$L^2$-cohomology of hyperplane complements

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\( \mathcal{A} \) a collection of affine hyperplanes in \( \mathbb{C}^n \)

**Definition**

\( \text{rk}(\mathcal{A}) \), the *rank*, of \( \mathcal{A} \) is the maximal codimension of a nonempty intersection of hyperplanes in \( \mathcal{A} \). Usually denote \( \text{rk}(\mathcal{A}) \) by \( l \). \( \mathcal{A} \) is *essential* if \( \text{rk}(\mathcal{A}) = n \).

\[
\Sigma(\mathcal{A}) := \bigcup_{H \in \mathcal{A}} H \\
M(\mathcal{A}) := \mathbb{C}^n - \Sigma(\mathcal{A})
\]
Fact

\( H^*(\mathbb{C}^n, \Sigma) \) vanishes except in dimension \( l (= \text{rk}(A)) \).
In fact, \( \Sigma \sim \bigvee S^{l-1} \).

The number \( \alpha(A) \)

\[
\alpha(A) := \dim H_i(\mathbb{C}^n, \Sigma) := b_i(\mathbb{C}^n, \Sigma)
= \text{the number of spheres in the wedge}
\]
Example

Suppose $\mathcal{A}_R$ is an essential hyperplane arrangement in $\mathbb{R}^n$. It divides $\mathbb{R}^n$ into convex regions.

- $\dim H_n(\mathbb{R}^n, \Sigma(\mathcal{A}_R))$ is the number of bounded components of $\mathbb{R}^n - \Sigma(\mathcal{A}_R)$.
- If $\mathcal{A}$ is complexification of $\mathcal{A}_R$, then

$$\left(\mathbb{C}^n, \Sigma(\mathcal{A})\right) \sim \left(\mathbb{R}^n, \Sigma(\mathcal{A}_R)\right).$$

- So $\alpha(\mathcal{A})$ is the number of bounded components of $\mathbb{R}^n - \mathcal{A}_R$. 

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Main Theorem

Suppose $\mathcal{A}$ is an arrangement of a finite number of affine hyperplanes in $\mathbb{C}^n$ with $\text{rk}(\mathcal{A}) = l$. Then the $L^2$-Betti numbers of $M(\mathcal{A})$ (the complement of the hyperplanes) are all 0, except in dimension $l$, where

$$\beta_i(M(\mathcal{A})) = \alpha(\mathcal{A}).$$

Here $\beta_i(\ )$ denotes the $i^{\text{th}}$ $L^2$-Betti number (to be defined later).
A similar theorem

The following is well-known.

**Theorem**

Suppose $L$ is a “generic” flat complex line bundle over $M(\mathcal{A})$. Then $H^*(M(\mathcal{A}); L)$ vanishes except in dimension $l$ and

$$\dim_{\mathbb{C}} H^l(M(\mathcal{A}); L) = \alpha(\mathcal{A}).$$

**Basic idea**

If $L$ is a flat line bundle over $S^1$ giving a nonconstant local coefficient system, then $H^*(S^1; L) = 0$ for $* = 0, 1$. 

Similarly, the basic idea for the Main Theorem is that the $L^2$-Betti numbers of $S^1$ vanish.
\( L^2 \)-Betti numbers

The regular representation

\( \pi \) is a countable discrete gp.

\[
L^2_\pi := \{ f : \pi \to \mathbb{C} \mid \sum_{x \in \pi} |f(x)|^2 < \infty \},
\]

where the sum is over all \( x \in \pi \).

\( L^2_\pi \) is a Hilbert space with Hermitian inner product:

\[
f \cdot f' := \sum_{x \in \pi} f(x) \overline{f'(x)}.\]

There are unitary \( \pi \)-actions on \( L^2_\pi \) by left or right translation.
Definition

