Lecture 5

Mike Davis

July 7, 2006

1. Hecke – von Neumann algebras
   - Hecke algebras
   - von Neumann algebra version
   - Idempotents

2. L^2_q-cohomology

joint work with
Jan Dymara, Tadeusz Januszkiewicz and Boris Okun

Recall

(W, S) is a Coxeter system, Σ the associated cx.
$W(t) =$ the growth series, $\rho =$ its radius of convergence

Goal

Define $L^2_q\mathcal{H}^*(\Sigma)$, $q \in (0, \infty)$.

- Change the definition of the inner product. W will no longer act orthogonally.
- Change the von Neumann algebra.

Properties of $L^2_q\mathcal{H}^*(\Sigma)$

- When $q = 1$, it is ordinary $L^2\mathcal{H}^*(\Sigma)$.
- There are "weighted L^2-Betti numbers," $L^2_q b_i(\Sigma)$ (= "von Neumann dim of $L^2_q\mathcal{H}^i(\Sigma)$ w.r.t. a Hecke algebra").
- The "weighted L^2-Euler characteristic," $\chi_q(\Sigma)$ is $= 1/W(q)$.
- If Φ is a bldg with chamber transitive automorphism gp G and "thickness" q, then the $L^2_q b_i(\Sigma)$ are the L^2-Betti numbers of Φ w.r.t. G.
- If $q < \rho$, then $L^2_q\mathcal{H}^*(\Sigma)$ vanishes except in dimension 0 (like ordinary cohomology).
- If $q > \rho^{-1}$, then $L^2_q\mathcal{H}^*(\Sigma)$ "looks like" cohomology with compact supports.

Definitions

$$\mathbf{R}^{(W)} := \{\text{finitely supported functions } \mathbf{R} \to W\}$$
$$:= \text{the } \mathbf{R}\text{-vector space on } W \text{ with basis } (e_w)_{w \in W}$$
$$\mathbf{R}W := \mathbf{R}^{(W)} \text{ with its structure as the gp algebra}$$

A "Hecke algebra" is a deformation of $\mathbf{R}W$ depending on q.

Proposition (Exercise in Bourbaki)

$\exists!$ *algebra structure on* $\mathbf{R}^{(W)}$ *s.t.*

$$e_s e_w = \begin{cases} e_{sw}, & \text{if } l(sw) > l(w); \\ qe_{sw} + (q-1)e_w, & \text{if } l(sw) < l(w), \end{cases}$$

$\forall\ s \in S$ *and* $w \in W$.

Definition

We denote this algebra $\mathbf{R}_q W$ and call it the *Hecke algebra* of W associated to the *parameter* q.

Remark

Given $i : S \to I$ s.t. $i(s) = i(s')$ whenever s, s' are conjugate and I-tuple $\mathbf{q}$, one can define the Hecke algebra with multiparameter $\mathbf{q}$. Similarly, given an I-tuple $\mathbf{t}$ of indeterminates there is a well-defined growth series $W(\mathbf{t})$.

An inner product on $\mathbf{R}^{(W)}$

On the standard basis define

$$\langle e_v, e_w \rangle_q := \begin{cases} q^{l(w)}, & \text{if } v = w, \\ 0, & \text{if } v \neq w. \end{cases}$$

$L^2_q(W) :=$ completion of $\mathbf{R}^{(W)}$.

The anti-involution $*$

$e_w \to e_{w^{-1}}$ extends to a linear endomorphism $*$ of $\mathbf{R}_q W$, i.e.,

$$\left(\sum a_w e_w\right)^* := \sum a_{w^{-1}} e_w .$$

$\mathbf{R}_q W$ is an algebra of operators on $L^2_q(W)$. (Actually, there are two algebras – multiplication on left or right.)

The next proposition shows that it satisfies the necessary conditions to be completable to a von Neumann algebra of operators. ($\mathbf{R}_q W$ is a C^* algebra.) The proof is a straightforward series of computations.

