on left or right.)

The next proposition shows that it satisfies the necessary conditions to be completable to a von Neumann algebra of operators. ($\mathbb{R}_q W$ is a $C^*$ algebra.) The proof is a straightforward series of computations.
Proposition (Dymara)

The inner product defined above and the involution \( * \) give \( R_q W \) the structure of a “Hilbert algebra,” i.e., the following properties hold:

- \((xy)^* = y^* x^*\),
- \(\langle x, y \rangle_q = \langle y^*, x^* \rangle_q\),
- \(\langle xy, z \rangle_q = \langle y, x^* z \rangle_q\),
- for any \( x \in R_q W \), left translation by \( x \), \( L_x : R_q W \to R_q W \), defined by \( L_x(y) = xy \), is continuous,
- the products \( xy \) over all \( x, y \in R_q W \) are dense in \( R_q W \).
Hecke – von Neumann algebra

\[ \mathcal{N}_q = \text{the weak completion of } \mathbb{R}_q \mathcal{W} \]
\[ := \{ \mathbb{R}_q \mathcal{W}-\text{equivariant bounded linear operators on } L^2_q(\mathcal{W}) \} \]

von Neumann trace

- For \( \varphi \in \mathcal{N}_q \),
  \[ \text{tr}_{\mathcal{N}_q}(\varphi) = \langle \varphi(e_1), e_1 \rangle_q. \]
- For \( \Phi = (\varphi_{ij}) \in M_m(\mathcal{N}_q) \),
  \[ \text{tr}_{\mathcal{N}_q}(\Phi) = \sum \text{tr}_{\mathcal{N}_q}(\varphi_{ii}). \]
von Neumann dimension

- Given a $R_qW$-stable, closed subspace, $V \subset \bigoplus L^2_q(W)$, let $p_V : \bigoplus L^2_q(W) \to \bigoplus L^2_q(W)$ be orthogonal projection onto $V$.
- Define
  \[ \dim_{N_q} V = \text{tr}_{N_q}(p_V) \in [0, \infty). \]

Some idempotents in $N_q$

For $T \subset S$, define

\[ a_T := \frac{1}{W_T(q)} \sum_{w \in W_T} e_w, \]
\[ h_T := \frac{1}{W_T(q^{-1})} \sum_{w \in W_T} (-1)^{l(w)} q^{-l(w)} e_w. \]
Sample calculations

\( \rho_T := \text{radius of convergence of } W_T(t) \)

- \( \tilde{a}_T := \sum_{w \in W} e_w \)
- If \( q < \rho_T \),
  \( \langle \tilde{a}_T, \tilde{a}_T \rangle_q = W_T(q) < \infty \).
- If \( s \in T \), \( \tilde{a}_T e_s = q \tilde{a}_T \).
- If \( w \in W_T \),
  \( \tilde{a}_T e_w = q^{l(w)} \tilde{a}_T \).
- \( \tilde{a}_T \) is bounded iff \( q < \rho_T \).
- \( (\tilde{a}_T)^2 = W_T(q) \tilde{a}_T \)

- \( \tilde{h}_T := \sum_{w \in W_T} (-1)^{l(w)} q^{-l(w)} e_w \)
- If \( q > 1/\rho_T \),
  \( \langle \tilde{h}_T, \tilde{h}_T \rangle_q = W_T(q^{-1}) < \infty \).
- If \( s \in T \), \( \tilde{h}_T e_s = -\tilde{h}_T \).
- If \( w \in W_T \),
  \( \tilde{h}_T e_w = (-1)^{l(w)} \tilde{h}_T \).
- \( \tilde{h}_T \) is bounded iff \( q > 1/\rho_T \).
- \( (\tilde{h}_T)^2 = W_T(q^{-1}) \tilde{h}_T \)
$a_T = \frac{1}{W_T(q)} \tilde{a}_T$ and $h_T = \frac{1}{W_T(q^{-1})} \tilde{h}_T$.

If $W_T$ is finite and $q = 1$, then $a_T$ is averaging over $W_T$ and $h_T$ is “alternation” over $W_T$.