$\mathcal{N}_\pi$ is the von Neumann algebra on $\pi$-equivariant bounded linear operators on $L^2\pi$. If $\varphi \in \mathcal{N}_\pi$, then

$$\text{tr}(\varphi) := \varphi(1) \cdot 1.$$ 

If $\Phi = (\varphi_{ij})$ is a $n \times n$ matrix with entries in $\mathcal{N}_\pi$, then

$$\text{tr}(\Phi) := \sum \text{tr}(\varphi_{ii}).$$

Similarly, if $F$ is a $\pi$-equivariant, bounded linear endomorphism on the direct sum of $n$ copies of $L^2\pi$, then $\text{tr}(F) := \text{tr}(\Phi)$ where $\Phi$ is any matrix representing $F$. 
Definition (von Neumann dimension)

Suppose $V$ is a $\pi$-stable closed subspace of $\bigoplus L^2\pi$ and $p_V : \bigoplus L^2\pi \to \bigoplus L^2\pi$ is orthogonal projection onto $V$. Put

$$\dim_\pi V := \text{tr}(p_V).$$

- $\dim_\pi V$ is a nonnegative real number.
- It is $= 0$ iff $V = 0$.
- Also, $\dim_\pi L^2\pi = 1$. 
(Co)homology with local coefficients

- $X$ a CW complex
- $\tilde{X}$ its universal cover
- $C_i(\tilde{X})$ the cellular $i$-chains on $\tilde{X}$
- $\pi = \pi_1(X)$. Suppose $M$ is a $\pi$-module.

\[
C_i(X; M) := C_i(\tilde{X}) \otimes_{\pi} M
\]
\[
C^i(X; M) := \text{Hom}_{\pi}(C_i(\tilde{X}), M)
\]

are the (co)chains with *local coefficients in $M$,*

$H_*(X; M)$ and $H^*(X; M)$ are the corresponding (co)homology groups.
To fix ideas, let’s stick to cohomology.

At first approximation $L^2$-cohomology means local coefficients in $L^2\pi$, i.e., $H^*(X; L^2\pi)$.

$C^*(X; L^2\pi)$ is a Hilbert space but $H^*(X; L^2\pi)$ need not be. Ker $\delta$ is a closed subspace but Im $\delta$ need not be.

Define

$$H^*(X; L^2\pi) := \text{Ker } \delta / \text{Im } \delta.$$

$H^*(X; L^2\pi)$ is a closed, $\pi$-stable subspace of $C^*(X; L^2\pi)$. (It is $= \text{Ker } \delta \cap (\text{Im } \delta)\perp$.)
If $X$ is a finite complex, then $C^i(X; L^2\pi)$ is a direct sum of finitely many copies of $L^2\pi$ (one for each $i$-cell of $X$).

So the closed, $\pi$-stable subspace $\mathcal{H}^i(X; L^2\pi)$ has a well-defined von Neumann dimension called the $i^{th}$ $L^2$-Betti number

$$\beta_i(X) := \dim_{\pi} \mathcal{H}^i(X; L^2\pi).$$

If $X$ is a finite complex then $C^*(X; L^2\pi)$ can be identified with the square summable cochains on $\tilde{X}$ (denoted by $L^2 C^*(\tilde{X})$). The corresponding (reduced) cohomology groups are denoted $L^2\mathcal{H}^*(\tilde{X})$. 
Example (The $L^2$-Betti numbers of $S^1$ vanish.)

$X = S^1$, $\tilde{X} = \mathbb{R}^1$, $\pi = \mathbb{Z}$.

A 0-cochain is a function on $\text{Vert}(\mathbb{R}^1)$ (=$\mathbb{Z}$); it is a cocycle iff it is constant. It is $L^2$ $\iff$ (constant $= 0$). Hence,

$H^0(S^1; L^2\mathbb{Z}) = 0 \implies H^0(X; L^2\pi) = 0 \implies \beta_0(S^1) = 0$.

A 1-chain is a function on $\text{Edge}(\mathbb{R}^1)$; it is a cycle iff it is constant. It is $L^2$ $\iff$ (constant $= 0$). Hence,

$H_1(S^1; L^2\mathbb{Z}) = 0 \implies H_1(X; L^2\pi) = 0 \implies \beta_1(S^1) = 0$. 

Corollary

All $L^2$-Betti numbers of $S^1 \times B$ vanish.

Proof.

K"unneth Formula.

