L^2-cohomology of hyperplane complements

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$\mathcal{A}$ a collection of affine hyperplanes in $\mathbf{C}^n$

Definition

rk($\mathcal{A}$), the *rank*, of $\mathcal{A}$ is the maximal codimension of a nonempty intersection of hyperplanes in $\mathcal{A}$. Usually denote rk($\mathcal{A}$) by l.
$\mathcal{A}$ is *essential* if $\mathrm{rk}(\mathcal{A}) = n$.

$$\Sigma(\mathcal{A}) := \bigcup_{H \in \mathcal{A}} H$$

$$M(\mathcal{A}) := \mathbf{C}^n - \Sigma(\mathcal{A})$$

Fact

$H^*(\mathbf{C}^n, \Sigma)$ vanishes except in dimension l $(= \mathrm{rk}(\mathcal{A}))$.
In fact, $\Sigma \sim \bigvee S^{l-1}$.

The number $\alpha(\mathcal{A})$

$$\begin{aligned}\alpha(\mathcal{A}) &:= \dim H_l(\mathbf{C}^n, \Sigma) \\ &:= b_l(\mathbf{C}^n, \Sigma) \\ &= \text{the number of spheres in the wedge}\end{aligned}$$

Example

Suppose $\mathcal{A}_{\mathbf{R}}$ is an essential hyperplane arrangement in $\mathbf{R}^n$. It divides $\mathbf{R}^n$ into convex regions.

- dim $H_n(\mathbf{R}^n, \Sigma(\mathcal{A}_{\mathbf{R}}))$ is the number of bounded components of $\mathbf{R}^n - \Sigma(\mathcal{A}_{\mathbf{R}})$.
- If $\mathcal{A}$ is complexification of $\mathcal{A}_{\mathbf{R}}$, then

$$(\mathbf{C}^n, \Sigma(\mathcal{A})) \sim (\mathbf{R}^n, \Sigma(\mathcal{A}_{\mathbf{R}})).$$

- So $\alpha(\mathcal{A})$ is the number of bounded components of $\mathbf{R}^n - \mathcal{A}_{\mathbf{R}}$.

Main Theorem

Suppose $\mathcal{A}$ is an arrangement of a finite number of affine hyperplanes in $\mathbf{C}^n$ with $\mathrm{rk}(\mathcal{A}) = l$. Then the L^2-Betti numbers of $M(\mathcal{A})$ (the complement of the hyperplanes) are all 0, except in dimension l, where

$$\beta_l(M(\mathcal{A})) = \alpha(\mathcal{A}).$$

Here $\beta_i(\ )$ denotes the i^{th} L^2-Betti number (to be defined later).

A similar theorem

The following is well-known.

Theorem

Suppose L is a "generic" flat complex line bundle over $M(\mathcal{A})$. Then $H^*(M(\mathcal{A}); L)$ vanishes except in dimension l and

$$\dim_{\mathbf{C}} H^l(M(\mathcal{A}); L) = \alpha(\mathcal{A}).$$

Basic idea

If L is a flat line bundle over S^1 giving a nonconstant local coefficient system, then $H^*(S^1; L) = 0$ for $* = 0, 1$.

Similarly, the basic idea for the Main Theorem is that the L^2-Betti numbers of S^1 vanish.

L^2-Betti numbers

The regular representation

π is a countable discrete gp.

- $$L^2\pi := \{f : \pi \to \mathbf{C} \mid \sum |f(x)|^2 < \infty\},$$
  where the sum is over all $x \in \pi$.
- $L^2\pi$ is a Hilbert space with Hermitian inner product:
  $$f \cdot f' := \sum_{x \in \pi} f(x)\overline{f'(x)}.$$
- There are unitary π-actions on $L^2\pi$ by left or right translation.