- $a_T$ is bounded iff $q < \rho_T$
- $a_T^* = a_T$
- $(a_T)^2 = a_T$
- $a_s = \frac{1}{1+q}(1 + e_s)$

- $h_T$ is bounded iff $q > 1/\rho_T$
- $h_T^* = h_T$
- $(h_T)^2 = h_T$
- $h_s = \frac{q}{1+q}(1 - q^{-1}e_s)$

Subspaces of $L^2_q(W)$

- $A_s := \text{Im } a_s$ and $H_s := \text{Im } h_s$.
- $A_T := \bigcap_{s \in T} A_s$ and $H_T := \bigcap_{s \in T} H_s$. 
Lemma

\(A_s \text{ and } H_s \text{ are orthogonal complements.}\)

Proof.

\[a_s + h_s = \frac{1}{1+q}(1 + e_s) + \frac{q}{1+q}(1 - q^{-1}e_s) = 1\]

Exercise

\[A_T = \text{Im } a_T, \text{ if } q < \rho_T \text{ and is 0 otherwise.}\]
\[H_T = \text{Im } h_T, \text{ if } q > 1/\rho_T \text{ and is 0 otherwise.}\]
Lecture 4: Weighted $L^2$ cohomology

Hecke – von Neumann algebras

$L^2_q$-cohomology

Hecke algebras
von Neumann algebra version
Idempotents

**Dimensions**

$$\dim_{\mathcal{N}_q} A_T = \text{tr}_{\mathcal{N}_q} a_T = \frac{1}{W_T(q)}, \quad \text{if } q < \rho_T$$

$$\dim_{\mathcal{N}_q} H_T = \text{tr}_{\mathcal{N}_q} h_T = \frac{1}{W_T(q^{-1})}, \quad \text{if } q > 1/\rho_T.$$ 

**Proof.**

$$\text{tr}_{\mathcal{N}_q} a_T = \langle e_1 a_T, e_1 \rangle_q = \langle a_T, e_1 \rangle_q = \frac{1}{W_T(q)}.$$

Similarly, for $H_T$. 

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Lecture 5
Review from last time

Spherical subsets and cosets

\[ S := \{ T \subset S \mid W_T \text{ is finite} \} = \{ \text{spherical subsets} \} \]

\[ WS := \coprod_{T \in S} W/W_T = \{ \text{spherical cosets} \} \]

Complexes \( L, K \) and \( \Sigma \)

Put

\[ K := |S| \quad \Sigma := |WS|. \]

\( S_{>\emptyset} \) is an abstract simp cx. The corresponding geometric cx is denoted \( L \) and called the *nerve* of \( (W, S) \). \( L \) is a finite simp cx.
## Facts about $K$

- $K$ is the cone on the barycentric subdivision of $L$ ($\emptyset \in S$ is a common vertex of every simplex in $K$).
- For each $T \in S$, put $K_T := |S_{\geq T}|$ and for each $s \in S$, $K_s := K_{\{s\}}$.

## Facts about $\Sigma$

- $\Sigma$ is contractible.
- $\Sigma$ has a cell structure with one $W$-orbit of $|T|$-cells for each spherical subset $T$. The link of each vertex in this structure is isomorphic to $L$.
- So, for example, if $L \cong S^{n-1}$, then $\Sigma$ is an $n$-mfld.
∀ k-simplex $\sigma$ in $\Sigma$, let $e_\sigma \in C_k(\Sigma)$ be its characteristic function.

Define an inner product on $C_k(\Sigma)$ by

$$\langle e_\sigma, e_\tau \rangle_q := \begin{cases} q^{l(w(\sigma))}, & \text{if } \sigma = \tau, \\ 0, & \text{otherwise.} \end{cases}$$

where $w(\sigma)$ is the shortest $w \in W$ s.t. $w^{-1}\sigma \in K$.

$L_q^2C_k(\Sigma) = L_q^2C^k(\Sigma) :=$ completion of $C_k(\Sigma)$

$L_q^2C^*(\Sigma)$ is a $\mathcal{N}_q$-module and $\delta : L_q^2C^k(\Sigma) \to L_q^2C^{k+1}(\Sigma)$ is a map of $\mathcal{N}_q$-modules.
N.B. The adjoint of $\delta$ is not the usual boundary map, rather the formula for it involves $q$’s. Put $\partial_q := \delta^*$. 

**Definitions**

\[
L^2_q H^k(\Sigma) := \ker \delta / \text{Im} \delta \\
L^2_q H_k(\Sigma) := H_k((L^2 C_*(\Sigma), \partial_q)) \\
L^2_q b_k(\Sigma) := \dim L^2_q H^k(\Sigma) \\
L^2_q \chi(\Sigma) := \sum (-1)^k L^2_q b_k(\Sigma)
\]
Theorem (Dymara)

\[ L^2_q \chi(\Sigma) = \frac{1}{W(q)} \]

Proof.