Remark

Same is true for any $S^1$-bundle where $\pi_1(\text{fiber})$ goes injectively into $\pi_1(\text{total space})$. Also true for any mapping torus.
Rough idea of proof of theorems

Suppose $\mathcal{U} = \{U_i\}$ is a cover of $X$ by connected open subsets and $\mathcal{V}$ is a subcover s.t.

- $\mathcal{V} = \{U_i \in \mathcal{U} \mid \pi_1(U_i) \neq 1\}$.
- $\forall \sigma \in N(\mathcal{U}), \pi_1(U_{\sigma}) \to \pi_1(X) (= \pi)$ is injective.
  (Here $N(\mathcal{U})$ denotes the nerve of $\mathcal{U}$ and $U_{\sigma} = U_{i_1} \cap \cdots \cap U_{i_k}$, where $\sigma = \{i_1, \ldots, i_k\}$.)
- $\forall \sigma \in N(\mathcal{U}) - N(\mathcal{V}), U_{\sigma}$ is contractible.
- $\forall \sigma \in N(\mathcal{V}), U_{\sigma} = S^1 \times$ (something).
There is a Mayer-Vietoris spectral sequence converging to $H^*(X; L^2\pi)$ with $E_2$-term

$$E_2^{p,q} = H^p(N(U); H^q(U_\sigma; L^2(\pi_1(U_\sigma))),$$

where the coefficient system is the functor

$$\sigma \mapsto H^q(U_\sigma; L^2(\pi_1(U_\sigma))).$$

Hypotheses $\implies E_2^{p,q}$ is concentrated on the bottom row $q = 0$ and

$$E_2^{p,0} = H^p(N(U), N(V)) \otimes L^2\pi.$$

(Here we are ignoring terms with vanishing $L^2$-Betti numbers.)

So, $\beta_p(X) = b_p(N(U), N(V)).$
More on hyperplane arrangements

- $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{C}^n$ and $\Sigma$ is the union of hyperplanes.
- A *subspace of* $\mathcal{A}$ is a nonempty intersection of hyperplanes in $\mathcal{A}$.
- $L(\mathcal{A})$ (= $L$) is the poset of subspaces of $\mathcal{A}$.
- $\mathcal{A}$ is *central* if $L(\mathcal{A})$ has a minimum.
- Given $G \in L$, $\mathcal{A}_G := \{H \in \mathcal{A} \mid G \subset H\}$.
- A *small neighborhood* of a subspace $G$ is a convex tubular neighborhood $V$ s.t. $V \cap H = \emptyset$ for $H \in \mathcal{A} - \mathcal{A}_G$. (So, $\mathcal{A}_G$ is a central hyperplane arrangement in $V$.)
Choose a small neighborhood $V_H$ for each $H \in \mathcal{A}$. Put $\mathcal{V} := \{V_H\}_{H \in \mathcal{A}}$ and $\mathcal{V} := \bigcup V_H$.

**Proposition**

The spaces $\Sigma$, $\mathcal{V}$ and $|L|$ are homotopy equivalent. Each is homotopy equivalent to a wedge of $(l - 1)$-spheres (where $l = \text{rk}(\mathcal{A})$).

$\alpha(\mathcal{A})$ = the number of spheres.

**Proof of 1$^{\text{st}}$ sentence**

For each simplex $\sigma$ of $N(\mathcal{V})$, $V_{\sigma}$ is convex, hence contractible. So, $\Sigma \sim \mathcal{V} \sim N(\mathcal{V})$. $|L|$ is the geometric realization of poset $L$. It has an open cover with same nerve as $\mathcal{V}$ and with contractible intersections. So, $|L| \sim N(\mathcal{V})$. 
Proof of 2\textsuperscript{nd} sentence.

The proof is by induction on \( l \) and on Card(\( \mathcal{A} \)) using the “usual deletion-restriction argument.”

Choose \( H \in \mathcal{A} \). Put \( \mathcal{A}' := \mathcal{A} - \{H\} \), \( \mathcal{A}'' := \mathcal{A}|_H \).

\( \Sigma = \Sigma' \cup H \) and \( \Sigma' \cap H = \Sigma'' \).