Proposition (Dymara)

The inner product defined above and the involution $*$ *give* $\mathbf{R}_q W$ *the structure of a "Hilbert algebra," i.e., the following properties hold:*

- $(xy)^* = y^* x^*$,
- $\langle x, y \rangle_q = \langle y^*, x^* \rangle_q$,
- $\langle xy, z \rangle_q = \langle y, x^* z \rangle_q$,
- *for any* $x \in \mathbf{R}_q W$, *left translation by* x, $L_x : \mathbf{R}_q W \to \mathbf{R}_q W$, *defined by* $L_x(y) = xy$, *is continuous,*
- *the products* xy *over all* $x, y \in \mathbf{R}_q W$ *are dense in* $\mathbf{R}_q W$.

Hecke – von Neumann algebra

$$\mathcal{N}_q = \text{the weak completion of } \mathbf{R}_q W$$
$$:= \{\mathbf{R}_q W\text{-equivariant bounded linear operators on } L^2_q(W)\}$$

von Neumann trace

- For $\varphi \in \mathcal{N}_q$,
$$\mathrm{tr}_{\mathcal{N}_q}(\varphi) = \langle \varphi(e_1), e_1 \rangle_q .$$
- For $\Phi = (\varphi_{ij}) \in M_m(\mathcal{N}_q)$,
$$\mathrm{tr}_{\mathcal{N}_q}(\Phi) = \sum \mathrm{tr}_{\mathcal{N}_q}(\varphi_{ii}).$$

von Neumann dimension

- Given a $\mathbf{R}_q W$-stable, closed subspace, $V \subset \oplus L^2_q(W)$, let $p_V : \oplus L^2_q(W) \to \oplus L^2_q(W)$ be orthogonal projection onto V.
- Define

$$\dim_{\mathcal{N}_q} V = \mathrm{tr}_{\mathcal{N}_q}(p_V) \in [0, \infty).$$

Some idempotents in $\mathcal{N}_q$

For $T \subset S$, define

$$a_T := \frac{1}{W_T(q)} \sum_{w \in W_T} e_w,$$

$$h_T := \frac{1}{W_T(q^{-1})} \sum_{w \in W_T} (-1)^{l(w)} q^{-l(w)} e_w.$$

Sample calculations

$\rho_T :=$ radius of convergence of $W_T(t)$

- $\tilde{a}_T := \sum_{w\in W} e_w$
- If $q < \rho_T$, $\langle \tilde{a}_T, \tilde{a}_T\rangle_q = W_T(q) < \infty$.
- If $s \in T$, $\tilde{a}_T e_s = q\tilde{a}_T$.
- If $w \in W_T$, $\tilde{a}_T e_w = q^{l(w)}\tilde{a}_T$.
- $\tilde{a}_T$ is bounded iff $q < \rho_T$.
- $(\tilde{a}_T)^2 = W_T(q)\tilde{a}_T$

- $\tilde{h}_T := \sum_{w\in W_T} (-1)^{l(w)} q^{-l(w)} e_w$
- If $q > 1/\rho_T$, $\langle \tilde{h}_T, \tilde{h}_T\rangle_q = W_T(q^{-1}) < \infty$.
- If $s \in T$, $\tilde{h}_T e_s = -\tilde{h}_T$.
- If $w \in W_T$, $\tilde{h}_T e_w = (-1)^{l(w)}\tilde{h}_T$.
- $\tilde{h}_T$ is bounded iff $q > 1/\rho_T$
- $(\tilde{h}_T)^2 = W_T(q^{-1})\tilde{h}_T$

$$a_T = \frac{1}{W_T(q)}\tilde{a}_T \quad \text{and} \quad h_T = \frac{1}{W_T(q^{-1})}\tilde{h}_T.$$

If W_T is finite and $q = 1$, then a_T is averaging over W_T and h_T is "alternation" over W_T.

- a_T is bounded iff $q < \rho_T$
- $a_T^* = a_T$
- $(a_T)^2 = a_T$
- $a_s = \frac{1}{1+q}(1 + e_s)$

- h_T is bounded iff $q > 1/\rho_T$
- $h_T^* = h_T$
- $(h_T)^2 = h_T$
- $h_s = \frac{q}{1+q}(1 - q^{-1}e_s)$

Subspaces of $L^2_q(W)$

- $A_s := \operatorname{Im} a_s$ and $H_s := \operatorname{Im} h_s$.
- $A_T := \bigcap_{s\in T} A_s$ and $H_T := \bigcap_{s\in T} H_s$.

Lemma

A_s and H_s are orthogonal complements.

Proof.