The proof is along the line of Atiyah’s formula. The space of \( L^2_q \)-chains on the orbit of a cell of type \( T \) is \( \cong \) to \( H_T \). So,

\[
L^2_q \chi(\Sigma) = \sum_{T \in S} (-1)^{|T|} \dim_{\mathcal{N}_q} H_T = \sum_{T \in S} \frac{(-1)^{|T|}}{W_T(q^{-1})} = \frac{1}{W(q)}.
\]

The last equality was a formula for growth series proved last time.
1/W(q) is a rational function of q, e.g., if W is RA

\[
\frac{1}{W(q)} = \frac{h(-q)}{(1 + q)^n}
\]

It can change signs at the roots of the numerator. The smallest root is ρ and the largest root is ρ⁻¹.
Theorem

$L^2_q b^k(\Sigma)$ is a continuous function of $q$.

Theorem (Dymara)

If $q < \rho$, then $L^2_q \mathcal{H}^*(\Sigma)$ is concentrated in dimension 0 and is $\cong A_S$. Conversely, if $q > \rho$, $L^2_q \mathcal{H}^0(\Sigma) = 0$.

Idea for proof.

$\Sigma$ is contractible. Show a “standard” chain contraction is a bounded operator in range $q < \rho$. 
Theorem (Dymara)

Suppose $\Sigma$ is a $n$-dim pseudomfld and $q > \rho^{-1}$. Then $L^2_q\mathcal{H}_n(\Sigma) \cong H_S$. Conversely, if $q < \rho^{-1}$, $L^2_q\mathcal{H}_n(\Sigma) = 0$

Poincaré duality has the following form

Theorem (Dymara))

If $L \cong S^{n-1}$ (s.t. $\Sigma$ is an $n$-mfld), then

$$L^2_qb_k(\Sigma) = L^2_{1/q}b_{n-k}(\Sigma)$$
Corollary

Suppose $L \cong S^{n-1}$ and $q > \rho^{-1}$. Then $L^2_q\mathcal{H}_*(\Sigma)$ is concentrated in $\text{dim } n$. 
Example

If $\Sigma$ is a 2-mfld, then $L^2_q H^*(\Sigma)$ is concentrated in dim:

$$
\begin{cases}
0, & \text{if } q \leq \rho; \\
1, & \text{if } \rho < q < \rho^{-1}; \\
2, & \text{if } q \geq \rho^{-1}.
\end{cases}
$$

Suppose $K$ is a right-angled $k$-gon.

$$L^2_q \chi(\Sigma) = \frac{1}{W(q)} = \frac{q^2 + (2 - k)q + 1}{(1 + q)^2}$$

so,

$$\rho^{\pm 1} = \frac{(k - 2) \mp \sqrt{k^2 - 4k}}{2},$$

e.g. when $k = 5$, $\rho^{-1} = \frac{3 + \sqrt{5}}{2}$, $2 < \rho^{-1} < 3$. 
In general we can calculate $L^2_q \mathcal{H}_*(\Sigma)$ for $q < \rho$ and $q > \rho^{-1}$ (but not for $\rho < q < \rho^{-1}$).

Recall $A_T := L^2_q(W)a_T$. If $U \supset T$, then $A_U \subset A_T$. Put

$$A_{>T} := \sum_{U \supset T} A_U, \quad D_T := A_T/A_{>T}.$$
Decomposition Theorem

We have direct sum decompositions of $\mathcal{N}_q$-modules:

$$L_q^2 = \bigoplus_{T \in S} D_{S-T} \quad \text{if } q < \rho,$$

$$L_q^2 = \bigoplus_{T \in S} D_T \quad \text{if } q > \rho^{-1}.$$

For $q > \rho^{-1}$,

$$\dim_{\mathcal{N}_q} D_T = \sum_{U \in S_{\geq T}} \frac{(-1)^{|U-T|}}{W_U(q)} = \frac{W^T(q^{-1})}{W(q^{-1})}.$$
Main Theorem (DDJO)

Suppose \( q > \rho^{-1} \). Then

\[
L^2_q \mathcal{H}^* (\Sigma) = \bigoplus_{T \in S} H^* (K, K^S - T) \otimes D^T.
\]

Corollary

- If \( q < \rho \), then can : \( L^2_q \mathcal{H}^k (\Sigma) \to H^k (\Sigma; \mathbb{R}) \) is isomorphism.
- If \( q > \rho^{-1} \), then \( H^k_c (\Sigma; \mathbb{R}) \to L^2_q \mathcal{H}^k (\Sigma) \) is injective with dense image.

So, \( L^2_q \mathcal{H}^* (\ ) \) interpolates between ordinary cohomology and cohomology with compact supports.