Definition

$\mathcal{N}\pi$ is the *von Neumann algebra* on π-equivariant bounded linear operators on $L^2\pi$. If $\varphi \in \mathcal{N}\pi$, then

$$\text{tr}(\varphi) := \varphi(1) \cdot 1.$$

If $\Phi = (\varphi_{ij})$ is a $n \times n$ matrix with entries in $\mathcal{N}\pi$, then

$$\text{tr}(\Phi) := \sum \text{tr}(\varphi_{ii}).$$

Similarly, if F is a π-equivariant, bounded linear endomorphism on the direct sum of n copies of $L^2\pi$, then $\text{tr}(F) := \text{tr}(\Phi)$ where Φ is any matrix representing F.

Definition (von Neumann dimension)

Suppose V is a π-stable closed subspace of $\bigoplus L^2\pi$ and $p_V : \bigoplus L^2\pi \to \bigoplus L^2\pi$ is orthogonal projection onto V. Put

$$\dim_\pi V := \operatorname{tr}(p_V).$$

- $\dim_\pi V$ is a nonnegative real number.
- It is $= 0$ iff $V = 0$.
- Also, $\dim_\pi L^2\pi = 1$.

(Co)homology with local coefficients

- X a CW complex $\qquad \widetilde{X}$ its universal cover
- $C_i(\widetilde{X})$ the cellular i-chains on $\widetilde{X}$
- $\pi = \pi_1(X)$. $\qquad$ Suppose M is a π-module.
- $$C_i(X; M) := C_i(\widetilde{X}) \otimes_\pi M$$ $$C^i(X; M) := \mathrm{Hom}_\pi(C_i(\widetilde{X}), M)$$
- are the (co)chains with *local coefficients in M*,
- $H_*(X; M)$ and $H^*(X; M)$ are the corresponding (co)homology groups.

L^2-(co)homology

- To fix ideas, let's stick to cohomology.
- At first approximation L^2-cohomology means local coefficients in $L^2\pi$, i.e., $H^*(X; L^2\pi)$.
- $C^*(X; L^2\pi)$ is a Hilbert space but $H^*(X; L^2\pi)$ need not be. $\operatorname{Ker}\delta$ is a closed subspace but $\operatorname{Im}\delta$ need not be.
- Define
$$\mathcal{H}^*(X; L^2\pi) := \operatorname{Ker}\delta / \overline{\operatorname{Im}\delta}.$$
- $\mathcal{H}^*(X; L^2\pi)$ is a closed, π-stable subspace of $C^*(X; L^2\pi)$. (It is $= \operatorname{Ker}\delta \cap (\operatorname{Im}\delta)^\perp$.)

- If X is a finite complex, then $C^i(X; L^2\pi)$ is a direct sum of finitely many copies of $L^2\pi$ (one for each i-cell of X).
- So the closed, π-stable subspace $\mathcal{H}^i(X; L^2\pi)$ has a well-defined von Neumann dimension called the i^{th} *L^2-Betti number*

$$\beta_i(X) := \dim_\pi \mathcal{H}^i(X; L^2\pi).$$

If X is a finite complex then $C^*(X; L^2\pi)$ can be identified with the square summable cochains on $\widetilde{X}$ (denoted by $L^2C^*(\widetilde{X})$). The corresponding (reduced) cohomology groups are denoted $L^2\mathcal{H}^*(\widetilde{X})$.

Example (The L^2-Betti numbers of S^1 vanish.)

$X = S^1, \qquad \widetilde{X} = \mathbf{R}^1, \qquad \pi = \mathbb{Z}$.
A 0-cochain is a function on Vert($\mathbf{R}^1$) ($= \mathbb{Z}$); it is a cocycle iff it is constant It is $L^2 \iff$ (constant $= 0$).) Hence,
$H^0(S^1; L^2\mathbb{Z}) = 0 \implies \mathcal{H}^0(X; L^2\pi) = 0 \implies \beta_0(S^1) = 0$.

A 1-chain is a function on Edge($\mathbf{R}^1$); it is a cycle iff it is constant. It is $L^2 \iff$ (constant $= 0$).) Hence,
$H_1(S^1; L^2\mathbb{Z}) = 0 \implies \mathcal{H}_1(X; L^2\pi) = 0 \implies \beta_1(S^1) = 0$.

Corollary

All L^2-Betti numbers of $S^1 \times B$ vanish.

Proof.