$a_s + h_s = \frac{1}{1+q}(1+e_s) + \frac{q}{1+q}(1-q^{-1}e_s) = 1$ □

Exercise

$A_T = \operatorname{Im} a_T$, if $q < \rho_T$ and is 0 otherwise.
$H_T = \operatorname{Im} h_T$, if $q > 1/\rho_T$ and is 0 otherwise.

Dimensions

$$\dim_{\mathcal{N}_q} A_T = \operatorname{tr}_{\mathcal{N}_q} a_T = \frac{1}{W_T(q)}, \quad \text{if } q < \rho_T$$
$$\dim_{\mathcal{N}_q} H_T = \operatorname{tr}_{\mathcal{N}_q} h_T = \frac{1}{W_T(q^{-1})}, \quad \text{if } q > 1/\rho_T.$$

Proof.

$$\operatorname{tr}_{\mathcal{N}_q} a_T = \langle e_1 a_T, e_1\rangle_q = \langle a_T, e_1\rangle_q = \frac{1}{W_T(q)}.$$
Similarly, for H_T. □

Review from last time

Spherical subsets and cosets

$$\mathcal{S} := \{T \subset S \mid W_T \text{ is finite}\} = \{\text{spherical subsets}\}$$

$$W\mathcal{S} := \coprod_{T \in \mathcal{S}} W/W_T = \{\text{spherical cosets}\}$$

Complexes L, K and Σ

Put

$$K := |\mathcal{S}| \qquad \Sigma := |W\mathcal{S}|.$$

$\mathcal{S}_{>\emptyset}$ is an abstract simp cx. The corresponding geometric cx is denoted L and called the *nerve* of (W, S). L is a finite simp cx.

Facts about K

- K is the cone on the barycentric subdivision of L ($\emptyset \in \mathcal{S}$ is a common vertex of every simplex in K).
- For each $T \in \mathcal{S}$, put $K_T := |\mathcal{S}_{\geq T}|$ and for each $s \in S$, $K_s := K_{\{s\}}$.

Facts about Σ

- Σ is contractible.
- Σ has a cell structure with one W-orbit of $|T|$-cells for each spherical subset T. The link of each vertex in this structure is isomorphic to L.
- So, for example, if $L \cong S^{n-1}$, then Σ is an n-mfld.

L^2_q-cohomology.

- $\forall$ k-simplex σ in Σ, let $e_\sigma \in C_k(\Sigma)$ be its characteristic function.
- Define an inner product on $C_k(\Sigma)$ by

$$\langle e_\sigma, e_\tau \rangle_q := \begin{cases} q^{l(w(\sigma))}, & \text{if } \sigma = \tau, \\ 0, & \text{otherwise.} \end{cases}$$

  where $w(\sigma)$ is the shortest $w \in W$ s.t. $w^{-1}\sigma \in K$.
- $L^2_q C_k(\Sigma) = L^2_q C^k(\Sigma) :=$ completion of $C_k(\Sigma)$
- $L^2_q C^*(\Sigma)$ is a $\mathcal{N}_q$-module and $\delta : L^2_q C^k(\Sigma) \to L^2_q C^{k+1}(\Sigma)$ is a map of $\mathcal{N}_q$-modules.

N.B. The adjoint of δ is not the usual boundary map, rather the formula for it involves q's. Put $\partial_q := \delta^*$.

Definitions

$$L^2_q\mathcal{H}^k(\Sigma) := \ker\delta/\overline{\operatorname{Im}\delta}$$

$$L^2_q\mathcal{H}_k(\Sigma) := \mathcal{H}_k((L^2C_*(\Sigma),\partial_q))$$

$$L^2_q b_k(\Sigma) := \dim L^2_q\mathcal{H}^k(\Sigma)$$

$$L^2_q\chi(\Sigma) := \sum(-1)^k L^2_q b_k(\Sigma)$$

Theorem (Dymara)

$L^2_q\chi(\Sigma) = \frac{1}{W(q)}$

Proof.

The proof is along the line of Atiyah's formula. The space of L^2_q-chains on the orbit of a cell of type T is $\cong$ to H_T. So,

$$L^2_q\chi(\Sigma) = \sum_{T\in\mathcal{S}} (-1)^{|T|} \dim_{\mathcal{N}_q} H_T = \sum_{T\in\mathcal{S}} \frac{(-1)^{|T|}}{W_T(q^{-1})}$$
$$= \frac{1}{W(q)}.$$

The last equality was a formula for growth series proved last time. □

$1/W(q)$ is a rational function of q, e.g., if W is RA

$$\frac{1}{W(q)} = \frac{h(-q)}{(1+q)^n}$$

It can change signs at the roots of the numerator. The smallest root is ρ and the largest root is ρ^{-1}.