If \( \mathcal{A} = \mathcal{A}'' \times \mathbf{C} \) done by induction on \( l \). Otherwise, \( \text{rk}(\mathcal{A}') = \text{rk}(\mathcal{A}) = \text{rk}(\mathcal{A}'') + 1 \) and

\[
\begin{array}{cccccc}
\longrightarrow & H_*(\Sigma') & \longrightarrow & H_*(\Sigma) & \longrightarrow & H_*(\Sigma, \Sigma') & \longrightarrow \\
& H_*(\mathcal{H}, \Sigma'') & \parallel & & & & \\
\end{array}
\]

By induction, \( \overline{H}_*(\Sigma') \) is concentrated in \( \text{dim } l - 1 \) and \( \overline{H}_*(\Sigma'') \) in \( \text{dim } l - 2 \). So \( \overline{H}_*(\Sigma) \) is concentrated in \( \text{dim } l - 1 \).
Extend \( \mathcal{V} \) to an open cover \( \mathcal{U} \) of \( \mathbb{C}^n \) by adding open balls in \( M(\mathcal{A}) := \mathbb{C}^n - \Sigma \).

Since each element of \( \mathcal{U} \) is convex,

\[
H^\ast(N(\mathcal{U}), N(\mathcal{V})) = H^\ast(\mathbb{C}^n, \Sigma).
\]

For each \( V_H \in \mathcal{V} \), put \( \hat{V}_H := V_H - H \). Define \( \hat{\mathcal{V}} := \{ \hat{V}_H \}_{H \in \mathcal{A}} \). Then \( \hat{\mathcal{V}} \) is open cover of \( V - \Sigma \).

Extend \( \hat{\mathcal{V}} \) to open cover \( \hat{\mathcal{U}} \) of \( M(\mathcal{A}) \) by adjoining the balls.

**Key point**

\[ N(\hat{\mathcal{V}}) = N(\mathcal{V}) \text{ and } N(\hat{\mathcal{U}}) = N(\mathcal{U}). \]
Lemma

Suppose $\mathcal{A}$ is a nonempty central arrangement. Then

$$M(\mathcal{A}) = S^1 \times (\text{something}).$$

Proof.

Can assume $\mathcal{A}$ is an arrangement of linear hyperplanes (through the origin). The Hopf bundle $M(\mathcal{A}) \to M(\mathcal{A})/S^1$ is trivial.
Main Theorem

The $L^2$-Betti numbers of $M(\mathcal{A})$ are all 0, except in dimension $l$, where $\beta_l(M(\mathcal{A})) = \alpha(\mathcal{A})$.

Proof.

- $\hat{U}$ is open cover of $M(\mathcal{A})$.
- $\forall \sigma \subset N(\hat{U}) - N(\hat{V}), U_\sigma \sim \ast$.
- $\forall \sigma \subset N(\hat{V}), U_\sigma = S^1 \times (\text{something})$.
- Use spectral sequence and fact that $H^*(N(\hat{U}), N(\hat{V})) = H^*\left(\mathbb{C}^n, \Sigma\right)$ to complete the proof.
Coxeter groups

- \((W, S)\) a Coxeter system.
- \(L := \{ T \subset S \mid \langle T \rangle \text{ is finite}\}; \) it is a simplicial cx.
- \(\exists\) a representation of \(W\) as a reflection group on \(\mathbb{C}^n\) s.t. \(W\) acts on an open convex set \(\Omega\) with finite isotropy subgps and freely on \(M\), the complement of the hyperplanes in \(\Omega\).

Artin groups

- \(A := \pi_1(M/W)\) is the associated Artin group.
- Conj. \(M/W \sim K(A, 1)\).
- If \(W\) is finite, \(M\) is complement of central arrangement and so has all \(L^2\)-Betti numbers \(= 0\).
Theorem (Davis-Leary)

\[ \beta_i(M/W) = b_i(\text{Cone}(L), L) \]
The pure symmetric automorphism group $P\Sigma_n$

- $P\Sigma_n \subset \text{Aut}(F_n)$.
- It acts on a contractible complex $M$ with isotropy subgps either trivial or free abelian.

Theorem (McCammond-Meier)

$$\beta_i(P\Sigma_{n+1}) = \begin{cases} 
0 & \text{if } i \neq n, \\
n^n & \text{if } i = n. 
\end{cases}$$