Künneth Formula. □

Remark

Same is true for any S^1-bundle where π_1(fiber) goes injectively into π_1(total space). Also true for any mapping torus.

Rough idea of proof of theorems

Suppose $\mathcal{U} = \{U_i\}$ is a cover of X by connected open subsets and $\mathcal{V}$ is a subcover s.t.

- $\mathcal{V} = \{U_i \in \mathcal{U} \mid \pi_1(U_i) \neq 1\}$.
- $\forall \sigma \in N(\mathcal{U})$, $\pi_1(U_\sigma) \to \pi_1(X)$ $(= \pi)$ is injective. (Here $N(\mathcal{U})$ denotes the nerve of $\mathcal{U}$ and $U_\sigma = U_{i_1} \cap \cdots \cap U_{i_k}$, where $\sigma = \{i_1, \ldots i_k\}$.)
- $\forall \sigma \in N(\mathcal{U}) - N(\mathcal{V})$, U_σ is contractible.
- $\forall \sigma \in N(\mathcal{V})$, $U_\sigma = S^1 \times$ (something).

There is a Mayer-Vietoris spectral sequence converging to $H^*(X; L^2\pi)$ with E_2-term

$$E_2^{p,q} = H^p(N(\mathcal{U}); H^q(U_\sigma; L^2(\pi_1(U_\sigma)),$$

where the coefficient system is the functor $\sigma \to H^q(U_\sigma; L^2(\pi_1(U_\sigma)).$

Hypotheses $\implies$ $E_2^{p,q}$ is concentrated on the bottom row $q = 0$ and

$$E_2^{p,0} = H^p(N(\mathcal{U}), N(\mathcal{V})) \otimes L^2\pi.$$

(Here we are iignoring terms with vanishing L^2-Betti numbers.) So, $\beta_p(X) = b_p(N(\mathcal{U}), N(\mathcal{V}))$.

More on hyperplane arrangements

- $\mathcal{A}$ is a hyperplane arrangement in $\mathbf{C}^n$ and Σ is the union of hyperplanes.
- A *subspace of* $\mathcal{A}$ is a nonempty intersection of hyperplanes in $\mathcal{A}$
- $L(\mathcal{A})$ $(= L)$ is the poset of subspaces of $\mathcal{A}$.
- $\mathcal{A}$ is *central* if $L(\mathcal{A})$ has a minimum.
- Given $G \in L$, $\mathcal{A}_G := \{H \in \mathcal{A} \mid G \subset H\}$.
- A *small neighborhood* of a subspace G is a convex tubular neighborhood V s.t. $V \cap H = \emptyset$ for $H \in \mathcal{A} - \mathcal{A}_G$. (So, $\mathcal{A}_G$ is a central hyperplane arrangement in V.)

Choose a small neighborhood V_H for each $H \in \mathcal{A}$. Put $\mathcal{V} := \{V_H\}_{H\in\mathcal{A}}$ and $V := \bigcup V_H$.

Proposition

The spaces Σ, V and $|L|$ are homotopy equivalent. Each is homotopy equivalent to a wedge of $(l-1)$-spheres (where $l = \mathrm{rk}(\mathcal{A})$).

$\alpha(\mathcal{A}) =$ the number of spheres.

Proof of 1st sentence

For each simplex σ of $N(\mathcal{V})$, V_σ is convex, hence contractible. So, $\Sigma \sim V \sim N(\mathcal{V})$. $|L|$ is the geometric realization of poset L. It has an open cover with same nerve as $\mathcal{V}$ and with contractible intersections. So, $|L| \sim N(\mathcal{V})$.

Proof of 2nd sentence.