Theorem

$L^2_q b^k(\Sigma)$ is a continuous function of q.

Theorem (Dymara)

If $q < \rho$, then $L^2_q\mathcal{H}^*(\Sigma)$ is concentrated in dimension 0 and is $\cong A_S$. Conversely, if $q > \rho$, $L^2_q\mathcal{H}^0(\Sigma) = 0$.

Idea for proof.

Σ is contractible. Show a "standard" chain contraction is a bounded operator in range $q < \rho$. □

Theorem (Dymara)

Suppose Σ *is a n-dim pseudomfld and* $q > \rho^{-1}$. *Then* $L^2_q\mathcal{H}_n(\Sigma) \cong H_S$. *Conversely, if* $q < \rho^{-1}$, $L^2_q\mathcal{H}_n(\Sigma) = 0$

Poincaré duality has the following form

Theorem (Dymara))

If $L \cong S^{n-1}$ *(s.t.* Σ *is an n-mfld), then*

$$L^2_q b_k(\Sigma) = L^2_{1/q} b_{n-k}(\Sigma)$$

Corollary

Suppose $L \cong S^{n-1}$ *and* $q > \rho^{-1}$. *Then* $L^2_q\mathcal{H}_*(\Sigma)$ *is concentrated in dim* n.

Example

If Σ is a 2-mfld, Then $L^2_q\mathcal{H}^*(\Sigma)$ is concentrated in dim:

$$\begin{cases} 0, & \text{if } q \leq \rho; \\ 1, & \text{if } \rho < q < \rho^{-1}; \\ 2, & \text{if } q \geq \rho^{-1}. \end{cases}$$

Suppose K is a right-angled k-gon.

$$L^2_q\chi(\Sigma) = \frac{1}{W(q)} = \frac{q^2 + (2-k)q + 1}{(1+q)^2} \text{ so,}$$

$$\rho^{\pm 1} = \frac{(k-2) \mp \sqrt{k^2 - 4k}}{2},$$

e.g. when $k = 5$, $\rho^{-1} = \frac{3+\sqrt{5}}{2}$, $\quad 2 < \rho^{-1} < 3$.

In general we can calculate $L^2_q\mathcal{H}_*(\Sigma)$ for $q < \rho$ and $q > \rho^{-1}$ (but not for $\rho < q < \rho^{-1}$).
Recall $A_T := L^2_q(W)a_T$. If $U \supset T$, then $A_U \subset A_T$. Put

$$A_{>T} := \sum_{U \supset T} A_U, \qquad D_T := A_T/A_{>T}.$$

Decomposition Theorem

We have direct sum decompositions of $\mathcal{N}_q$-modules:

$$L^2_q = \overline{\bigoplus_{T \in \mathcal{S}} D_{S-T}} \qquad \text{if } q < \rho,$$

$$L^2_q = \overline{\bigoplus_{T \in \mathcal{S}} D_T} \quad \text{if } q > \rho^{-1}.$$

For $q > \rho^{-1}$,

$$\dim_{\mathcal{N}_q} D_T = \sum_{U \in \mathcal{S}_{\geq T}} \frac{(-1)^{|U-T|}}{W_U(q)} = \frac{W^T(q^{-1})}{W(q^{-1})}.$$

Main Theorem (DDJO)

Suppose $q > \rho^{-1}$. Then

$$L^2_q\mathcal{H}^*(\Sigma) = \overline{\bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes D^T}.$$

Corollary

- *If $q < \rho$, then* can : $L^2_q\mathcal{H}^k(\Sigma) \to H^k(\Sigma; \mathbf{R})$ *is isomorphism.*
- *If $q > \rho^{-1}$, then $H^k_c(\Sigma; \mathbf{R}) \to L^2_q\mathcal{H}^k(\Sigma)$ is injective with dense image.*

So, $L^2_q\mathcal{H}^*(\ )$ interpolates between ordinary cohomology and cohomology with compact supports.