- The proof is by induction on l and on Card($\mathcal{A}$) using the "usual deletion-restriction argument."
- Choose $H \in \mathcal{A}$. Put $\mathcal{A}' := \mathcal{A} - \{H\}$, $\mathcal{A}'' := \mathcal{A}|_H$.
- $\Sigma = \Sigma' \cup H \quad$ and $\quad \Sigma' \cap H = \Sigma''$.
- If $\mathcal{A} = \mathcal{A}'' \times \mathbf{C}$ done by induction on l. Otherwise, $\text{rk}(\mathcal{A}') = \text{rk}(\mathcal{A}) = \text{rk}(\mathcal{A}'') + 1$ and

$$\begin{array}{ccccccc} \longrightarrow & H_*(\Sigma') & \longrightarrow & H_*(\Sigma) & \longrightarrow & H_*(\Sigma, \Sigma') & \longrightarrow \\ & & & & & \| & \\ & & & & & H_*(H, \Sigma'') & \end{array}$$

- By induction, $\overline{H}_*(\Sigma')$ is concentrated in dim $l - 1$ and $\overline{H}_*(\Sigma'')$ in dim $l - 2$. So $\overline{H}_*(\Sigma)$ is concentrated in dim $l - 1$.

- Extend $\mathcal{V}$ to an open cover $\mathcal{U}$ of $\mathbf{C}^n$ by adding open balls in $M(\mathcal{A})$ $(= \mathbf{C}^n - \Sigma)$.
- Since each element of $\mathcal{U}$ is convex,

$$H^*(N(\mathcal{U}), N(\mathcal{V})) = H^*(\mathbf{C}^n, \Sigma).$$

- For each $V_H \in \mathcal{V}$, put $\widehat{V}_H := V_H - H$. $\widehat{\mathcal{V}} := \{\widehat{V}_H\}_{H\in\mathcal{A}}$. Then $\widehat{\mathcal{V}}$ is open cover of $V - \Sigma$.
- Extend $\widehat{\mathcal{V}}$ to open cover $\widehat{\mathcal{U}}$ of $M(\mathcal{A})$ by adjoining the balls.

Key point

$N(\widehat{\mathcal{V}}) = N(\mathcal{V})$ and $N(\widehat{\mathcal{U}}) = N(\mathcal{U})$.

Lemma

Suppose $\mathcal{A}$ is a nonempty central arrangement. Then

$$M(\mathcal{A}) = S^1 \times (\textit{something}).$$

Proof.

Can assume $\mathcal{A}$ is an arrangement of linear hyperplanes (through the origin). The Hopf bundle $M(\mathcal{A}) \to M(\mathcal{A})/S^1$ is trivial. □

Main Theorem

The L^2-Betti numbers of $M(\mathcal{A})$ are all 0, except in dimension l, where $\beta_l(M(\mathcal{A})) = \alpha(\mathcal{A})$.

Proof.

- $\widehat{\mathcal{U}}$ is open cover of $M(\mathcal{A})$.
- $\forall \sigma \subset N(\widehat{\mathcal{U}}) - N(\widehat{\mathcal{V}})$, $U_\sigma \sim *$.
- $\forall \sigma \subset N(\widehat{\mathcal{V}})$, $U_\sigma = S^1 \times$ (something).
- Use spectral sequence and fact that $H^*(N(\widehat{\mathcal{U}}), N(\widehat{\mathcal{V}})) = H^*(\mathbf{C}^n, \Sigma)$ to complete the proof.

□

Coxeter groups

- (W, S) a Coxeter system.
- $L := \{T \subset S \mid \langle T \rangle \text{is finite}\}$; it is a simplicial cx.
- $\exists$ a representation of W as a reflection group on $\mathbf{C}^n$ s.t. W acts on an open convex set Ω with finite isotropy subgps and freely on M, the complement of the hyperplanes in Ω.

Artin groups

- $A := \pi_1(M/W)$ is the associated *Artin group*.
- **Conj.** $M/W \sim K(A, 1)$.
- If W is finite, M is complement of central arrangement and so has all L^2-Betti numbers $= 0$.

Theorem (Davis-Leary)

$$\beta_i(M/W) = b_i(\mathrm{Cone}(L), L)$$

The pure symmetric automorphism group $P\Sigma_n$

- $P\Sigma_n \subset \mathrm{Aut}(F_n)$.
- It acts on a contractible complex M with isotropy subgps either trivial or free abelian.

Theorem (McCammond-Meier)

$$\beta_i(P\Sigma_{n+1}) = \begin{cases} 0 & \text{if } i \neq n, \\ n^n & \text{if } i = n. \end{cases}$$